

RELATIVE HYPERBOLICITY, CLASSIFYING SPACES, AND LOWER ALGEBRAIC K-THEORY.

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ABSTRACT. For Γ a relatively hyperbolic group, we construct a model for the universal space among Γ -spaces with isotropy on the family \mathcal{VC} of virtually cyclic subgroups of Γ . We provide a recipe for identifying the maximal infinite virtually cyclic subgroups of Coxeter groups which are lattices in $O^+(n, 1) = \text{Isom}(\mathbb{H}^n)$. We use the information we obtain to explicitly compute the lower algebraic K-theory of the Coxeter group Γ_3 (a non-uniform lattice in $O^+(3, 1)$). Part of this computation involves calculating certain Waldhausen Nil-groups for $\mathbb{Z}[D_2]$, $\mathbb{Z}[D_3]$.

1. INTRODUCTION

Let Γ be a discrete group and let \mathcal{F} be a family of subgroups Γ . A Γ -CW-complex E is a model for the classifying space $E_{\mathcal{F}}(\Gamma)$ with isotropy on \mathcal{F} if the H -fixed point sets E^H are contractible for all $H \in \mathcal{F}$ and empty otherwise. It is characterized by the universal property that for every Γ -CW complex X whose isotropy groups are all in \mathcal{F} , one can find an equivariant continuous map $X \rightarrow E_{\mathcal{F}}(\Gamma)$ which is unique up to equivariant homotopy. The two extreme cases are $\mathcal{F} = \mathcal{ALL}$, where $E_{\mathcal{F}}(\Gamma)$ can be taken to be a point, and $\mathcal{F} = \mathcal{TR}$, where $E_{\mathcal{F}}(\Gamma)$ is a model for $E\Gamma$.

For the family of finite subgroups, the space $E_{\mathcal{F}}(\Gamma)$ has nice geometric models for various classes of groups Γ . For instance, in the case where Γ is a discrete subgroup of a virtually connected Lie group [FJ93], where Γ is word hyperbolic group [MS02], an arithmetic group [BS73] [S79], the outer automorphism group of a free group [CV86], a mapping class groups [K83], or a one relator group [LS77]. For a thorough survey on classifying spaces, we refer the reader to Lück [Lu04].

One motivation for the study of these classifying spaces comes from the fact that they appear in the Farrell-Jones Isomorphism Conjecture about the algebraic K-theory of groups rings (see [FJ93]). Because of this conjecture the computations of the relevant K-groups can be reduced to the computation of certain equivariant homology groups applied to these classifying spaces for the family of finite groups \mathcal{FIN} and the family of virtually cyclic subgroups \mathcal{VC} (where a group is called *virtually cyclic* if it has a cyclic subgroup of finite index). In this paper we are interested in classifying spaces with isotropy in the family \mathcal{VC} of virtually cyclic subgroups.

We start out by defining the notion of an adapted family associated to a pair of families $\mathcal{F} \subset \mathcal{G}$ of subgroups. We then explain how, in the presence of an adapted family, a classifying space $E_{\mathcal{F}}(\Gamma)$ can be modified to obtain a classifying space $E_{\mathcal{G}}(\Gamma)$ for the larger family. In the situation we are interested in, the smaller family will be \mathcal{FIN} and the larger family will be \mathcal{VC} .

Of course, our construction is only of interest if we can find examples of groups where there is already a good model for $E_{\mathcal{FIN}}(\Gamma)$, and where an adapted family can

easily be found. For Γ a relatively hyperbolic group in the sense of Bowditch [Bo] (or equivalently relatively hyperbolic with the bounded coset penetration property in the sense of Farb [Fa98]), Dahmani has constructed a model for $E_{\mathcal{FLN}}(\Gamma)$. We show that the family consisting of all conjugates of peripheral subgroups, along with all maximal infinite virtually cyclic subgroups *not* conjugate into a peripheral subgroup, forms an adapted family for the pair $(\mathcal{FLN}, \mathcal{VC})$. Both the general construction, and the specific case of relatively hyperbolic groups, are discussed in Section 2 of this paper.

In order to carry out our construction of the classifying spaces for the family \mathcal{VC} for these groups, we need to be able to classify the maximal infinite virtually cyclic subgroups. We establish a systematic procedure to complete this classification for arbitrary Coxeter groups arising as lattices in $SO(n, 1)$. We next focus on the group Γ_3 , a Coxeter group which is known to be a non-uniform lattice in $SO(3, 1)$. In this specific situation, it is well known that the action of Γ_3 on \mathbb{H}^3 is a model for $E_{\mathcal{FLN}}(\Gamma_3)$, and that the group Γ_3 is hyperbolic relative to the cusp group (in this case the 2-dimensional crystallographic group $P4m$). Our construction now yields an 8-dimensional classifying space for $E_{\mathcal{VC}}(\Gamma_3)$. These results can be found in Section 3 of our paper.

Since the Farrell-Jones isomorphism conjecture is known to hold for lattices in $SO(n, 1)$, we can use our 8-dimensional classifying space for $E_{\mathcal{VC}}(\Gamma_3)$ to compute the lower algebraic K-theory of (the integral group ring of) Γ_3 . The computations are carried out in Section 4 of the paper, and yields an explicit result for $K_n(\mathbb{Z}\Gamma_3)$ when $n \leq -1$. The $\tilde{K}_0(\mathbb{Z}\Gamma_3)$ and $Wh(\Gamma_3)$ terms we obtain involve some Waldhausen Nil-groups.

In general, very little is known about Waldhausen Nil-groups. In Section 5, we provide a complete *explicit* determination of the Waldhausen Nil-groups that occur in $\tilde{K}_0(\mathbb{Z}\Gamma_3)$ and $Wh(\Gamma_3)$. The approach we take was suggested to us by F.T. Farrell, and combined with the computations in Section 4, yields the first example of a lattice in a semi-simple Lie group for which (1) the lower algebraic K-theory is *explicitly* computed, but (2) the relative assembly map induced by the inclusion $\mathcal{FLN} \subset \mathcal{VC}$ is *not* an isomorphism. The result of our computations can be summarized in the following:

Theorem 1.1. *Let $\Gamma_3 = O^+(3, 1) \cap GL(4, \mathbb{Z})$. Then the lower algebraic K-theory of the integral group ring of Γ_3 is given as follows:*

$$\begin{aligned} Wh(\Gamma_3) &\cong \bigoplus_{\infty} \mathbb{Z}/2 \\ \tilde{K}_0(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \bigoplus_{\infty} \mathbb{Z}/2 \\ K_{-1}(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z} \oplus \mathbb{Z}, \text{ and} \\ K_n(\mathbb{Z}\Gamma_3) &\cong 0, \text{ for } n < -1. \end{aligned}$$

where the expression $\bigoplus_{\infty} \mathbb{Z}/2$ refers to a countable infinite sum of $\mathbb{Z}/2$.

Finally, we note that most of the results in this paper apply in a quite general setting, and in particular to any Coxeter group that occurs as a lattice in $O^+(n, 1)$. Due to space constraints, we have only included the computations for the group Γ_3 . In a forthcoming paper, the authors will carry out the corresponding computations

for the lower algebraic K-theory of the remaining 3-simplex hyperbolic reflection groups.

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2. A MODEL FOR $E_{\mathcal{VC}}(\Gamma)$

Let Γ be a discrete group and \mathcal{F} be a family of subgroups of Γ closed under inclusion and conjugation, i.e. if $H \in \mathcal{F}$ then $gH'g^{-1} \in \mathcal{F}$ for all $H' \subset H$ and all $g \in \Gamma$. Some examples for \mathcal{F} are \mathcal{TR} , \mathcal{FIN} , \mathcal{VC} , and \mathcal{ALL} , which are the families consisting of the trivial group, finite subgroups, virtually cyclic groups, and all subgroups respectively.

Definition 2.1. Let Γ be any finitely generated group, and $\mathcal{F} \subset \tilde{\mathcal{F}}$ a pair of families of subgroups of Γ , we say that a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups of Γ is *adapted* to the pair $(\mathcal{F}, \tilde{\mathcal{F}})$ provided that:

- (1) For all $G, H \in \{H_\alpha\}_{\alpha \in I}$, either $G = H$, or $G \cap H \in \mathcal{F}$.
- (2) The collection $\{H_\alpha\}_{\alpha \in I}$ is *conjugacy closed* i.e. if $G \in \{H_\alpha\}_{\alpha \in I}$ then $gGg^{-1} \in \{H_\alpha\}_{\alpha \in I}$ for all $g \in \Gamma$.
- (3) Every $G \in \{H_\alpha\}_{\alpha \in I}$ is *self-normalizing*, i.e. $N_\Gamma(G) = G$.
- (4) For all $G \in \tilde{\mathcal{F}} \setminus \mathcal{F}$, there exists $H \in \{H_\alpha\}_{\alpha \in I}$ such that $G \leq H$.

Remark 2.2. *The collection $\{\Gamma\}$ consisting of just Γ itself is adapted to every pair $(\mathcal{F}, \tilde{\mathcal{F}})$ of families of subgroups of Γ . Our goal in this section is to show how, starting with a model for $E_{\mathcal{F}}(\Gamma)$, and a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups adapted to the pair $(\mathcal{F}, \tilde{\mathcal{F}})$, one can build a model for $E_{\tilde{\mathcal{F}}}(\Gamma)$.*

2.1. The Construction.

- (1) For each subgroup H of Γ , define the induced family of subgroups $\tilde{\mathcal{F}}_H$ of H to be $\tilde{\mathcal{F}}_H := \{F \cap H | F \in \tilde{\mathcal{F}}\}$. Note that if $g \in \Gamma$, conjugation by g maps H to $g^{-1}Hg \leq \Gamma$, and sends $\tilde{\mathcal{F}}_H$ to $\tilde{\mathcal{F}}_{g^{-1}Hg}$.

Let E_H be a model for the classifying space $E_{\tilde{\mathcal{F}}_H}(H)$ of H with isotropy in $\tilde{\mathcal{F}}_H$. Define a new space $E_{H;\Gamma} = \coprod_{\Gamma/H} E_H$. This space consists of the disjoint union of copies of E_H , with one copy for each left-coset of H in Γ . Note that E_H is contractible, but $E_{H;\Gamma}$ is not (since it is not path-connected).

