WHEN DO THE DIRECT SUMS OF MODULES INHERIT CERTAIN PROPERTIES?

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Abstract. It is of obvious interest to know whether an algebraic property of modules is preserved by direct sums of such modules. In this paper we provide a survey of this question for various classes of modules of interest. The question of inheritance of a property by direct sums of modules has been explored for the classes of (quasi-)injective modules and some of their generalizations as a motivation for further work. In the main part of this paper we provide latest results and developments on this question for the related classes of Baer, quasi-Baer, and Rickart modules. Examples are provided that delimit our results and explain the notions. Some open problems are listed at the end of the paper.

1. Introduction

For a long time, algebraists have been interested in finding out when do (certain) properties of modules (or of other algebraic structures) go over to finite or infinite direct sums of modules (or such structures)? Among other things, this quest has led one to the conditions needed for the property of (quasi-)injectivity of modules and some of its generalizations to go to direct sums of such modules. In this survey paper, starting from the notion of (quasi-)injectivity, we will consider this question about direct sums for some well-known classes of modules which generalize injective modules or are related to the notion of injectivity by other means. In particular, we will also consider this direct sum question for the related classes of Baer, quasi-Baer, and Rickart modules. While the question of when do the direct sums of modules with a property $\mathcal{P}$ inherit the property $\mathcal{P}$ has been satisfactorily settled for the classes of (quasi-)injective, (quasi-)continuous, and FI-extending modules, the problem of a satisfactory characterization of when is a direct sum of extending modules, extending, remains an open problem. There have been a number of attempts to solve this open problem but with limited success. By an interesting result of Chatters and Khuri, a ring is right extending right nonsingular if and only if it is Baer and right cononsingular. This result on rings was extended to a module theoretic setting by Rizvi and Roman after the introduction of relevant notions for modules. That development has allowed us to connect the study of the class of extending modules to that of Baer modules and vice versa, under suitable conditions. In this paper, we will consider relevant properties of the classes of Baer, quasi-Baer, and Rickart modules needed in our study. We will discuss conditions needed for a direct sum of Baer, quasi-Baer, and Rickart modules to be Baer, quasi-Baer, and Rickart, respectively. A major part of the paper will be devoted to results related to these latter classes of modules in an attempt to bring the reader up to date on the latest developments in this newer area of research. It is hoped that some of these results and other related investigations may lead to a satisfactory answer for the question of the direct sum problem for the class of extending modules mentioned earlier. It will also be of interest if a solution to any of the general open problems listed at the end of the paper can be obtained. As a general observation, we will note from the results presented that for a direct sum to inherit a property under our consideration, one often needs some sort of 'relative conditions' between each of its direct summands. In addition, it will be seen that often for a property $\mathcal{P}$ to go to 'infinite' direct sums of modules with $\mathcal{P}$, one may require some kind of finiteness or additional conditions.

After this introduction, in Section 2 we consider injective modules and some of their generalizations. Connections between the classes of injective, quasi-injective, (quasi-)continuous, extending, and FI-extending modules are explored. Explicit examples are given which show that direct sums of modules belonging to each of these classes (except for the class of FI-extending modules) do not belong to these respective classes, in general. Necessary and sufficient conditions and other results for the direct sum problem are provided.

Our focus in Section 3 is on the related notions of Baer, Quasi-Baer, and Rickart rings and modules. Connections between extending and Baer properties are mentioned for rings via Chatters-Khuri's result (Theorem 3.1) and for modules via Theorem 3.4. The basic properties of these notions are discussed and highlighted. It is shown that in general direct sums of Baer, Quasi-Baer, and Rickart modules do not inherit each of these properties. This is the main topic of our discussions in the remainder of this paper.

Section 4 is devoted to results on direct sums of Baer and quasi-Baer modules. The direct sum problem of Baer modules is quite difficult. It is shown that the direct sum of Baer modules inherits the Baer property only under special conditions. For the quasi-Baer module case, it is shown that every free module over a quasi-Baer ring is quasi-Baer. More generally, any direct sum of copies of a quasi-Baer module is quasi-Baer.

Section 5 consists of latest developments on the direct sum problem in the theory of Rickart modules. Relative Rickart and relative C_2 condition are introduced to study the direct sums of Rickart modules. We use these conditions to obtain specific results for direct sums of Rickart modules to be Rickart.

In Section 6, we consider the question of when are free R-modules Rickart or Baer. A number of well-known classes of rings R are characterized via Rickart or Baer properties of certain classes of free R-modules. In particular, a ring R is right hereditary (resp., right semihereditary) iff every (resp., every finitely generated) free right R-module is Rickart. A ring R is right hereditary and semiprimary (resp., right semihereditary and left Π-coherent) iff every (resp., every finitely generated) free right R-module is Baer. As a consequence, a new characterization of a Prüfer domain R is obtained via Rickart (or Baer) property of its finitely generated free modules. An example of a module M is included showing that $M^{(n)}$ is a Baer module while $M^{(n+1)}$ is not Baer. It is shown that a ring R is von Neumann regular iff every finitely generated free R-module is Rickart with C_3 condition.

Throughout this paper, R is a ring with unity and M is a unital right R-module. For a right R-module M, $S = End_R(M)$ denotes the endomorphism ring of M; thus M can be viewed as a left S- right R-bimodule. For $\varphi \in S$, $Ker\varphi$ and $Im\varphi$ stand for the kernel and the image of φ, respectively. The notations $N \subseteq M$, $N \leq M$, $N \trianglelefteq M$, $N \leq^{ess} M$, or $N \leq^{\oplus} M$ mean that N is a subset, a submodule, a fully invariant submodule, an essential submodule, or a direct summand of M, respectively. $M^{(n)}$ denotes the direct sum of n copies of M and $\mathsf{Mat}_n(R)$ denotes an $n \times n$ matrix ring over R. By $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of complex, real, rational, integer, and natural numbers, respectively. $E(M)$ denotes the injective hull of M and $\mathbb{Z}_n$ denotes the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$.

We also denote $r_M(I) = \{m \in M \mid Im = 0\}$, $r_S(I) = \{\varphi \in S \mid I\varphi = 0\}$ for $\emptyset \neq I \subseteq S$; $r_R(N) = \{r \in R \mid Nr = 0\}$, $l_S(N) = \{\varphi \in S \mid \varphi N = 0\}$ for $N \leq M$.

2. Injectivity and some of its generalizations

We begin with some basic definitions and examples.

Definition 2.1. Let M and N be right R-modules. M is called N-*injective* if, $\forall\ N' \leq N$ and $\forall\ \varphi : N' \to M$, $\exists\ \overline{\varphi} : N \to M$ such that $\overline{\varphi}|_{N'} = \varphi$. M is said to be *quasi-injective* if M is M-injective. M is called *injective* if M is N-injective for all right R-modules N.

It is easy to see that any vector space is an injective module over its base field, and every semisimple module is always quasi-injective. The following examples motivate this study about direct sums.

Example 2.2. Let $R = \prod_{i \in \mathbb{N}} \mathbb{F}$ be a product of fields $\mathbb{F}$ and let $M_i = \mathbb{F}_R$. Then $M = \bigoplus_{i \in \mathbb{N}} M_i$ is semisimple and $E(M) = \prod_{i \in \mathbb{N}} \mathbb{F}$, thus M is a quasi-injective R-module which is not injective.

Example 2.3. Let $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$ with $\mathbb{F}$ a field. Then $M = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{pmatrix}$ is an injective R-module and $N = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{F} \end{pmatrix}$ is a quasi-injective R-module. However, $M \oplus N = R$ is not a quasi-injective R-module.

Example 2.4. Consider $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$, where p is a prime number. Each of these is a quasi-injective $\mathbb{Z}$-module. However, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is not a quasi-injective $\mathbb{Z}$-module.

For the stronger notion of injective modules, the finite direct sums inherit the property.

Theorem 2.5. *$E = \bigoplus_{i=1}^{n} E_i$ is injective iff each E_i is injective for $1 \leq i \leq n$.*

Next, we introduce some finiteness conditions that will be required for results on infinite direct sums.

Definition 2.6. Let $\mathcal{I}$ be an index set, and $\{M_\alpha\}_{\alpha \in \mathcal{I}}$ be a family of R-modules.

- (A_1) For every choice of distinct $\alpha_i \in \mathcal{I}$ $(i \in \mathbb{N})$ and $m_i \in M_{\alpha_i}$, the ascending sequence $(\bigcap_{i \geq n} r_R(m_i))_{n \in \mathbb{N}}$ becomes stationary.
- (A_2) For every choice of $x \in M_\alpha$ $(\alpha \in \mathcal{I})$ and $m_i \in M_{\alpha_i}$ for distinct $\alpha_i \in \mathcal{I}$ $(i \in \mathbb{N})$ such that $r_R(m_i) \geq r_R(x)$, the ascending sequence $(\bigcap_{i \geq n} r_R(m_i))_{n \in \mathbb{N}}$ becomes stationary.
- (A_3) For every choice of distinct $\alpha_i \in \mathcal{I}$ $(i \in \mathbb{N})$ and $m_i \in M_{\alpha_i}$, if the sequence $(r_R(m_i))_{i \in \mathbb{N}}$ is ascending, then it becomes stationary.

Note that $A_1 \Rightarrow A_2 \Rightarrow A_3$: Reverse implications do not hold true in general [36].

For the case of a specific infinite direct sum of injective module we need condition (A_1) to get the following:

Proposition 2.7. *(Proposition 1.10,* [36]*) $M = \bigoplus_{\alpha \in \mathcal{I}} M_\alpha$ is injective if and only if each M_α is injective and (A_1) holds where $\mathcal{I}$ is an index set.*

For an infinite direct sum of quasi-injective modules to be quasi-injective, we also need the slightly weaker chain condition (A_2):

Proposition 2.8. *(Proposition 1.18,* [36]*) The following conditions are equivalent for a direct sum decomposition of a module $M = \bigoplus_{\alpha \in \mathcal{I}} M_\alpha$:*

(a) *M is quasi-injective;*
(b) *M_α is quasi-injective and $\bigoplus_{\beta\in\mathcal{I}-\{\alpha\}} M_\beta$ is M_α-injective for every $\alpha \in \mathcal{I}$;*
(c) *M_α is M_β-injective for all $\alpha, \beta \in \mathcal{I}$ and (A_2) holds.*

Corollary 2.9. *(Corollary 1.19, [36]) $\bigoplus_{i=1}^n M_i$ is quasi-injective iff M_i is M_j-injective for all $1 \le i, j \le n$. In particular, $M^{(n)}$ is quasi-injective iff M is quasi-injective.*

In recent years the notions of continuous, quasi-continuous, and extending modules have garnered a great deal of interest. The remainder of this section is devoted to the direct sum problem for these classes of modules.

Definition 2.10. Let M be a right R-module. Consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M.
(C_2) If a submodule L of M is isomorphic to a direct summand of M, then L is a direct summand of M.
(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called *extending* (or *CS*) if it satisfies (C_1), M is called *continuous* if it satisfies (C_1) and (C_2), and M is called *quasi-continuous* if it satisfies (C_1) and (C_3). A ring R is called *right extending* or *right (quasi-)continuous* if R_R is extending or (quasi-)continuous, respectively.

The notion of extending modules was generalized to that of FI-extending modules in [7].

Definition 2.11. A right R-module M is called *FI-extending* if every *fully invariant* submodule of M is essential in a direct summand of M. (When $M = R_R$, the fully invariant submodules are precisely the 2-sided ideals of R.) A ring R is right *FI-extending* if R_R is an FI-extending module, i.e., every *ideal* of R is essential in a right ideal direct summand of R.

Large classes of modules and rings are FI-extending, but not necessarily extending (for example, direct sums of uniform modules, the ring of upper triangular matrices over $\mathbb{Z}$, etc; see Example 2.13(v) and [7]).

Remark 2.12. The hierarchy of the notions we have considered until now, goes as follows: Injective ⇒ Quasi-injective ⇒ Continuous ⇒ Quasi-continuous ⇒ Extending ⇒ FI-extending.

Example 2.13. The following examples show that, in general, the reverse implications in Remark 2.12 are not true.

(i) $M = \bigoplus_{i\in\mathbb{N}} \mathbb{F}$ is a quasi-injective R-module but not an injective R-module where $R = \prod_{i\in\mathbb{N}} \mathbb{F}$ with $\mathbb{F}$ a field. More precisely, the injective hull of M is $E(M) = \prod_{i\in\mathbb{N}} \mathbb{F}$.
(ii) Let R be a ring which has only 3 right ideals but which is not left artinian. Then $M = R_R$ is continuous but not quasi-injective. Since if so, R will be right self-injective and hence quasi-Frobenius, a contradiction. (See page 337 in [17] for an explicit example.)
(iii) $\mathbb{Z}_\mathbb{Z}$ is quasi-continuous but not continuous ($\mathbb{Z}_\mathbb{Z} \cong n\mathbb{Z}$, but $n\mathbb{Z}$ is not a direct summand of $\mathbb{Z}_\mathbb{Z}$).
(iv) Let $\mathbb{F}$ be a field and let $R = \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{smallmatrix}\right)$. Then R_R is an extending module but is not quasi-continuous.

(v) The $\mathbb{Z}$-module $\bigoplus_{i=1}^{\infty}\mathbb{Z}$ is FI-extending but not extending. Further, let $R = \left(\begin{smallmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{smallmatrix}\right)$. Then R_R is FI-extending but not extending.

The module $M = R_R$ in Example 2.3 and the module $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ in Example 2.4 also exhibit direct sums of (quasi-)continuous modules which are not (quasi-)continuous. Yet, in each of these cases, the direct sum is extending. In the next two examples, we present a situation where similar direct sums are not even extending.

Example 2.14. $\mathbb{Z}_p, \mathbb{Z}_{p^3}$ are quasi-injective $\mathbb{Z}$-modules, where p is a prime number; consequently, each of these modules is quasi-injective hence (quasi-)continuous, and so extending. However, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is not an extending $\mathbb{Z}$-module.