- (2) Next, we define a Γ -action on the space $E_{H;\Gamma}$. Observe that each component of $E_{H;\Gamma}$ has a natural H -action; we want to “promote” this action to a Γ -action. By abuse of notation, let us denote by E_{gH} the component of $E_{H;\Gamma}$ corresponding to the coset $gH \in \Gamma/H$. Fix a collection $\{g_i H | i \in I\}$ of left-coset representatives, so that we now have an identification $E_{H;\Gamma} = \coprod_{i \in I} E_{g_i H}$. Now for $g \in \Gamma$, we define the g -action on $E_{H;\Gamma}$ as follows: g

maps each $E_{g_i H}$ to $E_{gg_i H} \cong E_{g_j H}$, for some $j \in I$. Recall that both $E_{g_i H}$ and $E_{g_j H}$ are copies of E_H , and that $g_j^{-1} g g_i \in H$; since H acts on E_H , we define the g -action from $E_{g_i H}$ to $E_{g_j H}$ to be:

$$\begin{array}{ccc} E_{g_i H} & \xrightarrow{g} & E_{g_j H} \\ \cong \downarrow & \circlearrowleft & \downarrow \cong \\ E_H & \xrightarrow{g_j^{-1} g g_i} & E_H \end{array}$$

that is the $g_j^{-1} g g_i$ -action via the identification $E_{g_i H} \cong E_H \cong E_{g_j H}$. Note that while the Γ -action one gets out if this depends on the choice of left-coset representatives, different choices of representatives yield conjugate actions, i.e. there exist a Γ -equivariant homeomorphism from $\coprod_{i \in I} E_{g_i H}$ and $\coprod_{i \in I} E_{g'_i H}$. Observe that the Γ -space $E_{H; \Gamma}$ is in fact nothing more than the Γ -space obtained by applying the Borel construction $\Gamma \times_H E_H$; for conciseness, we will denote this space by $Ind_H^\Gamma(E_H)$. We will be using the explicit description of $Ind_H^\Gamma(E_H)$ presented here in some of our proofs.

- (3) Now assume that we have a pair of families $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, and a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups of Γ adapted to the pair $(\mathcal{F}, \tilde{\mathcal{F}})$. Select a subcollection $\{\Lambda_\alpha\}_{\alpha \in \tilde{I}} \subseteq \{H_\alpha\}_{\alpha \in I}$ consisting of one subgroup from each conjugacy class in $\{H_\alpha\}_{\alpha \in I}$. We form the space \hat{X} by taking the join of X , a model for $E_{\mathcal{F}}(\Gamma)$, with the disjoint union of the spaces $Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha})$ defined as in (1), with the naturally induced Γ -action given in (2), i.e. $\hat{X} = X * \coprod_{\alpha \in \tilde{I}} Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha})$.

Theorem 2.3. *The space \hat{X} is a model for $E_{\tilde{\mathcal{F}}}(\Gamma)$.*

Proof. We start by noting that if a space A is contractible, and B is any space, then $A * B$ is contractible; in particular the space \hat{X} is contractible (since it is a join with the contractible space X).

We need to show two points:

- (1) \hat{X}^H is contractible if $H \in \tilde{\mathcal{F}}$.
- (2) $\hat{X}^H = \emptyset$ if $H \notin \tilde{\mathcal{F}}$.

Let us concentrate on the first point; assume $H \in \tilde{\mathcal{F}}$. Note that if $H \in \mathcal{F}$, then since X is a model for $E_{\mathcal{F}}(\Gamma)$, we have that X^H is contractible, and since $\hat{X}^H = (X * \coprod_{\alpha \in \tilde{I}} Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H = X^H * \coprod_{\alpha \in \tilde{I}} (Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H$, we conclude that \hat{X}^H is contractible.

Now assume $H \in \tilde{\mathcal{F}} \setminus \mathcal{F}$. From property (4) of an adapted family for the pair $(\tilde{\mathcal{F}}, \mathcal{F})$ (see Definition 2.1), there exists Λ_α for some $\alpha \in \tilde{I}$, such that H can be conjugated into Λ_α . We now make the following claim:

Claim 2.4. *$(Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H$ is contractible, and $(Ind_{\Lambda_\beta}^\Gamma(E_{\Lambda_\beta}))^H$, for all $\beta \neq \alpha$.*

Note that if we assumed Claim 2.4, we immediately get that \hat{X}^H is contractible. Indeed, since X is a model for $E_{\mathcal{F}}(\Gamma)$ and $H \notin \mathcal{F}$, $X^H = \emptyset$, and we obtain:

$$\hat{X}^H = (Ind_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H * X^H * \prod_{\beta \neq \alpha} (Ind_{\Lambda_\beta}^\Gamma(E_{\Lambda_\beta}))^H$$

$$= (\text{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H * \emptyset * \prod_{\beta \neq \alpha} \emptyset \cong (\text{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H.$$

Proof of Claim 2.4. We first show that, in $\text{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}) = \prod_{i \in I} E_{g_i \Lambda_\alpha}$, there exists a connected component that is fixed by H . Since H is conjugate to Λ_α by some $k \in \Gamma$, consider the Λ_α coset $g_i \Lambda_\alpha$ containing k . We claim that for all $h \in H$, we have $h \cdot g_i \Lambda_\alpha = g_i \Lambda_\alpha$. First observe the following easy criterion for an element $h \in \Gamma$ to map a coset $g_i \Lambda_\alpha$ to itself:

$$h \cdot g_i \Lambda_\alpha = g_i \Lambda_\alpha \iff g_i^{-1} h g_i \cdot \Lambda_\alpha = \Lambda_\alpha \iff g_i^{-1} h g_i \in \Lambda_\alpha.$$

Now in our specific situation, by the choice of $g_i \Lambda_\alpha$, we have $k = g_i \tilde{k} \in g_i \Lambda_\alpha$, where k conjugates h into Λ_α , and $\tilde{k} \in \Lambda_\alpha$. Since $g_i = k \tilde{k}^{-1}$, substituting we see that $g_i^{-1} h g_i = \tilde{k} (k^{-1} h k) \tilde{k}^{-1}$. But by construction, we have that $k^{-1} h k \in \Lambda_\alpha$, and since $\tilde{k} \in \Lambda_\alpha$, we conclude that $g_i^{-1} h g_i \in \Lambda_\alpha$. So by the criterion above, we see that indeed, every element of H maps the component $E_{g_i \Lambda_\alpha}$ to itself.

Our next step is to note that an element $g \in H$ with $g \neq 1$ can map *at most one* of the $E_{g_i \Lambda_\alpha}$ to itself. In fact, if $g \in H$ maps $E_{g_i \Lambda_\alpha}$ to itself and $E_{g_j \Lambda_\alpha}$ to itself, then from the definition of the action given in step (2) of the construction, we have that $H \subset g_i \Lambda_\alpha g_i^{-1}$, and $H \subset g_j \Lambda_\alpha g_j^{-1}$. From condition (1) in Definition 2.1, we have that either $g_i \Lambda_\alpha g_i^{-1} = g_j \Lambda_\alpha g_j^{-1}$, or $H \subset g_i \Lambda_\alpha g_i^{-1} \cap g_j \Lambda_\alpha g_j^{-1} \in \mathcal{F}$. But this second possibility can not occur, as $H \in \tilde{\mathcal{F}} \setminus \mathcal{F}$. Hence we have that $g_i \Lambda_\alpha g_i^{-1} = g_j \Lambda_\alpha g_j^{-1}$. Therefore $(g_j^{-1} g_i) \Lambda_\alpha (g_i^{-1} g_j) = \Lambda_\alpha$, that is, $g_j^{-1} g_i \in N_\Gamma(\Lambda_\alpha)$. But the self-normalizing condition (see condition (3) in Definition 2.1) forces $g_j^{-1} g_i \in \Lambda_\alpha$, thus $g_i \in g_j \Lambda_\alpha$, i.e. the cosets $g_i \Lambda_\alpha$, and $g_j \Lambda_\alpha$ had to be the same to begin with.

Up to this point, we know that the H -action on $\text{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha})$ maps the $E_{g_i \Lambda_\alpha}$ component to itself, and permutes all the others. This implies $(\text{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H = E_{g_i \Lambda_\alpha}^H$. But the H -action on $E_{g_i \Lambda_\alpha}$ coincides with the $g_i^{-1} H g_i$ -action on E_{Λ_α} (see step (2) of the construction). From step (1) of the construction, since $g_i^{-1} H g_i \in \tilde{\mathcal{F}}_{\Lambda_\alpha}$, and E_{Λ_α} is a model for $E_{\tilde{\mathcal{F}}_{\Lambda_\alpha}}(\Lambda_\alpha)$, we have that $E_{g_i \Lambda_\alpha}^H = E_{\Lambda_\alpha; \Gamma}^H$ is contractible. \square

This completes the verification of the first point, i.e. \hat{X}^H is contractible if $H \in \tilde{\mathcal{F}}$.

We now verify the second point, i.e. $\hat{X}^H = \emptyset$ if $H \notin \tilde{\mathcal{F}}$. But this is considerably easier. In fact, if $H \notin \tilde{\mathcal{F}}$, then $H \notin \mathcal{F}$, and $X^H = \emptyset$; so let us focus on the H -action on the individual $E_{\Lambda_\alpha; \Gamma}$. By the discussion on the proof of Claim 2.4, the H -action on $E_{\Lambda_\alpha; \Gamma}$ will have empty fixed point set provided that H can not be conjugated into Λ_α . From the separability condition (1) and conjugacy closure condition (2) in the definition of an adapted family (see Definition 2.1), H can be conjugated into at most one of the Λ_α ; let k^{-1} be the conjugating element, i.e. $k^{-1} H k \subseteq \Lambda_\alpha$. Then H fixes precisely one component $E_{\Lambda_\alpha; \Gamma}$, namely the component corresponding to $E_{k \Lambda_\alpha}$. Furthermore, the H -action on $E_{k \Lambda_\alpha} \cong E_{\Lambda_\alpha}$ is via the $k^{-1} H k$ -action on E_{Λ_α} which is a model for $E_{\tilde{\mathcal{F}}_{\Lambda_\alpha}}(\Lambda_\alpha)$. Since $H \notin \tilde{\mathcal{F}}$, then $k^{-1} H k \notin \tilde{\mathcal{F}}$, and this implies $k^{-1} H k \notin \tilde{\mathcal{F}}_{\Lambda_\alpha}$, therefore $E_{k \Lambda_\alpha}^H = \emptyset$. This immediately gives $E_{\Lambda_\alpha; \Gamma}^H = \emptyset$, and hence $\hat{X}^H = \emptyset$. This completes the verification of the second point, and hence the proof of Theorem 2.3. \square

Remark 2.5. *As we mentioned in Remark 2.2, the collection $\{\Gamma\}$ consisting of just Γ itself is adapted to every pair $(\mathcal{F}, \tilde{\mathcal{F}})$ of families of subgroups. Looking at*

our construction of \widehat{X} , and applying it to $\{\Gamma\}$, we get \widehat{X} is the join of $E_{\mathcal{F}}(\Gamma)$ and $E_{\widetilde{\mathcal{F}}}(\Gamma)$. While this is indeed an $E_{\widetilde{\mathcal{F}}}(\Gamma)$, the construction is not useful, since it needs an $E_{\widetilde{\mathcal{F}}}(\Gamma)$ to produce an $E_{\widetilde{\mathcal{F}}}(\Gamma)$. In order to be useful, the construction requires a “non-trivial” adapted family of subgroups.

2.2. Relatively hyperbolic groups. Our next goal is to exhibit an adapted family of subgroups in the special case where $\mathcal{F} = \mathcal{FTN}$, $\widetilde{\mathcal{F}} = \mathcal{VC}$, and Γ is a relatively hyperbolic group (in the sense of Bowditch [Bo]). We refer the reader to Bowditch for the definition, and will content ourselves with mentioning that the following classes of groups are relatively hyperbolic:

- (1) free products of finitely many groups (relative to the factors),
- (2) geometrically finite isometry groups of Hadamard manifolds of pinched negative curvature (relative to maximal parabolic subgroups),
- (3) CAT(0)-groups with isolated flats (relative to the flat stabilizers), by recent work of Hruska-Kleiner [HK05],
- (4) fundamental groups of spaces obtained via strict relative hyperbolizations (relative to the fundamental groups of the subspaces the hyperbolization is taken relative to), by a recent paper of Belegardek [Bel].