Example 2.15. Direct sums of extending modules are not extending in general:
(i) Let $M = \bigoplus_{i=1}^{\infty}\mathbb{Z}$. Then M is not an extending $\mathbb{Z}$-module, while the domain $\mathbb{Z}$ is uniform and hence extending;
(ii) Let $R = \mathbb{Z}[X]$. Thus R is a commutative Noetherian domain (hence quasi-continuous), but $R \oplus R$ is not an extending R-module.

These examples illustrate that some extra conditions will be required for a direct sum of (quasi-)continuous or extending modules to be (quasi-)continuous or extending, respectively. The next two results address this question for the classes of continuous and quasi-continuous modules, respectively.

Theorem 2.16. *(Theorem 3.16,* [36]*) The following conditions are equivalent for a module $M = \bigoplus_{\alpha\in\mathcal{I}} M_\alpha$:*

(a) *M is continuous;*
(b) *M is quasi-continuous and the M_α are continuous;*
(c) *M_α is continuous and M_β-injective for all $\alpha \neq \beta$, and (A_2) holds.*

Proof. See also [40] and Theorem 8 in [39]. □

Theorem 2.17. *(Theorem 2.13,* [36]*) Let $\{M_\alpha\}_{\alpha\in\mathcal{I}}$ be a family of quasi-continuous modules. Then the following conditions are equivalent:*

(a) *$M = \bigoplus_{\alpha\in\mathcal{I}} M_\alpha$ is quasi-continuous;*
(b) *$\bigoplus_{\beta\in\mathcal{I}-\{\alpha\}} M_\beta$ is M_α-injective for every $\alpha \in \mathcal{I}$;*
(c) *M_α is M_β-injective for all $\alpha \neq \beta \in \mathcal{I}$ and (A_2) holds.*

Proof. See also [40] and Theorem 7 in [39]. □

Corollary 2.18. *(Theorems 12 and 13,* [38]*) Let $M = \bigoplus_{i\in\mathcal{I}} M_i$ where $\mathcal{I}$ is an index set. If $\mathcal{I}$ is finite or R is right Noetherian, then M is continuous if and only if each M_i is continuous and M_j-injective for all $j \neq i \in \mathcal{I}$.*

Example 2.19. (i) Any direct sum of simple modules is quasi-injective, but the direct sum of their injective hulls need not be injective.
(ii) Let R be a domain, and $A_i = E(R)$ $(i = 0, 1, \dots)$. Then $\bigoplus_{i=0}^{\infty} A_i$ need not be quasi-continuous. Otherwise, $\bigoplus_{i=1}^{\infty} A_i$ is A_0-injective, hence $\bigoplus_{i=1}^{\infty} A_i$ is injective as $R \subset A_0$, thus $E(R)$ is Σ-injective, and R is a right Ore domain.

Proposition 2.20. *(Lemma 7.9,* [13]*) Let M_1 and M_2 be extending modules. Then $M = M_1 \oplus M_2$ is extending if and only if every closed submodule K of M with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.*

Proposition 2.21. *(Proposition 7.10,* [13]*) Let $M = M_1 \oplus \cdots \oplus M_n$ such that M_i is M_j-injective for all $1 \leq i \neq j \leq n$. Then M is extending iff each M_i is extending.*

In Proposition 2.21, the assumption that M_i is M_j-injective for all $1 \leq i \neq j \leq n$ is a sufficient but not necessary condition for the direct sum of extending modules to be extending (e.g, $\mathbb{Z} \oplus \mathbb{Z}$ is extending, but $\mathbb{Z}$ is not $\mathbb{Z}$-injective.)

To obtain conditions for a direct sum of extending modules to be extending, Mohamed and Müller renamed a notion studied by Oshiro earlier and called it Ojectivity to honor Oshiro for his contributions to this study.

Definition 2.22. For M, N right R-modules, N is M-ojective if for any submodule $X \leq M$ and any homomorphism $\varphi : X \to N$, there exist decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ together with homomorphisms $\varphi_1 : M_1 \to N_1$ and $\varphi_2 : N_2 \to M_2$, such that φ_2 is one-to-one, and for $x = m_1 + m_2$ and $\varphi(x) = n_1 + n_2$ one has $n_1 = \varphi_1(m_1)$ and $m_2 = \varphi_2(n_2)$. If N is M-ojective and M is N-ojective, we say that M and N are mutually ojective.

Example 2.23. The $\mathbb{Z}$-module $\mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^{n+1}}$ is self-ojective but not self-injective.

Theorem 2.24. *(Theorem 10,* [37]*) Let $M = M_1 \oplus M_2$. Then M_i is extending and is M_j-ojective for $1 \leq i \neq j \leq 2$ if and only if for any closed submodule N, we have $M = N \oplus M_1' \oplus M_2'$ with $M_i' \leq M_i$ for $1 \leq i \leq 2$.*

Recall that a decomposition $M = \bigoplus_{i \in \mathcal{I}} M_i$ is called *exchangeable* if for any summand N of M, we have $M = \bigoplus_{i \in \mathcal{I}} M_i' \oplus N$ with $M_i' \leq M_i$.

Theorem 2.25. *(Theorem 11,* [37]*) Let $n \geq 2$ be an integer and let $M = \bigoplus_{i=1}^{n} M_i$. Then the following conditions are equivalent:*

(a) *M is extending and the decomposition is exchangeable;*
(b) *M_i is extending, and $M_1 \oplus \cdots \oplus M_{i-1}$ and M_i are mutually ojective for $2 \leq i \leq n$.*

Theorem 2.26. *(Theorem 13,* [37]*) Let $M = M_1 \oplus \cdots \oplus M_n$, where the M_i are uniform. Then M is extending and the decomposition is exchangeable if and only if M_i is M_j-ojective for all $i \neq j$.*

More recently, another generalization of relative injectivity was introduced, which is a necessary rather than a sufficient condition for a direct sum of extending modules to be extending. This is in contrast to some of the other generalizations we have considered so far.

Definition 2.27. ([48]) We say that M_2 is *relatively $*$-injective* with respect to M_1 if, $\forall\ K \leq^{ess} M_1$ and $\varphi : K \to M_2$, there exist homomorphisms $\alpha, \alpha', \beta, \beta'$ such that the diagram below commutes:

$$\begin{array}{ccccc} & & M_2 & \overset{\alpha'}{\to} & M_1 \\ & & \uparrow \varphi & & \uparrow \beta' \\ 0 & \to & K & \hookrightarrow & M_1, \\ & & \downarrow \varphi & & \downarrow \beta \\ & & M_2 & \overset{\alpha}{\to} & M_2 \end{array} \quad \text{i.e.} \quad \alpha\varphi = \beta, \alpha'\varphi = \beta'$$

and either (α, β) or (α', β') is nontrivial.

Example 2.28. $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^l}$ are relative $*$-injective.

A result due to Osofsky (Corollary 7.4, [13]) states that, for a uniserial module M with unique composition series $0 < U < V < M$, $M \oplus (V/U)$ is not extending, although both summands are uniform (in fact, one is simple). It can be checked that M and V/U are not relatively $*$-injective.

When $M_1 = M_2$ the definition of relative $*$-injectivity yields the following.

Definition 2.29. We say that M is quasi-$*$-injective if, $\forall\ K \leq^{ess} M$, $\forall\ \varphi : K \to M$, $\exists\ \alpha \neq 0$, $\beta \neq 0$ such that $\alpha\varphi = \beta$ on K:

$$\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow \varphi & & \downarrow \beta \\ & & M & \stackrel{\alpha}{\to} & M \end{array}$$

Example 2.30. $\mathbb{Z}_{\mathbb{Z}}$ is quasi-$*$-injective, but it is not quasi-injective.

We have the following result which is of interest to our present investigations.

Proposition 2.31. *If $M = M_1 \oplus M_2$ is extending then M_i is extending and relatively M_j-$*$-injective for any $1 \leq i \neq j \leq 2$.*

We mention that this condition is, however, not sufficient for direct sums of extending modules to inherit the property.

Finally, we consider the class of FI-extending modules. Recall that every fully invariant submodule of such a module is essential in a direct summand. The class of fully invariant submodules of a module includes some of its most well-known submodules such as its socle, its singular submodule, the Jacobson radical, or the second singular submodule etc. One interesting property of the class of FI-extending modules, is that it is *closed under direct sums without any additional requirements.* This also provides a motivation for the study of this notion.

Theorem 2.32. *(Theorem 1.3, [7]) Every direct sum of FI-extending modules is always FI-extending.*

Corollary 2.33. *(Corollary 1.4, [7]) Let M be a direct sum of extending (e.g., uniform) modules. Then M is FI-extending.*

In view of Corollary 2.33, we obtain that in any direct sum of extending modules, every fully invariant submodule will always be essential in a direct summand without any additional requirements.

So far we have discussed some of the cases when specific direct sums of modules with a property generalizing injectivity, inherit that property. We conclude this section by a result of Matlis and Papp showing that if the direct sum of any family of injective R-modules is injective then the ring has to be right noetherian.

Theorem 2.34. *(Theorem 1.11, [36]) R is right noetherian if and only if $\bigoplus_{i\in\mathcal{I}} E_i$ is injective for any index set $\mathcal{I}$ and any family of injective modules $\{E_i\}_{i\in\mathcal{I}}$.*

3. Baer, Quasi-Baer, and Rickart Modules

Kaplansky introduced the notion of Baer rings in 1955 [26] which was extended to that of quasi-Baer rings by Clark in 1967 [12]. These two notions have their roots in functional analysis. A ring R is called *(quasi-)Baer* if the right annihilator of any nonempty subset (two-sided ideal) of R is generated by an idempotent as a right ideal. Examples of Baer rings include (any product of) domains, the Boolean ring of all subsets of a given set, and the ring of all bounded operators on a Hilbert space. Examples of quasi-Baer rings include any prime ring and any n-by-n upper triangular matrix ring over a domain. A prime ring with a nonzero singular ideal is quasi-Baer but not Baer. Similarly, an n-by-n upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not Baer, e.g., $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$.

In 1980, Chatters and Khuri proved a very useful result. Recall that a ring R is called *right nonsingular* if $\{t \in R \mid r_R(t) \leq^{ess} R_R\} = 0$. R is called *right cononsingular* if any right ideal, with zero left annihilator, is essential in R_R.

Theorem 3.1. *(Theorem 2.1,* [9]*) A ring R is right nonsingular, right extending if and only if R is a right cononsingular Baer ring.*

The notions of Baer and quasi-Baer rings were extended to a general module theoretic setting using the endomorphism ring of a module in 2004 [44].

Definition 3.2. ([44]) A right R-module M is called *Baer* if the right annihilator in M of any nonempty subset of $End_R(M)$ is a direct summand of M.

Example 3.3. Every semisimple module is a Baer module. R_R is a Baer module if R is a Baer ring. Every nonsingular injective (or extending) module is Baer. $\mathbb{Z}^{(\mathbb{N})}$ $(\cong \mathbb{Z}[x])$ is a Baer $\mathbb{Z}$-module, while $\mathbb{Z}^{(\mathbb{R})}$ is not a Baer $\mathbb{Z}$-module.

Recall that a module M is said to have the *strong summand intersection property* (SSIP) if the intersection of any family of direct summands is a direct summand of M. Examples include Baer modules and indecomposable modules. The $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$ has the SSIP, while the $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{R})}$ has not the SSIP.

It was proved in [44] that similar to Theorem 3.1, there are close connections between extending modules and Baer modules. A module M is called $\mathcal{K}$*-nonsingular* if, for all $\varphi \in End_R(M), Ker\varphi \leq^{ess} M$ implies $\varphi = 0$. M is called $\mathcal{K}$*-cononsingular* if, for all $N \leq M, l_S(N) = 0$ implies $N \leq^{ess} M$.

The next useful result explicitly characterizes an extending module in terms of a Baer module analogous to the ring case in Theorem 3.1. This result may help translate results on (direct sums of) Baer modules to those on (direct sums of) extending modules and vice-versa.

Theorem 3.4. *(Theorem 2.12,* [44]*) A module M is $\mathcal{K}$-nonsingular and extending iff M is a $\mathcal{K}$-cononsingular Baer module.*

Proposition 3.5. *The following statements hold true:*

(i) *Every direct summand of a Baer module is a Baer module.*
(ii) *Every Baer module has the SSIP.*
(iii) *The endomorphism ring of a Baer module is a Baer ring.*
(iv) *A finitely generated $\mathbb{Z}$-module M is Baer iff M is semisimple or torsion-free.*

Proof. See Theorem 2.17, Proposition 2.22, Theorem 4.1, and Proposition 2.19 in [44]. □

A module M is said to be *retractable* if, for every $0 \neq N \leq M$, $Hom_R(M, N) \neq 0$. Note that every free R-module is retractable. The next result shows that the property of retractability passes to arbitrary direct sums of copies of a retractable module.

Lemma 3.6. *(Lemma 2.8,* [47]*) Let $\{M_i\}_{i\in\mathcal{I}}$ be a class of retractable modules. Then $\bigoplus_{i\in\mathcal{I}} M_i$ is retractable.*

We recall a result of Khuri (Theorem 3.2, [29]).

Theorem 3.7. *Let M_R be nonsingular and retractable. Then $End_R(M)$ is a right extending ring if and only if M is a extending module.*

The evident close connections between Baer and extending modules (Theorem 3.4) suggest that a similar result of Theorem 3.7 could possibly hold true for the case of Baer modules. In the following we show that this is the case:

Proposition 3.8. *(Proposition 4.6,* [44]*) Let M be a retractable module. If the endomorphism ring of M is a Baer ring then M is a Baer module.*

Theorem 3.9. *(Proposition 2.22,* [44] *and Theorem 2.5,* [47]*) The following conditions are equivalent for a module M:*
(a) *M is a Baer module;*
(b) *M has the SSIP and $Ker\varphi \leq^{\oplus} M$ for all $\varphi \in S$;*
(c) *$End_R(M)$ is a Baer ring and M is quasi-retractable.*

Recall that a module M is called *quasi-retractable* if, for any left ideal I of $End_R(M)$ and $0 \neq r_M(I)$, $Hom(M, r_M(I)) \neq 0$. Note that every retractable module is quasi-retractable. Next, the following example (due to Chatters) exhibits an R-module which is quasi-retractable but not retractable.