Theorem 2.6. *Let Γ be a group which is hyperbolic relative to subgroups $\{H_i\}_{i=1}^k$, in the sense of Bowditch [Bo] (or equivalently, a relatively hyperbolic group with the bounded coset penetration property in the sense of Farb [Fa98]). Consider the collection of subgroups of Γ consisting of:*

- (1) All conjugates of H_i (these will be called peripheral subgroups).
- (2) All maximal virtually infinite cyclic subgroups V such that $V \not\subseteq gH_i g^{-1}$, for all $i = 1, \dots, k$, and for all $g \in \Gamma$.

Then this is an adapted family for the pair $(\mathcal{FTN}, \mathcal{VC})$.

Proof. We first observe that our collection of subgroups clearly satisfies condition (4) for an adapted family (i.e. every virtually cyclic subgroup is contained in one of our subgroups). Furthermore, since the collection is conjugacy closed by construction, we see that this collection of subgroups satisfies condition (2) for an adapted family. We are left with establishing conditions (1) and (3).

Let us focus on condition (1): any two subgroups in our collection have finite intersection. A consequence of relative hyperbolicity is that any two peripheral subgroups intersect in a finite group (see Section 4 in Bowditch [Bo]). Hence to establish (1), it is sufficient to show:

- if V_1, V_2 are a pair of distinct maximal infinite virtually cyclic subgroups which *do not* lie inside peripheral subgroups, then $V_1 \cap V_2$ is finite, and
- if V is a maximal infinite virtually cyclic subgroup which *does not* lie in a peripheral subgroup, then it intersects each H_i in a finite subgroup.

To see the first of these two cases, assume that V_1, V_2 are as above, and that $|V_1 \cap V_2| = \infty$. Choose an element $g \in V_1 \cap V_2$ of infinite order, and note that g is *hyperbolic*, in the sense that it *cannot* be conjugated into one of the cusps. Osin has established (Theorem 4.3 in [Os1]) that the \mathbb{Z} -subgroup $\langle g \rangle$ generated by such a g lies in a *unique* maximal infinite virtually cyclic subgroup. This immediately forces $V_1 = V_2$.

Now consider the second of these two cases, and assume $V \cap H_i$ is infinite. Then picking an element $g \in V \cap H_i$ of infinite order, we observe that g is of parabolic

type (since it lies in H_i). On the other hand, V contains a finite index subgroup isomorphic to \mathbb{Z} generated by an element h of hyperbolic type. Now consider the intersection $\langle h \rangle \cap \langle g \rangle$, and observe that this intersection is non-empty (since both $\langle h \rangle$ and $\langle g \rangle$ are finite index subgroups in V), hence contains an element which is simultaneously of hyperbolic type and of parabolic type. But this is impossible, giving us a contradiction.

This leaves us with establishing property (3) of an adapted family. We first note that in a relatively hyperbolic group, the peripheral subgroups are self-normalizing (see Section 4 in Bowditch [Bo]). So we merely need to establish that the maximal infinite virtually cyclic subgroups V of hyperbolic type are self-normalizing. But Osin has established (Theorem 1.5 and Corollary 1.7 in [Os2]) that if $g \in \Gamma$ is hyperbolic and has infinite order, and if V is the unique maximal virtually infinite cyclic subgroup containing g , then V has the property that $V \cap gVg^{-1}$ is *finite* for every $g \in \Gamma - V$. Since V is infinite, this immediately implies that $V = N_\Gamma(V)$. This establishes property (3), and completes the proof of the Theorem. \square

Remark 2.7. (1) *Osin has defined a notion of relative hyperbolicity in terms of relative Dehn functions. The results we cite in the proof of Theorem 2.6 make use of his definition. However, in a previous paper, Osin has established that for finitely generated groups, his notion of relative hyperbolicity coincides with Bowditch's definition (Theorem 1.5 in [Os1]). As such, his results apply to the setting in which we are interested.*

(2) *We can always view a hyperbolic group Γ as a group which is hyperbolic relative to the trivial subgroup $\{1\}$. In this specific case, our construction for the classifying space $E_{\mathcal{VC}}(\Gamma)$ coincides with the classifying space constructed by Juan-Pineda and Leary in [JL].*

(3) *In Theorem 2.3, note that it is important that we are using Bowditch's notion of relative hyperbolic group. If we were instead using Farb's notion without the bounded coset penetration property (known as weak relative hyperbolicity), the peripheral subgroups are no longer self-normalizing. Indeed, weak relative hyperbolicity is preserved if one replaces the peripheral subgroups by subgroups of finite index (while in contrast, relative hyperbolicity is not).*

3. THE MAXIMAL INFINITE VIRTUALLY CYCLIC SUBGROUPS OF Γ_3

Let Γ_3 be the subgroup of $O^+(3, 1)$ that preserves the standard integer lattice $\mathbb{Z}^4 \subset \mathbb{R}^{3,1}$, that is, $\Gamma_3 = O^+(3, 1) \cap GL(4, \mathbb{Z})$.

Since Γ_3 is a subgroup of the discrete group $GL(4, \mathbb{Z})$, it is also a discrete group of $O^+(3, 1)$. The group Γ_3 is a hyperbolic Coxeter, noncompact, 3-simplex reflection group with fundamental domain its defining Coxeter 3-simplex Δ^3 (see [R94, pg. 301]).

The group Γ_3 is part of a nice family of discrete subgroups of isometries of hyperbolic n -space for which the Farrell-Jones Isomorphism Conjecture in lower algebraic K -theory holds, that is $H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{K}\mathbb{Z}^{-\infty}) \cong K_n(R\Gamma_3)$ for $n < 2$ (see [Or04, Theorem 2.1]). One of our intentions in this paper is to use this result and our model for $E_{\mathcal{VC}}(\Gamma)$ constructed in Section 2 to explicitly compute the lower algebraic K -theory of the integral group ring $\mathbb{Z}\Gamma_3$. In order to accomplish this task, we must first classify up to isomorphism the family \mathcal{VC} of all virtually cyclic subgroups of Γ_3 . At this point we will like to direct the reader to [Or04] for more

information on the relatively hyperbolic groups $\Gamma_n = O^+(n, 1) \cap GL(n + 1, \mathbb{Z})$, for $n = 3, \dots, 9$.

We now proceed to classify up to conjugacy all maximal virtually infinite cyclic subgroups of Γ_3 of hyperbolic type. The infinite virtually cyclic subgroups of parabolic type (or cusp groups) are virtually infinite cyclic subgroups of the cusp group $P4m$, a 2-dimensional crystallographic group. These groups have already been classified by Pearson in [Pe98, Lemma 2.3]. For the maximal virtually infinite cyclic subgroups of hyperbolic type, our approach to the classification problem is *geometric*, as opposed to previous approaches which were algebraic in nature.

Lemma 3.1. *Let $Q \leq \Gamma_3$ be a infinite virtually cyclic subgroup of Γ_3 of hyperbolic type. Then there exist a geodesic $\gamma \subset \mathbb{H}^3$ such that $Q \leq \text{Stab}_{\Gamma_3}(\gamma)$.*

Proof. Q is infinite virtually cyclic, hence contains an infinite cyclic subgroup H of finite index. Since Q is of hyperbolic type, H stabilizes some geodesic γ in \mathbb{H}^3 . We want to show that $Q \cdot \gamma = \gamma$. Let $g \in Q$ satisfy $g \cdot \gamma \neq \gamma$, and let $g \cdot \gamma = \gamma'$. Note that $gHg^{-1} \leq Q$ stabilizes γ' . Since H and gHg^{-1} are both of finite index in Q , their intersection is an infinite cyclic subgroup, call it K . Since γ and γ' are distinct geodesics and K acts by isometries, we get $|K| < \infty$, contradicting $K \cong \mathbb{Z}$. \square

Note that a subgroup of the type $\text{Stab}_{\Gamma_3}(\gamma)$ is always virtually cyclic. Lemma 3.1 now reduces the problem of classifying maximal virtually infinity cyclic subgroups of Γ_3 of hyperbolic type to the more geometric question of finding stabilizers of geodesics $\gamma \subset \mathbb{H}^3$.

In order to do this, first we observe that the Γ_3 -action on \mathbb{H}^3 induces a tessellation of \mathbb{H}^3 by copies of the fundamental domain Δ^3 of Γ_3 . This tessellation is determined by a collection of totally geodesic copies of \mathbb{H}^2 lying in \mathbb{H}^3 (each of them corresponding to the faces of the Δ^3) intersecting in a family of geodesics (corresponding to the 1-skeleton of Δ^3).

Now the stabilizers of the geodesics will depend on the behavior of the geodesic; more precisely, will depend on the intersection of the geodesic with the tessellation of \mathbb{H}^3 by copies of the fundamental domain Δ^3 . Denote by $p : \mathbb{H}^3 \rightarrow \mathbb{H}^3/\Gamma_3 \cong \Delta^3$ the canonical projection from \mathbb{H}^3 to the fundamental domain Δ^3 . We first establish two easy lemmas.

Lemma 3.2. *If $\text{Stab}_{\Gamma_3}(\gamma)$ is infinite, then $p(\gamma) \subset \Delta^3$ is periodic.*

Proof. This follows from the fact that if $\text{Stab}_{\Gamma_3}(\gamma)$ is virtually infinite cyclic, then it contains an element of infinite order, which must act on γ by translations. If $g \in \Gamma_3$ is this element, then $p(x) = p(g \cdot x) \in \Delta^3$, forcing periodicity of $p(\gamma)$. \square

Lemma 3.3. *Let $\gamma \subset \mathbb{H}^3$ be an arbitrary geodesic, $x \in \gamma$ an arbitrary point, and $g \in \text{Stab}_{\Gamma_3}(\gamma) \subset \Gamma_3$ an arbitrary element. Then we have $p(x) = p(g \cdot x)$.*

Note that Lemma 3.3 is immediate, since any two points (for instance x , and $g \cdot x$) have the same image in Δ^3 provided they differ by an element in Γ_3 (by the definition of the fundamental domain).

Definition 3.4. For γ any geodesic in \mathbb{H}^3 , we say that:

- (1) γ is of *type I*, if γ is equal to the intersection of two of the totally geodesic copies of \mathbb{H}^2 inside \mathbb{H}^3 .

- (2) γ is of *type II*, if γ lies entirely within one of the totally geodesic copies of $\mathbb{H}^2 \subset \mathbb{H}^3$, but not lying in the intersection of two of the \mathbb{H}^2 's.
- (3) γ is of *type III*, if γ does not lie within one of the totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$ arising from the tessellation.

Note that the type of a geodesic can easily be seen in terms of its image under the projection map p . Indeed, geodesics of *type I* are those for which $p(\gamma)$ lies in the 1-skeleton of Δ^3 , those of *type II* have $p(\gamma)$ lying in $\partial\Delta^3$, but not in the 1-skeleton of Δ^3 , and those of *type III* have non-trivial intersection with $\text{Int}(\Delta^3)$.

We now make the following easy observation: given any geodesic $\gamma \subset \mathbb{H}^3$, invariant under the isometric action of a Coxeter group Γ on \mathbb{H}^3 , there exists a short exact sequence:

$$0 \rightarrow \text{Fix}_\Gamma(\gamma) \rightarrow \text{Stab}_\Gamma(\gamma) \rightarrow \text{Isom}_{\Gamma,\gamma}(\mathbb{R}) \rightarrow 0,$$

where $\text{Fix}_\Gamma(\gamma) \leq \Gamma$ is the subgroup of Γ that fixes γ pointwise, and $\text{Isom}_{\Gamma,\gamma}(\mathbb{R})$ is the induced action of $\text{Stab}_\Gamma(\gamma)$ on γ (identified with an isometric copy of \mathbb{R}). Furthermore we have:

- (1) the group $\text{Isom}_{\Gamma,\gamma}(\mathbb{R})$, being a discrete cocompact subgroup of the isometry group of \mathbb{R} , has to be isomorphic to \mathbb{Z} or D_∞ .
- (2) the group $\text{Fix}_\Gamma(\gamma) \leq \Gamma$ acts trivially on the geodesic γ , hence can be identified with a finite Coxeter group acting on the unit normal bundle to a point $p \in \gamma$.

In particular, since Γ_3 is a Coxeter group, we can use this short exact sequence to get an easy description of stabilizers of type II and type III geodesics.

Proposition 3.5. *Let γ be a geodesic of type III, with $\text{Stab}_{\Gamma_3}(\gamma)$ infinite virtually cyclic, then $\text{Stab}_{\Gamma_3}(\gamma)$ is isomorphic to either \mathbb{Z} or D_∞ .*

Proof. Since $p(\gamma) \cap \text{Int}(F) \neq \emptyset$, we have that γ enters the interior of a fundamental domain in \mathbb{H}^3 . If $g \in \Gamma_3$ is arbitrary, then the fundamental domain and its g -translate have disjoint interiors; this forces $\text{Fix}_{\Gamma_3}(\gamma) = 0$. From the short exact sequence mentioned above, we immediately obtain $\text{Stab}_{\Gamma_3}(\gamma) \cong \mathbb{Z}$ or D_∞ , as desired. \square

Proposition 3.6. *If γ is of type II, and $\text{Stab}_{\Gamma_3}(\gamma)$ is virtually infinite cyclic, then $\text{Stab}_{\Gamma_3}(\gamma)$ is either $\mathbb{Z} \times \mathbb{Z}/2$ or $D_\infty \times \mathbb{Z}/2$.*

Proof. Let us consider the short exact sequence:

$$0 \rightarrow \text{Fix}_{\Gamma_3}(\gamma) \rightarrow \text{Stab}_{\Gamma_3}(\gamma) \rightarrow \text{Isom}_{\Gamma_3,\gamma}(\mathbb{R}) \rightarrow 0$$

where $\text{Isom}_{\Gamma_3,\gamma}(\mathbb{R})$ is either \mathbb{Z} or D_∞ . Now let us focus on $\text{Fix}_{\Gamma_3}(\gamma)$. Note that since γ is of type II, it lies in one of the totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$, but does not lie in the intersection of two such \mathbb{H}^2 . We now have that $\text{Fix}_{\Gamma_3}(\gamma) \cong \mathbb{Z}/2$, given by the reflection in the \mathbb{H}^2 containing γ . Thus $\text{Stab}_{\Gamma_3}(\gamma)$ fits into one of the short exact sequences:

- (1) $0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Stab}_{\Gamma_3}(\gamma) \rightarrow \mathbb{Z} \rightarrow 0$
- (2) $0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Stab}_{\Gamma_3}(\gamma) \rightarrow D_\infty \rightarrow 0$

We next proceed to show that $\text{Stab}_{\Gamma_3}(\gamma)$ is either isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$ or $D_\infty \times \mathbb{Z}/2$ according to whether we are in case (1) or (2). Let us consider case (1), and pick $g \in \text{Stab}_{\Gamma_3}(\gamma)$ that acts via translation along γ . Consider the $\mathbb{H}^2 \subset$

\mathbb{H}^3 containing γ , and note that $g(\mathbb{H}^2)$ is another one of the totally geodesic \mathbb{H}^2 containing γ . But since γ is of type II there is a unique such totally geodesic \mathbb{H}^2 , hence $g(\mathbb{H}^2) = \mathbb{H}^2$, i.e. g leaves the totally geodesic \mathbb{H}^2 invariant. This immediately implies that the subgroup generated by g commutes with the reflection in the \mathbb{H}^2 , yielding that $\text{Stab}_{\Gamma_3}(\gamma) \cong \mathbb{Z} \times \mathbb{Z}/2$, as desired. The argument in case (2) is nearly identical; pick a pair $g, h \in \text{Stab}_{\Gamma_3}(\gamma)$ whose image are the generators for D_∞ , and observe that g, h must be reflections in a pair of \mathbb{H}^2 's both of which are perpendicular to γ . This implies that the hyperplanes are both orthogonal to the totally geodesic \mathbb{H}^2 containing γ , and hence that g, h both map the \mathbb{H}^2 to itself. This yields that g, h both commute with reflection in the corresponding \mathbb{H}^2 , and hence that $\text{Stab}_{\Gamma_3}(\gamma) \cong D_\infty \times \mathbb{Z}/2$. \square

To finish the classification of maximal virtually infinite cyclic subgroup of Γ_3 , we are left with the study of stabilizers of geodesic of type I. Recall that geodesics γ of type I are those for which $p(\gamma)$ lies in the 1-skeleton of Δ^3 . In this situation, the short exact sequence is not too useful for our purposes. Instead, we switch our viewpoint, and appeal instead to Bass-Serre theory [S80].

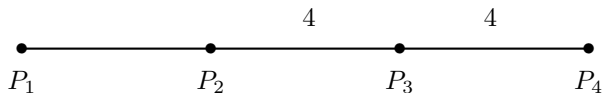


FIGURE 1. The Coxeter graph of $\Gamma_3 = [3, 4, 4]$

Recall that the group Γ_3 is a Coxeter group with Coxeter graph given in Figure 1. The fundamental domain for the Γ_3 -action on \mathbb{H}^3 is a 3-simplex Δ^3 in \mathbb{H}^3 , with one ideal vertex. The group $\Gamma_3 \leq \text{Isom}(\mathbb{H}^3)$ is generated by a reflection in the hyperplanes (totally geodesic \mathbb{H}^3) extending the four faces of the 3-simplex. Each reflection corresponds to a generator, and the faces of the 3-simplex are labelled P_1, \dots, P_4 according to the corresponding generator. Note that the angles between the faces can be read off from the Coxeter graph. The possible type I geodesics correspond to geodesics extending the intersections of pairs of faces (and hence, there are at most six such geodesics).

Proposition 3.7. *Let γ be the geodesic extending the intersection of the two hyperplanes $P_1 \cap P_3$, and let η be the geodesic extending $P_1 \cap P_2$. Then $\text{Stab}_{\Gamma_3}(\gamma) \cong D_2 \times D_\infty$, and $\text{Stab}_{\Gamma_3}(\eta) \cong D_3 \times D_\infty$. Furthermore, for the geodesics extending the remaining edges in the one skeleton, the stabilizers are finite.*

Proof. Our procedure for identifying the stabilizers of the geodesics of type I relies on the observation that when these stabilizers act cocompactly on the geodesics, Bass-Serre theory gives us an easy description of the corresponding stabilizer. Indeed, if the quotient of the geodesic in the fundamental domain is a segment, then the stabilizer of the geodesic is an amalgamation of the vertex stabilizers, amalgamated over the edge stabilizer. Let us carry out this procedure in the specific case of Γ_3 . Notice that three of the six edges in the 1-skeleton of the fundamental domain Δ^3 are actually geodesic rays (since one of the vertices is an ideal vertex), and hence their geodesic extensions will have finite stabilizers. This leaves us with three edges in the 1-skeleton to worry about, namely those corresponding to the

edges $P_1 \cap P_2$, $P_1 \cap P_3$, and $P_1 \cap P_4$. Let us first focus on the edge corresponding to the intersection $P_1 \cap P_4$. We claim that the geodesic extending this edge projects into a non-compact segment inside Δ^3 . Indeed, if one considers the link of the vertex $P_1 \cap P_2 \cap P_4$ (a 2-dimensional sphere), we note that it has a natural action by a parabolic Coxeter group, namely the stabilizer of this vertex. Corresponding to this action is a tessellation of S^2 by geodesic triangles, where the vertices of the triangles correspond to the directed edges from the given vertex to (translates) of the remaining vertices from the fundamental domain. In other words, each vertex of the tessellation of S^2 comes equipped with a *label* identifying which directed edge in the fundamental domain it corresponds to. But now in the tessellation of the vertex $P_1 \cap P_2 \cap P_4$, it is easy to see that the vertex antipodal to the one corresponding to the edge $P_1 \cap P_4$, has a label indicating that it projects to the edge $P_2 \cap P_4$. This immediately allows us to see that the geodesic ray extending the edge $P_1 \cap P_4$ does *not* project to a closed loop in Δ^3 , and hence has finite stabilizer.



FIGURE 2

So we are left with identifying the stabilizers of the two geodesics γ and η . Let us consider the geodesic γ , and observe that the segment that is being extended joins the vertex $P_1 \cap P_2 \cap P_3$ to the vertex $P_1 \cap P_3 \cap P_4$. When looking at the tilings of S^2 one obtains at each of these two vertices, we find that in both cases, the label of the vertex corresponding to the edge $P_1 \cap P_3$ is antipodal to a vertex with the same labeling. This tells us that the geodesic γ projects precisely to the segment $P_1 \cap P_3$ in the fundamental domain Δ^3 . Now the subgroup of $Stab_{\Gamma_3}(\gamma)$ that fixes the vertex $P_1 \cap P_2 \cap P_3$ is the subgroup of the Coxeter group stabilizing the vertex, that additionally fixes the pair of antipodal vertices in the tiling of S^2 corresponding to γ . But it is immediate by looking at the tiling that this subgroup at each of the two vertices is just $D_2 \times \mathbb{Z}_2$, where in both cases the D_2 stabilizer of the edge $P_1 \cap P_3$ injects into the first factor, and the \mathbb{Z}_2 -factor is generated by the reflection of S^2 that interchanges the given antipodal points. This yields that $Stab_{\Gamma_3}(\gamma)$ is the amalgamation of two copies of $D_2 \times \mathbb{Z}_2$ along the D_2 factors, giving us $D_2 \times D_\infty$ (see Figure 2 for the graph of groups). An identical analysis in the case of the geodesic η yields that the stabilizer is an amalgamation of two copies of $D_3 \times \mathbb{Z}_2$ along the D_3 factors, yielding that the stabilizer is $D_3 \times D_\infty$ (see Figure 2). This completes the proof of Proposition 3.8. \square

4. THE COMPUTATIONS OF $Wh_n(\Gamma_3), n < 2$

In this section we briefly recall the Isomorphism Conjecture in lower algebraic K -theory (the interested reader should refer to [FJ93], [DL98]), and compute the homology groups $H_n^{\Gamma_3}(E_{\mathcal{V}\mathcal{C}}(\Gamma_3); \mathbb{K}\mathbb{Z}^{-\infty}) \cong Wh_n(\Gamma_3)$ for $n < 2$.