Example 3.10. (Example 3.4, [28]) Let K be a subfield of complex numbers $\mathbb{C}$. Let R be the ring $\left[\begin{smallmatrix} K & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right]$. Then R is a right nonsingular right extending ring. Consider the module $M = eR$ where $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then M is projective, extending, and nonsingular (as it is a direct summand of R), hence is Baer by Theorem 3.4. Thus M is quasi-retractable, by Theorem 3.9. But M is not retractable, since the endomorphism ring of M, which is isomorphic to K, consists of isomorphisms and the zero endomorphism; on the other hand, M is not simple, and retractability implies that there exist nonzero endomorphisms which are not onto.

Proposition 3.11. *(Proposition 4.6,* [44]*) The endomorphism ring of a free module F_R is a Baer ring if and only if F_R is a Baer module.*

Definition 3.12. ([44]) A right R-module M is called *quasi-Baer* if for all $N \trianglelefteq M$, $l_S(N) = Se$ for some $e^2 = e \in S = End_R(M)$.

Example 3.13. All semisimple modules are quasi-Baer; all Baer and quasi-Baer rings are quasi-Baer modules, viewed as modules over themselves. Baer modules are obviously quasi-Baer modules. Every finitely generated abelian group is quasi-Baer. Every direct sum of copies of a quasi-Baer module is a quasi-Baer module.

The class of quasi-Baer modules is strictly larger than that of Baer modules, as the next example will show.

Example 3.14. $\mathbb{Z}^{(\mathbb{R})}$ is a quasi-Baer module, but is not a Baer module.

Similar to Theorems 3.1 and 3.4, we show that there are also close connections between the class of FI-extending modules and that of quasi-Baer modules ([44]). A module M is called *FI-$\mathcal{K}$-nonsingular* if, for all $I \trianglelefteq S$ such that $r_M(I) \leq^{ess} eM$ for $e^2 = e \in S$, we have $r_M(I) = eM$. M is called *FI-$\mathcal{K}$-cononsingular* if, for every $N \trianglelefteq^{\oplus} M$ and $N' \trianglelefteq N$ such that $\varphi(N') \neq 0$ for all $\varphi \in End_R(M)$, we have $N' \leq^{ess} N$.

Theorem 3.15. *(Theorem 3.10,* [44]*) A module M is FI-$\mathcal{K}$-nonsingular and FI-extending iff M is quasi-Baer and FI-$\mathcal{K}$-cononsingular.*

Corollary 3.16. *(Corollary 3.16,* [44]*) A ring R is right FI-extending and right FI-$\mathcal{K}$-nonsingular if and only if R is quasi-Baer and right FI-$\mathcal{K}$-cononsingular.*

Proposition 3.17. *(Proposition 3.8,* [44]*) The following equivalences hold true for a module M and $S = End_R(M)$.*

(i) *M is FI-$\mathcal{K}$-nonsingular if and only if, for all $I \trianglelefteq S$, $r_M(I) \leq^{ess} eM$ for $e^2 = e \in S$, implies $I \cap Se = 0$;*
(ii) *M is FI-$\mathcal{K}$-cononsingular if and only if, for all $N \trianglelefteq M$, $r_M(l_S(N)) \leq^{\oplus} M$ implies $N \leq^{ess} r_M(l_S(N))$.*

Proposition 3.18. *(Theorem 3.17 and Theorem 4.1,* [44]*) The following hold true:*
(i) *Every direct summand of a quasi-Baer module is a quasi-Baer module.*
(ii) *The endomorphism ring of a quasi-Baer module is a quasi-Baer ring.*

Example 3.19. Let $M = \mathbb{Z}_{p^\infty}$, considered as a $\mathbb{Z}$-module. Then it is well-known that $End_{\mathbb{Z}}(M)$ is the ring of p-adic integers. Since the ring of p-adic integers is a commutative domain, it is a (quasi-)Baer ring. However M is not a (quasi-)Baer module.

Proposition 3.20. *(Proposition 4.7,* [44]*) Let M be retractable. Then M is quasi-Baer if and only if $End_R(M)$ is a quasi-Baer ring.*

We remark that Example 3.27 also exhibits that a direct sum of two quasi-Baer modules need not be quasi-Baer, in general.

Next, we consider the notion of a right Rickart ring, introduced independently by Maeda and Hattori in 1960. Maeda defined a ring R to be *right Rickart* if the right annihilator of an element is generated by an idempotent, as a right ideal. Hattori called a ring a *right p.p. ring* if all of its principal right ideals are projective. Examples include von Neumann regular rings, Baer rings, right (semi-)hereditary rings, and $End_R(R^{(\mathcal{I})})$ with R a right hereditary ring and $\mathcal{I}$ an index set. It was discovered that the notion of a right Rickart ring coincides with that of a right p.p. ring. Motivated by the work on Baer modules [44] and the definition of a right Rickart ring we introduce the notion of a Rickart module. We do this by utilizing the endomorphism ring of a module, similar to the case of Baer modules.

Definition 3.21. ([45]) A right R-module M is called *Rickart* if the right annihilator in M of any single element of $End_R(M)$ is a direct summand of M.

Example 3.22. Every semisimple module is a Rickart module. R_R is a Rickart module if R is a right Rickart ring. Every Baer module is a Rickart module. Every projective right R-module over a right hereditary ring R is a Rickart module. The free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathcal{I})}$, for any index set $\emptyset \neq \mathcal{I}$, is Rickart, while $\mathbb{Z}^{(\mathcal{I})}$ is not a Baer $\mathbb{Z}$-module if $\mathcal{I}$ is uncountable. In particular, $\mathbb{Z}^{(\mathbb{N})}$ ($\cong \mathbb{Z}[x]$) is a Rickart (and Baer) $\mathbb{Z}$-module, while $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart but not a Baer $\mathbb{Z}$-module. In general, if R is a right hereditary ring which is not Baer then every free R-module is Rickart but not Baer. On the contrary, $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_4$ are injective and quasi-injective $\mathbb{Z}$-modules, respectively, neither of which is a Rickart $\mathbb{Z}$-module.

Note that an indecomposable Rickart module is a Baer module by Theorem 3.9. The next example provides another explicit instance when a Rickart module is not a Baer module.

Example 3.23. Let $T = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}_2 \mid a_n \text{ is eventually constant}\}$ and $I = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}_2 \mid a_n = 0 \text{ eventually}\} = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Now, consider the ring $R = \begin{pmatrix} T & T/I \\ 0 & T/I \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} (1,1,\dots) & 0+I \\ 0 & 0+I \end{pmatrix} \in R$. Then $M = eR = \begin{pmatrix} T & T/I \\ 0 & 0 \end{pmatrix}$ is a Rickart module, but is not Baer because $r_M\left(\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \begin{pmatrix} 0 & T/I \\ 0 & 0 \end{pmatrix}$ is not a direct summand of M where $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right) \subseteq End_R(M)$.