The *Farrell and Jones Isomorphism Conjecture in algebraic K-theory*, reformulated in terms of the Davis and Lück functor $\mathbb{K}R^{-\infty}$ (see [DL98]), states that the assembly map $H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{K}R^{-\infty}) \longrightarrow K_n(R\Gamma)$ is an isomorphism for all $n \in \mathbb{Z}$.

The main point of the validity of this conjecture is that it allows the computations of the groups of interest $K_n(R\Gamma)$ from the values of $\mathbb{K}R^{-\infty}(\Gamma/H)$ on the groups $H \in \mathcal{VC}$.

The pseudo-isotopic version of the Farrell and Jones Conjecture is obtained by replacing the algebraic K -theory spectrum by the functors \mathcal{P}_* , $\mathcal{P}_*^{\text{diff}}$, which map from the category of topological spaces X to the category of Ω -*SPECTRA*. The functor $\mathcal{P}_*(?)$ (or $\mathcal{P}_*^{\text{diff}}(?)$) maps the space X to the Ω -spectrum of stable topological (or smooth) pseudo-isotopies of X (see [FJ93, Section 1.1]).

The relation between $\mathcal{P}_*(?)$ and lower algebraic K -theory is given by the work of Anderson and Hsiang [AH77, Theorem 3]. They show

$$\pi_j(\mathcal{P}_*(X)) = \begin{cases} Wh(\mathbb{Z}\pi_1(X)), & j = -1 \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)), & j = -2 \\ K_{j+2}(\mathbb{Z}\pi_1(X)), & j \leq -3. \end{cases}$$

The main result in [FJ93] is that the Isomorphism Conjecture is true for the pseudo-isotopy and smooth pseudo-isotopy functors when $\pi_1(X) = \Gamma$ is a discrete cocompact subgroup of a virtually connected Lie group. This result together with the identification given by Anderson and Hsiang of the lower homotopy groups of the pseudo-isotopy spectrum and the lower algebraic K -theory implies the following Theorem (see [FJ93, Section 1.6.5, and Theorem 2.1]):

Theorem 4.1 (Farrell, F.T. and Jones, L.E). *Let Γ be a cocompact discrete subgroup of a virtually connected Lie group. Then the assembly map*

$$H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty}) \longrightarrow K_n(\mathbb{Z}\Gamma)$$

is an isomorphism for $n \leq 1$ and a surjection for $n = 2$.

Farrell and Jones also proved Theorem 4.1 for discrete cocompact groups, acting properly discontinuously by isometries on a simply connected Riemannian manifold M with everywhere non-positive curvature ([FJ93, Proposition 2.3]). Berkove, Farrell, Pineda, and Pearson extended this result to discrete groups, acting properly discontinuously on hyperbolic n -spaces via isometries, whose orbit space has finite volume (but not necessarily compact), (see [BFPP00, Theorem A]). In particular this result is valid for Γ a hyperbolic, non-cocompact, n -simplex reflection group.

Therefore for $\Gamma = \Gamma_3$, it follows that

$$H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{K}\mathbb{Z}^{-\infty}) \cong Wh_n(\mathbb{Z}\Gamma_3), \quad \text{for } n < 2.$$

Hence to compute the lower algebraic K -theory of the integral group ring Γ_3 it suffices to compute for $n < 2$, the homotopy groups

$$H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{K}\mathbb{Z}^{-\infty}).$$

These computations are feasible using the Atiyah-Hirzebruch type spectral sequence described by Quinn in [Qu82, Theorem 8.7] for the pseudo-isotopy spectrum \mathcal{P} (see Section 5):

$$E_{p,q}^2 = H_p(E_{\mathcal{F}}(\Gamma)/\Gamma; \{Wh_q(\mathbb{Z}\Gamma_\sigma)\}) \implies Wh_{p+q}(\Gamma),$$

where

$$Wh_q(F) = \begin{cases} Wh(F), & q = 1 \\ \tilde{K}_0(\mathbb{Z}F), & q = 0 \\ K_n(\mathbb{Z}F), & q \leq -1. \end{cases}$$

All the information need it to compute the E^2 term is encoded in $E_{\mathcal{V}\mathcal{C}}(\Gamma_3)/\Gamma_3$ and the algebraic K -groups of the finite subgroups and maximal infinite virtually cyclic subgroups of Γ_3 .

We now proceed to give a proof of our main Theorem 1.1. Recall that this Theorem states that the lower algebraic K -theory of the integral group ring of Γ_3 is given as follows:

$$\begin{aligned} Wh(\Gamma_3) &\cong \bigoplus_{\infty} \mathbb{Z}/2 \\ \tilde{K}_0(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \bigoplus_{\infty} \mathbb{Z}/2 \\ K_{-1}(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z} \oplus \mathbb{Z}, \quad \text{and} \\ K_n(\mathbb{Z}\Gamma_3) &\cong 0, \quad \text{for } n < -1. \end{aligned}$$

Proof. Since Γ_3 is hyperbolic relative to $P4m$, then the fundamental domain Δ^3 has one cusp with cusp subgroup $P4m$ (a 2-dimensional crystallographic group), which is the unique maximal parabolic subgroup of Γ_3 (up to conjugacy).

Let $X_0 \subset \mathbb{H}^3$ be the space obtained by truncating the cusp, that is, we remove from \mathbb{H}^3 a countable collection of (open) horoballs, in a Γ_3 -equivariant way, where the horoballs B_i are based at the points $\{p_i\} \subset \partial^\infty \mathbb{H}^3 = S^2$ which are fixed points of the subgroups $G \leq \Gamma_3$ which are conjugate to $P4m$. Now let $X = X_0/\Gamma_3$, i.e. $X = \Delta^3 \setminus B_i$, where B_i is the open horoball from our countable collection that is based at the ideal vertex of Δ^3 .

Note that X satisfies the requirements to be a model for $E_{\mathcal{F}\mathcal{I}\mathcal{N}}(\Gamma_3)/\Gamma_3$. X has five faces (with stabilizers: $1, \mathbb{Z}/2$), nine edges (with stabilizers: $\mathbb{Z}/2, D_2, D_3, D_4$), and six vertices (with stabilizers: $D_2, D_4, D_6, \mathbb{Z}/2 \times D_4, \mathbb{Z}/2 \times S_4$).

Let $\mathcal{V}\mathcal{C}_\infty$ be a collection of subgroups of Γ_3 , consisting of one representative from each conjugacy class of maximal virtually cyclic subgroups of Γ_3 of hyperbolic type. Note that for each $H \in \mathcal{V}\mathcal{C}_\infty$, a model for $E_{\mathcal{V}\mathcal{C}}(H)$ can be taken to be a single point with the obvious H -action. Now let $E_{\mathcal{V}\mathcal{C}}(P4m)$ be the classifying space for $P4m$ constructed by Alves and Ontaneda in [AO03] ($E_{\mathcal{V}\mathcal{C}}(P4m)$ is a 4-dimensional CW-complex, for the isotropy groups we refer the reader to [AO03]), and define the Γ_3 -space $Y = \coprod_{H \in \mathcal{V}\mathcal{C}_\infty} Ind_H^{\Gamma_3} (*) \coprod Ind_{P4m}^{\Gamma_3}(E_{\mathcal{V}\mathcal{C}}(P4m))$. Form the space:

$$\widehat{X} = X * Y = X * \left[\left(\prod_{H \in \mathcal{V}\mathcal{C}_\infty} Ind_H^{\Gamma_3} (*) \right) \prod Ind_{P4m}^{\Gamma_3}(E_{\mathcal{V}\mathcal{C}}(P4m)) \right]$$

From Theorem 2.3 and Theorem 2.6, we have that \widehat{X} satisfies the requirements to be a model for $E_{\mathcal{V}\mathcal{C}}(\Gamma_3)/\Gamma_3$. \widehat{X} is an 8 dimensional Γ_3 -CW complex with isotropy groups:

- (1) **Isotropy of the 0-cells:** The stabilizers of the 0-cells are either finite or maximal infinite virtually cyclic of hyperbolic type:

- (a) *Finite subgroups:* $\mathbb{Z}/2, D_2, D_4, D_6, \mathbb{Z}/2 \times D_4, \mathbb{Z}/2 \times S_4$.
 - (b) *Maximal infinite virtually cyclic subgroups of hyperbolic type:* $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}/2, D_\infty \times \mathbb{Z}/2, D_2 \times D_\infty \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2, D_3 \times D_\infty \cong D_6 *_{D_3} D_6$.
- (2) **Isotropy of the 1-cells:** The stabilizers of the 1-cells are either finite or infinite virtually cyclic of parabolic type:
- (a) *Finite subgroups:* $1, \mathbb{Z}/2, D_2, D_3, D_4, D_6$.
 - (b) *Infinite virtually cyclic subgroups of parabolic type:* $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}/2, D_\infty \times \mathbb{Z}/2$.
- (3) **Isotropy of the 2-cells:** The stabilizers of the 2-cells are either finite or infinite virtually cyclic of parabolic type:
- (a) *Finite subgroups:* $1, \mathbb{Z}/2, D_2, D_3, D_4$.
 - (b) *Infinite virtually cyclic subgroups of parabolic type:* \mathbb{Z}, D_∞ .
- (4) **Isotropy of the 3-cells and 4-cells:** $1, \mathbb{Z}/2$.
- (5) **Isotropy of the 5-cells, 6-cells, 7-cells and 8-cells:** Trivial.

The complex that gives the homology of $E_{\mathcal{VC}}(\Gamma_3)/\Gamma_3$ with local coefficients $\{Wh_q(F_\sigma)\}$ has the form

$$\bigoplus_{\sigma^8} Wh_q(F_{\sigma^8}) \rightarrow \cdots \rightarrow \bigoplus_{\sigma^2} Wh_q(F_{\sigma^2}) \rightarrow \bigoplus_{\sigma^1} Wh_q(F_{\sigma^1}) \rightarrow \bigoplus_{\sigma^0} Wh_q(F_{\sigma^0}),$$

where σ^i denotes the cells in dimension i , and $Wh_q(F_{\sigma^i})$ occurs in the summand as many times as the numbers of conjugacy classes of the subgroup F_{σ^i} in Γ_3 . The homology of this complex will give us the entries for the E^2 -term of the spectral sequence. We now proceed to analyze this complex for each of the following cases: $q < -1, q = -1, 0, 1$.

$q < -1$. Carter showed in [C80] that $K_q(\mathbb{Z}F) = 0$ when F is a finite group. Farrell and Jones showed in [FJ95] that $K_q(\mathbb{Z}Q) = 0$ when Q is a infinite virtually cyclic group. Hence the whole complex consists of zero terms and we obtain $E_{p,q}^2 = 0$ for $q < -1$.