Recall that a module M is said to have the *summand intersection property* (SIP) if the intersection of any two direct summands is a direct summand of M. M is said to satisfy D_2 *condition* if, $\forall N \leq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$.

Proposition 3.24. *The following statements hold true:*

(i) *Every direct summand of a Rickart module is a Rickart module.*
(ii) *Every Rickart module satisfies D_2 condition.*
(iii) *Every Rickart module is $\mathcal{K}$-nonsingular.*
(iv) *Every Rickart module has the SIP.*
(v) *The endomorphism ring of a Rickart module is a right Rickart ring.*

Proof. See Theorem 2.7 and Propositions 2.11, 2.12, 2.16, and 3.2 in [32]. □

Theorem 3.25. *(Proposition 2.11 and Theorem 3.9,* [32]*) The following conditions are equivalent for a module M and $S = End_R(M)$:*

(a) *M is a Rickart module;*
(b) *M satisfies D_2 condition, and $Im\varphi$ is isomorphic to a direct summand of M for any $\varphi \in S$;*
(c) *S is a right Rickart ring and M is k-local-retractable.*

Recall that a module M is called *k-local-retractable* if $r_M(\varphi) = r_S(\varphi)(M)$ for any $\varphi \in S = End_R(M)$. Note that every free R-module is k-local-retractable.

Proposition 3.26. *(Corollary 5.3,* [32]*) The endomorphism ring of a free module F_R is a right Rickart ring if and only if F_R is a Rickart module.*

Until now, we have been focused on developing the notions and presenting basic properties of Baer, quasi-Baer, and Rickart modules. As we see from Propositions 3.5(i), 3.18(i), and 3.24(i), every direct summand each of Baer, quasi-Baer, and Rickart modules inherits the respective property. It is, therefore, natural to ask if these properties go to their respective direct sums? The next three examples and the following proposition show that this is not the case, in general. In fact, the Baer and Rickart properties are not inherited by even a direct sum of copies of such modules. In contrast, we will see in Corollary 4.15 that a direct sum of copies of a quasi-Baer module does, in fact, inherit the quasi-Baer property. The focus of our investigations in Sections 4, 5, and 6, will be these direct sum questions.

Example 3.27. It is easy to see that $\mathbb{Z}$ and $\mathbb{Z}_p$ are both Baer $\mathbb{Z}$-modules where p is a prime number in $\mathbb{N}$. However, the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_p$ is not Rickart: Consider the endomorphism $\varphi \in End_R(M)$ defined by $\varphi : (m, \overline{n}) \mapsto (0, \overline{m})$, then $Ker\varphi = p\mathbb{Z} \oplus \mathbb{Z}_p \leq^{ess} M$ is not a direct summand of M.

Example 3.28. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$, $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the modules $e_1R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2R = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ are Baer R-modules (since $End(e_1R) \cong \mathbb{Z} \cong End(e_2R)$). But $M = R_R$ is not a Rickart module. Since $End_R(M) \cong R$, the only direct summands of M are: $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & n \\ 0 & 1 \end{smallmatrix}\right)\mathbb{Z}$ where $n \in \mathbb{Z}$. Consider $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in End_R(M)$. Then $r_M\left(\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)\mathbb{Z}$ is not a direct summand of M.

Example 3.29. $M = \mathbb{Z}[x]$ is a Baer $\mathbb{Z}[x]$-module, but $M \oplus M$ is not a Rickart $\mathbb{Z}[x]$-module (more details in Example 6.8).

Our next result extends Example 3.27 to arbitrary modules and motivates our study.

Proposition 3.30. *(Proposition 2.1,* [33]*) If M is an indecomposable Rickart module which has a nonzero maximal submodule N, then $M \oplus (M/N)$ is not a Baer module, while M and M/N are Baer modules.*

4. Direct sums of Baer and quasi-Baer modules

In this section we will show results on direct sums of Baer and quasi-Baer modules and list conditions which allow for such direct sums to inherit these properties. Recall that a sufficient condition for a finite direct sum of extending modules to be extending is that each direct summand be relatively injective to all others (see [22] or Proposition 7.10 in [13]). We prove that an analogue holds true also for the case of Baer modules.

Definition 4.1. (Definition 1.3, [47], see also [32]) A module M is called *N-Rickart* (or *relatively Rickart to N*) if, for every homomorphism $\varphi : M \to N$, $Ker\varphi \leq^{\oplus} M$.

Theorem 4.2. *(Theorem 3.19,* [47]*) Let $\{M_i\}_{1\leq i\leq n}$ be a class of Baer modules where $n \in \mathbb{N}$. Assume that, for any $i \neq j$, M_i and M_j are relative Rickart and relative injective. Then $\bigoplus_{i=1}^n M_i$ is a Baer module.*

The preceding result was improved further by reducing the requirement of relative injectivity to a smaller subset of a finite index set $\mathcal{I}$ in [33] as follows.

Proposition 4.3. *(Proposition 2.14,* [33]*) Assume that there exists an ordering $\mathcal{I} = \{1, 2, \cdots, n\}$ for a class of Baer R-modules $\{M_i\}_{i\in\mathcal{I}}$ such that M_i is M_j-injective for all $i < j \in \mathcal{I}$. Then $\bigoplus_{i=1}^n M_i$ is a Baer module if and only if M_i is M_j-Rickart for all $i, j \in \mathcal{I}$.*

Using Proposition 4.3, we obtain the following useful consequence. First recall that in view of Theorem 3.4, every nonsingular extending module is Baer.

Theorem 4.4. *(Theorem 2.16,* [33]*) Let M be a nonsingular extending module. Then M and $E(M)$ are relatively Rickart to each other and $E(M) \oplus M$ is a Baer module.*

Remark 4.5. In the hypothesis of Theorem 4.4, it suffices to have that $E(M)$ be $\mathcal{K}$-nonsingular instead of M to be nonsingular. Since the $\mathcal{K}$-nonsingularity of $E(M)$ is inherited by M (see Proposition 2.18, [46]), the hypothesis of Theorem 4.4 can be improved to "if M is extending and $E(M)$ is $\mathcal{K}$-nonsingular then $E(M) \oplus M$ is a Baer module".

Next example shows that the extending condition for the module M in Theorem 4.4 is not superfluous.

Example 4.6. Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Then the ring A is commutative, von Neumann regular, and Baer. Consider $R = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant}\}$, a subring of A. Then R is a von Neumann regular ring which is not a Baer ring (see Example 7.54, [30]). Note that $M = R_R$ is not extending, but is a nonsingular Rickart module. On the other hand, the injective hull, $E(M) = A$, is an injective Rickart R-module. In this case, $E(M)$ is M-injective and M-Rickart, but M is not $E(M)$-Rickart: For $\varphi = (1, 0, 1, 0, \cdots, 1, 0, \cdots) \in Hom_R(M, E(M))$, $Ker\varphi$ is not a direct summand of M. Thus, $E(M) \oplus M$ is not a Rickart module.

The nonsingular condition for the module M in Theorem 4.4 is not superfluous (and can not be weakened to $\mathcal{K}$-nonsingularity) as the next example shows.

Example 4.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_p$ where p is a prime number. Then M is not nonsingular but is $\mathcal{K}$-nonsingular extending. However, $E(M) = \mathbb{Z}_{p^\infty}$ is not a Rickart $\mathbb{Z}$-module. Thus, $E(M) \oplus M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is not a Rickart $\mathbb{Z}$-module.

Corollary 4.8. *(Corollary 2.20,* [33]*) If M is a nonsingular extending module then $E(M)^{(n)} \oplus M$ is a Baer module for any $n \in \mathbb{N}$.*

Definition 4.9. A module M is called (*finitely*) Σ-*Rickart* if every (finite) direct sum of copies of M is a Rickart module. A (finitely) Σ-Baer module and a (finitely) Σ-extending module are defined similarly.

We remark that every right (semi)hereditary ring R is precisely (finitely) Σ-Rickart as a right R-module (see Theorems 6.2 and 6.16). Also, if M is a finitely generated retractable module and if $End_R(M)$ is a right (semi)hereditary ring then M is a (finitely) Σ-Rickart module (see Proposition 6.22 and Corollary 6.29).

The next corollary provides a rich source of examples of Baer modules (hence, of Rickart modules).

Corollary 4.10. *(Corollary 2.22,* [33]*) If M is a nonsingular finitely Σ-extending module then M and $E(M)$ are finitely Σ-Baer modules, and $E(M)^{(m)} \oplus M^{(n)}$ is a Baer (hence, Rickart) module for any $m, n \in \mathbb{N}$.*

An explicit application of Theorem 4.4 and Corollary 4.10 is exhibited in the next examples.

Example 4.11. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ as a right R-module. Since M is a nonsingular finitely Σ-extending module, $R^{(m)} \oplus M^{(n)} = E(M)^{(m)} \oplus M^{(n)}$ is a Baer R-module for any $m, n \in \mathbb{N}$ by Corollary 4.10. (Compare to Example 5.11.)

Example 4.12. Consider $M = \mathbb{Z}^{(n)}$ as a right $\mathbb{Z}$-module for any $n \in \mathbb{N}$. Then M is a nonsingular extending $\mathbb{Z}$-module and $E(M) = \mathbb{Q}^{(n)}$. Thus, from Theorem 4.4, $E(M) \oplus M = \mathbb{Q}^{(n)} \oplus \mathbb{Z}^{(n)}$ is a Baer $\mathbb{Z}$-module. In particular, $\mathbb{Q}^{(m)} \oplus \mathbb{Z}^{(n)}$ is a Baer $\mathbb{Z}$-module for $m, n \in \mathbb{N}$. We remark that $\mathbb{Z}$ is a nonsingular finitely Σ-extending $\mathbb{Z}$-module.

Note that for any $n \in \mathbb{N}$, $\mathbb{Z}^{(n)}$ is an extending and Baer $\mathbb{Z}$-module, $\mathbb{Z}^{(\mathbb{N})}$ is a Baer but not an extending $\mathbb{Z}$-module (Page 56, [13]), and $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart but neither a Baer nor an extending $\mathbb{Z}$-module (Remark 2.28, [32]).

Theorem 4.13. *(Theorem 3.18,* [44]*) Let M_1 and M_2 be quasi-Baer modules. If we have the property $\psi(x) = 0 \ \forall \ 0 \neq \psi \in Hom(M_i, M_j)$ implies $x = 0$ $(i \neq j$, $i, j = 1, 2)$ then $M_1 \oplus M_2$ is quasi-Baer.*

Proposition 4.14. *(Proposition 3.19,* [44]*) $\bigoplus_{i \in \mathcal{I}} M_i$ is quasi-Baer if M_i is quasi-Baer and subisomorphic to (i.e. isomorphic to a submodule of) M_j for all $i \neq j \in \mathcal{I}$ where $\mathcal{I}$ is an index set.*

Corollary 4.15. *A module M is quasi-Baer if and only if $M^{(\mathcal{I})}$ is a quasi-Baer module for any nonempty index set $\mathcal{I}$.*

Corollary 4.16. *(Corollary 3.20,* [44]*) A free module over a quasi-Baer ring is a quasi-Baer module.*

We can point out now that we have a general method of producing quasi-Baer modules that are not Baer modules.

Example 4.17. An infinitely generated free module M over a non-Dedekind commutative domain R is not a Baer R-module. On the other hand, since M is a free R-module over a quasi-Baer ring, M is a quasi-Baer module.

Next, we provide a complete characterization for an arbitrary direct sum of (quasi-)Baer modules to be (quasi-)Baer, provided that each module is fully invariant in the direct sum (see Proposition 2.4.15 in [49]).

Proposition 4.18. *(Proposition 3.20, [47]) Let $M_i \trianglelefteq \bigoplus_{j\in\mathcal{I}} M_j$, $\forall\ i \in \mathcal{I}$, $\mathcal{I}$ is an arbitrary index set. Then $\bigoplus_{j\in\mathcal{I}} M_j$ is a (quasi-)Baer module if and only if M_i is a (quasi-)Baer module for all $i \in \mathcal{I}$.*

5. Direct sums of Rickart modules

In this section we provide and exemplify a number of relative conditions for Rickart modules. These conditions are then used to obtain results for direct sums of Rickart modules to inherit the Rickart property. Recall that a module M is called *N-Rickart* (or *relatively Rickart to N*) if, for every homomorphism $\varphi : M \to N$, $Ker\varphi \leq^{\oplus} M$. In particular, a right R-module M is Rickart iff M is M-Rickart.

Our next characterization extends Proposition 3.24(i).

Theorem 5.1. *(Theorem 2.6, [33]) Let M and N be right R-modules. Then M is N-Rickart if and only if for any direct summand $M' \leq^{\oplus} M$ and any submodule $N' \leq N$, M' is N'-Rickart.*

Proposition 5.2. *(Proposition 2.9, [33]) Let $\{M_i\}_{i\in\mathcal{I}}$ and N be right R-modules. Then the following implications hold:*

- (i) *If N has the SIP, then N is $\bigoplus_{i\in\mathcal{I}} M_i$-Rickart if and only if N is M_i-Rickart for all $i \in \mathcal{I}$, $\mathcal{I} = \{1, 2, \cdots, n\}$.*
- (ii) *If N has the SSIP, then N is $\bigoplus_{i\in\mathcal{I}} M_i$-Rickart if and only if N is M_i-Rickart for all $i \in \mathcal{I}$, $\mathcal{I}$ is an arbitrary index set.*
- (iii) *If N has the SSIP, then N is $\prod_{i\in\mathcal{I}} M_i$-Rickart if and only if N is M_i-Rickart for all $i \in \mathcal{I}$, $\mathcal{I}$ is an arbitrary index set.*

Corollary 5.3. *(Corollary 2.10, [33]) For each $i \in \mathcal{I} = \{1, 2, \cdots, n\}$, M_i is $\bigoplus_{j\in\mathcal{I}} M_j$-Rickart if and only if M_i is M_j-Rickart for all $j \in \mathcal{I}$.*

While from Corollary 5.3 M_i is $\bigoplus_{j=1}^{n} M_j$-Rickart if M_i is M_j-Rickart for all $1 \leq j \leq n$, the next example shows that $\bigoplus_{i=1}^{n} M_i$ may not be M_j-Rickart even though M_i is M_j-Rickart for all $1 \leq i \leq n$.