$q = -1$. Again using Carter's result in [C80], $K_{-1}(\mathbb{Z}F) = 0$, for all the finite subgroups which occur as stabilizers of the n -cells, with $n = 2, \dots, 8$. For $n = 2$, we have 2-cells with stabilizers which are virtually infinite cyclic subgroups of parabolic type. Bass in [Bas68] showed that $K_n(\mathbb{Z}Q)$ for $n < 0$ vanishes if $Q = \mathbb{Z}$, or $Q = D_\infty$. Therefore $E_{p,-1}^2 = 0$ for $p \geq 2$. Also $K_{-1}(\mathbb{Z}F) = 0$, for all finite subgroups which are stabilizers of the 1-cells except for $F = D_6$; we have two 1-cells with stabilizer $F = D_6$, for which $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [Pe98, pg. 274]). For $n = 1$, we also have 1-cells with stabilizers which are infinite virtually cyclic subgroups of parabolic type. Pearson in [Pe98] showed that $K_n(\mathbb{Z}Q)$ for $n < 0$ vanishes if $Q = \mathbb{Z} \times \mathbb{Z}/2$ or $Q = D_\infty \times \mathbb{Z}/2$. It follows that for $p = 0, 1$ the complex may have non-zero terms in dimension one and zero:

$$\cdots \rightarrow 0 \rightarrow \bigoplus_{\sigma^1} K_{-1}(D_6) \rightarrow \left[\bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} K_{-1}(\mathbb{Z}F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_{\infty}} K_{-1}(\mathbb{Z}Q_{\sigma^0}) \right]$$

If F_{σ^0} is one of the finite subgroups $\mathbb{Z}/2$, D_2 , or D_4 , then $K_{-1}(\mathbb{Z}F) = 0$ (see [C80]). As we mentioned earlier, $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [Pe98]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, then Ortiz in [Or04, pg. 350]) showed that $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = 0$, and $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) = \mathbb{Z}$.

If $Q_{\sigma^0} \in \mathcal{VC}_{\infty}$, then Q_{σ^0} is one of the groups: \mathbb{Z} , D_{∞} , $\mathbb{Z} \times \mathbb{Z}/2$, $D_{\infty} \times \mathbb{Z}/2$, $D_2 \times D_{\infty} \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $D_3 \times D_{\infty} \cong D_6 *_{D_3} D_6$. As we mentioned earlier, the groups $K_n(\mathbb{Z}Q)$ for $n < 0$ vanish if $Q = \mathbb{Z}$, or $Q = D_{\infty}$ (see [Bas68]). For the groups with $\mathbb{Z}/2$ summands, we also have that $K_n(\mathbb{Z}Q) = 0$ if $n < 0$, (see [Pe98, pg. 272]).

The other two cases are groups of the form $Q = Q_0 *_{\mathbb{Z}/p} Q_1$. In [FJ95] Farrell and Jones show that if Q is infinite virtually cyclic, then $K_n(\mathbb{Z}Q)$ is zero for $n < -1$ and that $K_{-1}(\mathbb{Z}Q)$ is generated by the images of $K_{-1}(\mathbb{Z}F)$ where F ranges over all finite subgroups $F \subset Q$. Since $Wh_q(D_2) = 0$ for $q < 0$, and $Wh_q(D_2 \times \mathbb{Z}/2) = 0$ for $q < 0$ (see [LS00]), then for $Q = D_2 \times D_{\infty} \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, it follows that $K_{-1}(\mathbb{Z}Q) = 0$. Since $K_{-1}(\mathbb{Z}D_3) = 0$, and $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [Or04], [Pe98]), then for $Q = D_3 \times D_{\infty} \cong D_6 *_{D_3} D_6$, it follows that $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$.

Since there is only one conjugacy class for each of the subgroups of Γ_3 occurring as stabilizers of the 1-cells and the 0-cells, then the complex that gives the homology groups $H_0(\hat{X}; \{K_{-1}(\mathbb{Z}F_{\sigma})\})$, and $H_1(\hat{X}; \{K_{-1}(\mathbb{Z}F_{\sigma})\})$ yields the following exact sequence

$$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \rightarrow 0.$$

Hence after working through the exact sequence, we have that that the E^2 term for \hat{X} has the following entries for $q = -1$: $E_{p,q}^2 = 0$ for $p \geq 1$, and $E_{0,-1}^2 = \mathbb{Z} \oplus \mathbb{Z}$.

q = 0. It is well known that $\tilde{K}_0(\mathbb{Z}F) = 0$ when F is any of the finite subgroups that occur as stabilizers of the n -cells, for $n = 1, \dots, 8$ (see for example [Re76], [Ro94]). For $n = 1, 2$, we have 1-cells and 2-cells with stabilizers which are infinite virtually cyclic subgroups of parabolic type. In [Pe98] Pearson showed that $\tilde{K}_0(\mathbb{Z}Q) = 0$ if $Q = \mathbb{Z}$, D_{∞} , $\mathbb{Z} \times \mathbb{Z}/2$, $D_{\infty} \times \mathbb{Z}/2$, therefore $E_{p,0}^2 = 0$ for $p \geq 1$. For $p = 0$ the complex may have non-zero terms in dimension zero and the resulting homology is:

$$H_0(\hat{X}; \{\tilde{K}_0(\mathbb{Z}F_{\sigma})\}) = \bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} \tilde{K}_0(\mathbb{Z}F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_{\infty}} \tilde{K}_0(\mathbb{Z}Q_{\sigma^0}).$$

If F_{σ^0} is one of the finite subgroups $\mathbb{Z}/2$, D_2 , D_4 , or D_6 , then $\tilde{K}_0(\mathbb{Z}F) = 0$ (see [Re76]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, Ortiz in [Or04, pg. 351] showed that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = \mathbb{Z}/4$, and $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) = \mathbb{Z}/4$.

If Q_{σ^0} is one of the maximal infinite virtually cyclic subgroups \mathbb{Z} , D_{∞} , $\mathbb{Z} \times \mathbb{Z}/2$, or $D_{\infty} \times \mathbb{Z}/2$, then $\tilde{K}_0(\mathbb{Z}Q) = 0$ (see [Pe98]). For the remaining subgroups, using [CP02, Lemma 3.8], we have that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_2; B_1, B_2)$, where $B_i = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2$ for $i = 1, 2$. For $Q = D_6 *_{D_3} D_6$, we have that $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_3; C_1, C_2)$, where $C_i = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$ -bimodule generated by $D_6 \setminus D_3$ for $i = 1, 2$. The Nil-groups NK_0 appearing in these computations are the Waldhausen's Nil-groups.

Hence we obtain that

$$E_{0,0}^2 = \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus NK_0(\mathbb{Z}D_2; B_1, B_2) \oplus NK_0(\mathbb{Z}D_3; C_1, C_2).$$

$\mathbf{q} = \mathbf{1}$. Oliver in [O89] showed that $Wh(F) = 0$ when F is any of the finite subgroups that occur as stabilizers of the n -cells, for $n = 1, \dots, 8$. For $n = 1, 2$, we have 1-cells and 2-cells with stabilizers which are virtually infinite cyclic subgroups of parabolic type. In [Pe98] Pearson show that $Wh(Q) = 0$ if $Q = \mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}/2, D_\infty \times \mathbb{Z}/2$, therefore $E_{p,1}^2 = 0$ for $p \geq 1$. As before for $p = 0$ the complex may have non-zero terms in dimension zero and the resulting homology is:

$$H_0(\widehat{X}; \{Wh(F_\sigma)\}) = \bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} Wh(F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_\infty} Wh(Q_{\sigma^0}).$$

If F_{σ^0} is one of the finite subgroups $\mathbb{Z}/2, D_2, D_4$, or D_6 , then $Wh(F) = 0$ (see [O89]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, Ortiz in [Or04, pg. 352] showed that $Wh(\mathbb{Z}/2 \times D_4) = 0$, and $Wh(\mathbb{Z}/2 \times S_4) = 0$.

If Q_{σ^0} is one of the maximal infinite virtually cyclic subgroups $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}/2$, or $D_\infty \times \mathbb{Z}/2$, then $Wh(Q) = 0$ (see [Pe98]). For the remaining subgroups, using [CP02, Lemma 3.8], we have that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $Wh(Q) \cong NK_1(\mathbb{Z}D_2; B_1, B_2)$, where $B_i = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2$ for $i = 1, 2$. For $Q = D_6 *_{D_3} D_6$, we have that $Wh(Q) \cong NK_1(\mathbb{Z}D_3; C_1, C_2)$, where $C_i = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$ -bimodule generated by $D_6 \setminus D_3$ for $i = 1, 2$. The Nil-groups NK_1 appearing in these computations are the Waldhausen's Nil-groups. It follows that

$$E_{0,1}^2 = NK_1(\mathbb{Z}D_2; B_1, B_2) \oplus NK_1(\mathbb{Z}D_3; C_1, C_2).$$

Hence the spectral sequence collapses at E^2 , giving the following preliminary results on the algebraic K -groups $Wh_n(\Gamma_3)$ for $n < 2$.

$$\begin{aligned} Wh(\Gamma_3) &\cong NK_1(\mathbb{Z}D_2; B_1, B_2) \oplus NK_1(\mathbb{Z}D_3; C_1, C_2) \\ \tilde{K}_0(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus NK_0(\mathbb{Z}D_2; B_1, B_2) \oplus NK_0(\mathbb{Z}D_3; C_1, C_2) \\ K_{-1}(\mathbb{Z}\Gamma_3) &\cong \mathbb{Z} \oplus \mathbb{Z}, \text{ and} \\ K_n(\mathbb{Z}\Gamma_3) &\cong 0, \text{ for } n < -1. \end{aligned}$$

where $B_i = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2$ for $i = 1, 2$, and $C_i = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$ -bimodule generated by $D_6 \setminus D_3$ for $i = 1, 2$.

Finally, to conclude the proof, we observe that the Waldhausen Nil-groups that appear above are isomorphic to:

$$NK_1(\mathbb{Z}D_2; B_1, B_2) \cong NK_0(\mathbb{Z}D_2; B_1, B_2) \cong \bigoplus_{\infty} \mathbb{Z}/2$$

$$NK_1(\mathbb{Z}D_3; C_1, C_2) \cong NK_0(\mathbb{Z}D_3; C_1, C_2) \cong 0$$

The computation of these Nil-groups will be carried out in the next section. \square

5. COMPUTING THE WALDHAUSEN NIL-GROUPS.

In this section we implement an approach suggested to us by F.T. Farrell, and provide explicit computations for the Waldhausen Nil-groups appearing in the previous section. The results of this section can be summarized in the following two theorems:

Theorem 5.1. For $i = 0, 1$, $NK_i(\mathbb{Z}D_3; C_1, C_2) \cong 0$ where $C_j = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$ -bimodule generated by $D_6 \setminus D_3$ for $j = 1, 2$

Theorem 5.2. For $i = 0, 1$, $NK_i(\mathbb{Z}D_2; B_1, B_2) \cong \bigoplus_{\infty} \mathbb{Z}/2$, where $B_j = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2$ for $j = 1, 2$

The proof of these theorems will be based on the following observations:

(1) There exists a *surjection*:

$$2 \cdot NK_i(\mathbb{Z}[D_k]) \rightarrow NK_i(\mathbb{Z}D_k; M_1, M_2)$$

from the direct sum of two copies of the Bass Nil-groups $NK_i(\mathbb{Z}[D_k])$ to the Waldhausen Nil-groups for $i = 0, 1$ ($k = 2, 3$). M_1, M_2 are suitably defined bi-modules that specialize to B_1, B_2 and C_1, C_2 when $k = 2, 3$ respectively.

(2) The surjection above is an *injection* on each individual copy of $NK_i(\mathbb{Z}[D_k])$.

Let us now outline how Theorems 5.1 and 5.2 follow from these observations. In the case where $k = 3$, we know that $NK_i(\mathbb{Z}[D_3]) \cong 0$ ($i = 0, 1$), and the surjectivity of the map immediately tells us that the Waldhausen Nil-groups $NK_i(\mathbb{Z}D_3; C_1, C_2)$ vanish. On the other hand, when $k = 2$, we establish that $NK_i(\mathbb{Z}[D_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2$ ($i = 0, 1$), an infinite countable sum of $\mathbb{Z}/2$. This is done by establishing that $NK_i(\mathbb{Z}[D_2]) \cong NK_{i+1}(\mathbb{F}_2[\mathbb{Z}/2])$ ($i = 0, 1$), and appealing to computations of Madsen [Mad95] for the latter groups. Once we have this result for $NK_i(\mathbb{Z}[D_2])$, the surjectivity and injectivity statements above combine to force $NK_i(\mathbb{Z}D_2; B_1, B_2) \cong \bigoplus_{\infty} \mathbb{Z}/2$ ($i = 0, 1$).

Let us start by recalling some well-known facts about infinity virtually cyclic groups. If Γ is a virtually infinite cyclic group not of the form $G \rtimes \mathbb{Z}$, then Γ always maps epimorphically onto the infinite dihedral group D_{∞} with finite kernel (see [FJ95, Lemma 2.5]). Using this epimorphism, Farrell and Jones in [FJ95] constructed a stratified fiber bundle $\rho_E : E \rightarrow X$, where E is a closed manifold with $\pi_1 E = \Gamma$ and such that, for each point $x \in X$, $\pi_1 \rho_E^{-1}(x) = G_x$ or $G_x \rtimes \mathbb{Z}$, where G_x is a finite group.

Next, we briefly describe the control space X as constructed by Farrell and Jones in [FJ95, Section 2].

Recall the model for the real hyperbolic \mathbb{H}^2 discussed in [FJ86, Section 2], i.e., $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid -1 < y < 1\}$, with the Riemannian metric defined in [FJ95, pg. 23].

Let $S\mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the tangent sphere bundle consisting of unit length vectors, and let $S^+\mathbb{H}^2 \rightarrow \mathbb{H}^2$ be the subbundle of $S\mathbb{H}^2 \rightarrow \mathbb{H}^2$ whose fiber over $x \in \mathbb{H}^2$ is $S_x^+\mathbb{H}^2$ (the *asymptotic northern hemisphere* defined in [FJ86, 0.12]). The natural action of D_{∞} on \mathbb{H}^2 determined by $D_{\infty} \subseteq \text{Isom}(\mathbb{H}^2)$ induces an action of D_{∞} on $S^+\mathbb{H}^2$.

The orbit space $S^+\mathbb{H}^2/D_{\infty}$ has a stratification with six strata. The lowest strata are H, V_0, V_1 , the intermediate strata are B_0, B_1 and T is the top stratum. Using the canonical quotient map

$$p : S^+\mathbb{H}^2 \rightarrow S^+\mathbb{H}^2/D_{\infty},$$

these strata are defined in [FJ95, pg. 24].

Here the key point is to note that H is diffeomorphic to the circle S^1 while V_0 and V_1 are both diffeomorphic to \mathbb{R} . Also $p(\partial S^+\mathbb{H}^2)$ is diffeomorphic to the cylinder

$S^1 \times \mathbb{R}$, and H cuts this manifold into two connected components B_0 and B_1 , i.e.

$$B_0 \amalg B_1 = p(\partial S^+ \mathbb{H}^2) - H.$$

Note that each pair $(B_i \cup H, H)$, $i = 0, 1$, is diffeomorphic to the pair $(S^1 \times [0, +\infty), S^1 \times 0)$. Finally, the top stratum is the complement in $S^+ \mathbb{H}^2 / D_\infty$ of the union $V_0 \cup V_1 \cup p(\partial S^+ \mathbb{H}^2)$.

The control space X is defined to be the quotient space of $S^+ \mathbb{H}^2 / D_\infty$ obtained by identifying the stratum H to a single point $*$. Let

$$\rho : S^+ \mathbb{H}^2 / D_\infty \rightarrow X$$

denote the canonical quotient map. Then the stratification of $S^+ \mathbb{H}^2 / D_\infty$ induces a stratification of X whose six strata are the images of the strata in $S^+ \mathbb{H}^2 / D_\infty$ under ρ . Since

$$\rho : S^+ \mathbb{H}^2 / D_\infty - H \rightarrow X - *$$

is a homeomorphism, we identify via ρ the strata in $S^+ \mathbb{H}^2 / D_\infty$ different from H with the corresponding strata in X different from $*$.

This concludes the description of the control space X . For specific details and the construction of the stratified fiber bundle $\rho_E : E \rightarrow X$ mentioned earlier, we refer the reader to [FJ95, Section 2].

Farrell and Jones also proved in [FJ95, Theorem 2.6] that the group homeomorphism

$$\pi_i(\mathcal{A}) : \pi_i \mathcal{P}_*(E; \rho_E) \longrightarrow \pi_i \mathcal{P}_*(E)$$

is an epimorphism for every integer i . Here $\mathcal{P}_*(E)$ is the spectrum of stable topological pseudoisotopies on E , $\mathcal{P}_*(E, \rho_E)$ is the spectrum of those stable pseudoisotopies which are controlled over X via ρ_E , and \mathcal{A} is the ‘assembly’ map.

Recall that Quinn [Qu82, Theorem 8.7] constructed a spectral $E_{s,t}^n$, abutting to $\pi_{s+t} \mathcal{P}_*(E; \rho_E)$ with $E_{s,t}^2 = H_s(X; \pi_t \mathcal{P}_*(\rho_E))$. Here $\pi_q \mathcal{P}_*(\rho_E)$, $q \in \mathbb{Z}$, denotes the stratified system of abelian groups over X where the group above $x \in X$ is $\pi_q(\mathcal{P}_*(\rho_E^{-1}(x)))$. Note that by Anderson and Hsiang’s result (see [AH77, Theorem 3] and Section 4), $\pi_i \mathcal{P}_{*+2}(\rho_E^{-1}(x)) = K_i(\mathbb{Z}\pi_1(\rho_E^{-1}(x)))$ for $i \leq -1$, $\tilde{K}_0(\mathbb{Z}\pi_1(\rho_E^{-1}(x)))$, for $i = 0$, and $Wh(\mathbb{Z}\pi_1(\rho_E^{-1}(x)))$ for $i = 1$.

It is important to emphasize that for each $x \in X$, the fundamental group $\pi_1 \rho_E^{-1}(x)$ of the fiber is either finite or a semidirect product $F \rtimes \mathbb{Z}$ where F is a finite subgroup of Γ (see [FJ95, Remark 2.6.8]).

These facts together with the immediate consequences of the construction of the control space X and the stratified fiber bundle $\rho_E : E \rightarrow X$ are extremely useful in analyzing Quinn’s spectral sequence. This analysis is then used to complete the proof of the *surjectivity* part in Theorems 5.1 and 5.2.

Proof of Theorem 5.1. For $\Gamma = D_3 \times D_\infty \cong D_6 *_{D_3} D_6$, first note that the fundamental group F of a fiber of ρ_E , i.e. $\pi_1 \rho_E^{-1}(x)$ is either the group $D_3 \times \mathbb{Z}$ or some finite subgroup F of Γ . It is a fact that $K_q(\mathbb{Z}F)$ (for $q \leq -1$), $\tilde{K}_0(\mathbb{Z}F)$, and $Wh(F)$ all vanish when F is a finite subgroup of Γ except when $F = D_6$. This fact can be obtained by combining results from [C80], [Re76], [Ro94], [Or04], [Pe98], [CuR81], [O89] (see the arguments in Section 4). It is also a fact that $\tilde{K}_0(\mathbb{Z}D_6)$, $Wh(D_6)$ both vanish, and $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [Or04, p. 350], [Pe98, p. 274]). To compute the lower algebraic K -theory of the ring $\mathbb{Z}[D_3 \times \mathbb{Z}] = \mathbb{Z}[D_3][\mathbb{Z}]$, we use the

Fundamental Theorem of algebraic K -theory (see [Bas68]). The following are the results found:

$$\begin{aligned} Wh(D_3 \times \mathbb{Z}) &\cong NK_1(\mathbb{Z}[D_3]) \oplus NK_1(\mathbb{Z}[D_3]), \\ \tilde{K}_0(\mathbb{Z}[D_3 \times \mathbb{Z}]) &\cong NK_0(\mathbb{Z}[D_3]) \oplus NK_0(\mathbb{Z}[D_3]), \\ K_q(\mathbb{Z}[D_3 \times \mathbb{Z}]) &\cong 0 \quad \text{for } q \leq -1, \end{aligned}$$

where the Nil-groups appearing in these computations are the Bass's Nil-groups.

Next, note that given the simple nature of the control space X , the spectral sequence collapses at E^2 and for all $s, t \in \mathbb{Z}$

$$E_{s,t}^2 = H_s(X; \pi_t \mathcal{P}_*(\rho_E)) \implies \pi_{s+t} \mathcal{P}_*(E; \rho_E)$$

Combining the above with the fact that $Wh_q(D_3)$ vanish for all $q \leq 1$, we have that for all $s, t \in \mathbb{Z}$

$$\begin{aligned} H_s(X; \pi_t \mathcal{P}_*(\rho_E)) &= H_s(*; Wh_t(D_3 \times \mathbb{Z})) \oplus H_s(\mathbb{R}; Wh_t(D_6)) \oplus H_s(\mathbb{R}; Wh_t(D_6)) \\ &= Wh_t(D_3 \times \mathbb{Z}) \oplus Wh_t(D_6) \oplus Wh_t(D_6). \end{aligned}$$

Consequently,

$$\begin{aligned} \pi_{-1} \mathcal{P}_*(E; \rho_E) &\cong NK_1(\mathbb{Z}[D_3]) \oplus NK_1(\mathbb{Z}[D_3]), \\ \pi_{-2} \mathcal{P}_*(E; \rho_E) &\cong NK_0(\mathbb{Z}[D_3]) \oplus NK_0(\mathbb{Z}[D_3]), \\ \pi_{-3} \mathcal{P}_*(E; \rho_E) &\cong K_{-1}(\mathbb{Z}[D_3 \times \mathbb{Z}]) \oplus K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[D_6]) \cong \mathbb{Z} \oplus \mathbb{Z}, \\ \pi_q \mathcal{P}_*(E; \rho_E) &= 0 \quad \text{when } q \leq -4, \end{aligned}$$

On the other hand using the results found in Section 4 for the lower algebraic K -theory of the integral group ring of $D_3 \times D_\infty$, we have that

$$\begin{aligned} \pi_{-1} \mathcal{P}_*(E) &\cong Wh(D_3 \times D_\infty) \cong NK_1(\mathbb{Z}D_3; C_1, C_2) \\ \pi_{-2} \mathcal{P}_*(E) &\cong \tilde{K}_0(\mathbb{Z}[D_3 \times D_\infty]) \cong NK_0(\mathbb{Z}D_3; C_1, C_2) \\ \pi_{-3} \mathcal{P}_*(E) &\cong K_{-1}(\mathbb{Z}[D_3 \times D_\infty]) \cong \mathbb{Z} \oplus \mathbb{Z} \\ \pi_q \mathcal{P}_*(E) &\cong K_q(\mathbb{Z}[D_3 \times D_\infty]) \cong 0 \quad \text{for } q \leq 4, \end{aligned}$$

where $C_i = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$ -bimodule generated by $D_6 \setminus D_3$ for $i = 1, 2$. Here the Nil-groups appearing in these computations are the Waldhausen's Nil-groups.

Combining the above with [FJ95, Theorem 2.6] we obtain for $i = 0, 1$ the desired epimorphism

$$2 \cdot NK_i(\mathbb{Z}[D_3]) \longrightarrow NK_i(\mathbb{Z}D_3, C_0, C_1) \longrightarrow 0$$

Since $NK_i(\mathbb{Z}[D_3]) = 0$ for $i = 0, 1$ (see [Ha87]), then it follows that for $i = 0, 1$, $NK_i(\mathbb{Z}D_3, C_0, C_1)$ completing the proof of Theorem 5.1. \square

Before proving Theorem 5.2, we prove the following Lemmas.

Lemma 5.3.

$$NK_0(\mathbb{Z}[D_2]) \cong NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_{\infty} \mathbb{Z}/2.$$

Proof. Write $\mathbb{Z}[D_2] = \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$ as $\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2]$, and consider the following Cartesian square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2] & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}/2] & \longrightarrow & \mathbb{F}_2[\mathbb{Z}/2] \end{array}$$

This Cartesian square yields the Mayer-Vietories sequence

$$\begin{aligned} \cdots \rightarrow NK_1(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_1(\mathbb{Z}[\mathbb{Z}/2]) &\rightarrow NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \rightarrow NK_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2]) \rightarrow \\ &\rightarrow NK_0(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_0(\mathbb{Z}[\mathbb{Z}/2]) \rightarrow \cdots \end{aligned}$$

Since $NK_i(\mathbb{Z}[\mathbb{Z}/2]) = 0$, for $i = 0, 1$ (see [Ha87]), we obtain that $NK_0(\mathbb{Z}[D_2]) \cong NK_1(\mathbb{F}_2[\mathbb{Z}/2])$. In [Mad95], Madsen showed that $NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_{\infty} \mathbb{Z}/2$, giving us the desired result. \square

Lemma 5.4.

$$NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_{\infty} \mathbb{Z}/2.$$

Proof. As before, consider the following Cartesian square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2] & \longrightarrow & \mathbb{Z}[\mathbb{Z}/2] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}/2] & \longrightarrow & \mathbb{F}_2[\mathbb{Z}/2] \end{array}$$

As mentioned in Lemma 5.3, since $NK_1(\mathbb{Z}[\mathbb{Z}/2])$ vanishes, the Mayer-Vietories sequence yields the epimorphism

$$NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \rightarrow NK_1(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2]) \rightarrow 0$$

Now, in [Mad95], Madsen has shown that $NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_{\infty} \mathbb{Z}/2$. Combining this result with the well-known fact that the Nil groups are either trivial or infinitely generated [F77], we conclude that up to isomorphism, either

$$NK_1(\mathbb{Z}[D_2]) \cong \bigoplus_{\infty} \mathbb{Z}/2$$

or the group is trivial. But the group $NK_1(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2])$ is known to be non-trivial (see [Bas68]), completing the proof of the Lemma. \square

Proof of Theorem 5.2. Let $\Gamma = D_2 \times D_{\infty} \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$. In this case note that the fundamental group F of a fiber of ρ_E , i.e. $\pi_1 \rho_E^{-1}(x)$ is either the group $D_2 \times \mathbb{Z}$ or some finite subgroup F of Γ . It is a fact that $K_i(\mathbb{Z}F)$ (for $i \leq -1$), $\tilde{K}_0(\mathbb{Z}F)$, and $Wh(F)$ all vanish when F is a finite subgroup of Γ . This fact can be obtained by combining results from [C80], [Re76], [Ro94], [Or04], [Pe98], [CuR81], [O89] (see Section 4). To compute the lower algebraic K -theory of the ring

$\mathbb{Z}[D_2 \times \mathbb{Z}] = \mathbb{Z}[D_2][\mathbb{Z}]$, we apply the Fundamental Theorem of algebraic K -theory to obtain:

$$\begin{aligned} Wh(D_2 \times \mathbb{Z}) &\cong NK_1(\mathbb{Z}[D_2]) \oplus NK_1(\mathbb{Z}[D_2]) \\ \tilde{K}_0(\mathbb{Z}[D_2 \times \mathbb{Z}]) &\cong NK_0(\mathbb{Z}[D_2]) \oplus NK_0(\mathbb{Z}[D_2]) \\ K_q(\mathbb{Z}[D_2 \times \mathbb{Z}]) &\cong 0 \quad \text{for } q \leq -1, \end{aligned}$$

where the Nil-groups appearing in these computations are the Bass's Nil-groups.

Once again, given the simple nature of the control space X in this case, the spectral sequence collapses at E^2 and for all $s, t \in \mathbb{Z}$

$$E_{s,t}^2 = H_s(X; \pi_t \mathcal{P}_*(\rho_E)) \implies \pi_{s+t} \mathcal{P}_*(E; \rho_E)$$

Combining the above with the fact that $Wh_q(D_2)$ and $Wh_q(D_2 \times \mathbb{Z}/2)$ are both trivial for all $q \leq 1$, we have that for all $s, t \in \mathbb{Z}$

$$H_s(X; \pi_t \mathcal{P}_*(\rho_E)) = Wh_t(D_2 \times \mathbb{Z})$$

Consequently, we obtain:

$$\begin{aligned} \pi_{-1} \mathcal{P}_*(E; \rho_E) &\cong NK_1(\mathbb{Z}[D_2]) \oplus NK_1(\mathbb{Z}[D_2]), \\ \pi_{-2} \mathcal{P}_*(E; \rho_E) &\cong NK_0(\mathbb{Z}[D_2]) \oplus NK_0(\mathbb{Z}[D_2]), \\ \pi_q \mathcal{P}_*(E; \rho_E) &= 0 \quad \text{when } q \leq -3. \end{aligned}$$

On the other hand, using the results found in Section 4 for the lower algebraic K -theory of $D_2 \times D_\infty$, we have that

$$\begin{aligned} \pi_{-1} \mathcal{P}_*(E) &\cong Wh(D_2 \times D_\infty) \cong NK_1(\mathbb{Z}D_2; B_1, B_2) \\ \pi_{-2} \mathcal{P}_*(E) &\cong \tilde{K}_0(\mathbb{Z}[D_2 \times D_\infty]) \cong NK_0(\mathbb{Z}D_2; B_1, B_2) \\ \pi_q \mathcal{P}_*(E) &\cong K_i(\mathbb{Z}[D_2 \times D_\infty]) \cong 0 \quad \text{for } q \leq -3, \end{aligned}$$

where $B_i = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2$ for $i = 1, 2$. Recall that the Nil-groups appearing in these computations are the Waldhausen's Nil-groups.

Combining the above with [FJ95, Theorem 2.6] we obtain for $i = 0, 1$ the epimorphism

$$2 \cdot NK_i(\mathbb{Z}[D_2]) \longrightarrow NK_i(\mathbb{Z}D_2, B_1, B_2) \longrightarrow 0$$

Next, recall that a monomorphism of group $\sigma : \Pi \rightarrow \Gamma$ induces transfer maps $\sigma^* : Wh(\Gamma) \rightarrow Wh(\Pi)$ and $\sigma^* : \tilde{K}_0(\mathbb{Z}\Gamma) \rightarrow \tilde{K}_0(\mathbb{Z}\Pi)$ provided the image of σ have finite index in Γ (see [FH78]).

Let $\Pi = G \times \mathbb{Z}$, where $G = D_2$. By the above paragraph, the monomorphism σ in the exact sequence

$$0 \longrightarrow \Pi \xrightarrow{\sigma} \Gamma \longrightarrow \mathbb{Z}/2 \longrightarrow$$

induces the transfer maps:

$$\sigma^* : Wh(G \times D_\infty) \longrightarrow Wh(G \times \mathbb{Z})$$

$$\sigma^* : \tilde{K}_0(\mathbb{Z}[G \times D_\infty]) \longrightarrow \tilde{K}_0(\mathbb{Z}[D_2 \times \mathbb{Z}])$$

Let τ be the standard involution of G given by $\tau(g) = g^{-1}$ for $g \in G$. This involution can be extended to an involution of $\mathbb{Z}[G \times \mathbb{Z}] = \mathbb{Z}[G][\mathbb{Z}]$ (see [FH68, pg. 209]). By abuse of notation, we will also denote this new involution by τ .

In the direct sum decomposition given by the *Fundamental Theorem of algebraic K -theory* of the algebraic K -groups of the ring $\mathbb{Z}[G][\mathbb{Z}]$:

$K_i(\mathbb{Z}[G][\mathbb{Z}]) \cong K_{i-1}(\mathbb{Z}[G]) \oplus K_i(\mathbb{Z}[G]) \oplus NK_i(\mathbb{Z}[G]) \oplus NK_i(\mathbb{Z}[G])$,
 τ_* interchanges the two direct summands $NK_i(\mathbb{Z}[G])$ (see [FH68, Proposition 20]).
 Now recall that $(\sigma^* \circ \sigma_*)(x) = x + \tau_*(x)$ for all $x \in Wh_i(D_2 \times \mathbb{Z})$, $i = 0, 1$; we will denote this map by ψ . This yields the following commutative diagram for $i = 0, 1$:

$$\begin{array}{ccc} Wh_i(D_2 \times \mathbb{Z}) & \xrightarrow{\sigma_*} & Wh_i(D_2 \times D_\infty) \\ & \searrow \psi & \downarrow \sigma_* \\ & & Wh_i(D_2 \times \mathbb{Z}) \end{array}$$

Using our earlier computations, we obtain for $i = 0, 1$ the commutative diagram:

$$\begin{array}{ccc} NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2]) & \xrightarrow{\sigma_*} & NK_i(\mathbb{Z}D_2, B_1, B_2) \\ & \searrow \psi & \downarrow \sigma_* \\ & & NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2]) \end{array}$$

We now define the map φ by:

$$\varphi : NK_i(\mathbb{Z}[D_2]) \xrightarrow{(\text{Id}, 0)} NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2])$$

Note that $\sigma^* \circ \sigma_*$ is injective on $\varphi(NK_i(\mathbb{Z}[D_2]))$. It follows that σ_* is injective on the first summand of the direct sum $NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2])$. Note that an identical argument can be used to show that $\sigma^* \circ \sigma_*$ is injective on the second summand (though it is easy to see that the image of the first and the second summand coincide).

For $i = 0, 1$, this gives us the desired monomorphism:

$$0 \longrightarrow NK_i(\mathbb{Z}[D_2]) \longrightarrow NK_i(\mathbb{Z}D_2, B_1, B_2).$$

Combining this monomorphism with the epimorphism found earlier, and applying Lemma 5.3 and Lemma 5.4, it follows that $i = 0, 1$

$$NK_i(\mathbb{Z}D_2, B_1, B_2) \cong \bigoplus_{\infty} \mathbb{Z}/2,$$

This completes the proof of Theorem 5.2. □

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