Example 5.4. Let $R = \mathbb{Z}[x]$ and let $M_1 = M_2 = N = \mathbb{Z}[x]$ be right R-modules. While M_i is N-Rickart for all $i = 1, 2$, $M_1 \oplus M_2$ is not N-Rickart: Consider $\varphi = (2, x) \in Hom_R(M_1 \oplus M_2, N)$. Then $Ker\varphi = (x, -2)R$ is not a direct summand of $M_1 \oplus M_2$.

In the next result, we present conditions under which $\bigoplus_{i=1}^{n} M_i$ is M_j-Rickart.

Theorem 5.5. *(Theorem 2.12, [33]) Assume that there exists an ordering $\mathcal{I} = \{1, 2, \cdots, n\}$ for a class of R-modules $\{M_i\}_{i\in\mathcal{I}}$ such that M_i is M_j-injective for all $i < j \in \mathcal{I}$. Then $\bigoplus_{i=1}^{n} M_i$ is N-Rickart if and only if M_i is N-Rickart for all $i \in \mathcal{I}$, for any right R-module N.*

Example 5.4 also exhibits that the one-sided relative injective condition in Theorem 5.5 is not superfluous. (See that M_1 is not M_2-injective in that example.)

Corollary 5.6. *(Corollary 2.13, [33]) Assume that there exists an ordering $\mathcal{I} = \{1, 2, \cdots, n\}$ for a class of R-modules $\{M_i\}_{i\in\mathcal{I}}$ such that M_i is M_j-injective for all $i < j \in \mathcal{I}$. Then $\bigoplus_{i=1}^{n} M_i$ is a Rickart module if and only if M_i is M_j-Rickart for all $i, j \in \mathcal{I}$.*

Definition 5.7. ([33]) A module M is called N-C_2 (or *relatively* C_2 *to* N) if any submodule $N' \leq N$ with $N' \cong M' \leq^{\oplus} M$ implies $N' \leq^{\oplus} N$. Hence, M has C_2 condition iff M is M-C_2.

We now provide another instance when $\bigoplus_{i=1}^{n} M_i$ is M_j-Rickart if M_i is M_j-Rickart for all $1 \leq i \leq n$ (cf. Corollary 5.6).

Theorem 5.8. *(Theorem 2.29,* [33]*) Let* $\{M_i\}_{i\in\mathcal{I}}$ *be a class of right* R*-modules where* $\mathcal{I} = \{1, 2, \cdots, n\}$*. Assume that* M_i *is* M_j*-*C_2 *for all* $i, j \in \mathcal{I}$*. Then* $\bigoplus_{i=1}^{n} M_i$ *is a Rickart module if and only if* M_i *is* M_j*-Rickart for all* $i, j \in \mathcal{I}$*.*

In the next example, we show that the relative C_2 condition in Theorem 5.8 and the one-sided relative injective condition in Corollary 5.6 are not superfluous.

Example 5.9. It is easy to see that $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are Rickart $\mathbb{Z}$-modules as each is a domain. $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are relatively Rickart to each other by Theorem 5.1 and because every $0 \neq \varphi \in Hom_R(\mathbb{Z}, \mathbb{Z}[\frac{1}{2}])$ is a monomorphism. Further, $\mathbb{Z}[\frac{1}{2}]$ is $\mathbb{Z}$-C_2, but $\mathbb{Z}$ is not $\mathbb{Z}[\frac{1}{2}]$-C_2 and $\mathbb{Z}[\frac{1}{2}]$ is not $\mathbb{Z}$-injective. For $\psi \in End_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z})$ defined by $\psi : (a, m) \mapsto (3a - m, 0)$, $Ker\psi = \{(m, 3m) \mid m \in \mathbb{Z}\} \not\leq^{\oplus} \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$, hence $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ is not a Rickart $\mathbb{Z}$-module.

Corollary 5.10. *(Corollary 2.31,* [33]*) Let* M *be a Rickart module with* C_2 *condition. Then any finite direct sum of copies of* M *is a Rickart module.*

Next example follows from Corollary 5.10.

Example 5.11. Consider $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ as a right R-module. Since M is a nonsingular quasi-injective R-module, M is a Rickart module with C_2 condition. Thus by Corollary 5.10, $M^{(n)}$ is a Rickart module for any $n \in \mathbb{N}$.

Theorem 5.12. *(Theorem 2.33,* [33]*) A module* $M = M_1 \oplus M_2$ *is Rickart if and only if* M_1 *and* M_2 *are Rickart modules,* M_1 *is* M_2*-Rickart, and for any* $\varphi \in End_R(M)$ *with* $Ker\varphi \cap M_1 = 0$*,* $Ker\varphi \leq^{\oplus} M$*.*

We remark that in Example 3.29 while $M_1 = \mathbb{Z}[x]$ is M_1-Rickart, it does not satisfy the last part of the statement of Theorem 5.12. Thus $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ is not a Rickart $\mathbb{Z}[x]$-module.

We conclude this section with providing a complete characterization for an arbitrary direct sum of Rickart modules to be Rickart, provided that each module is fully invariant in the direct sum.

Proposition 5.13. *(Proposition 2.34,* [33]*) Let* $M_i \trianglelefteq \bigoplus_{j\in\mathcal{I}} M_j$*,* $\forall\ i \in \mathcal{I}$*,* $\mathcal{I}$ *is an arbitrary index set. Then* $\bigoplus_{j\in\mathcal{I}} M_j$ *is a Rickart module if and only if* M_i *is a Rickart module for all* $i \in \mathcal{I}$*.*

6. Free Rickart and free Baer modules

The last section of our paper is devoted to the case of a special case of direct sums, namely that of free modules over the base ring. We will obtain conditions for the base ring such that free (and projective) modules over the base ring are Rickart modules. To obtain our first main result of this section (Theorem 6.2), we begin with the following well-known result of L. Small.

Theorem 6.1. *(Proposition 7.63,* [30]*)* R *is a right semihereditary ring iff* $\mathsf{Mat}_n(R)$ *is a right Rickart ring for all* $n \in \mathbb{N}$*.*

Theorem 6.2. *(Theorem 3.6,* [33]*) The following are equivalent for a ring R:*

(a) *every finitely generated free (projective) right R-module is a Rickart module;*
(b) $\mathsf{Mat}_n(R)$ *is a right Rickart ring for all* $n \in \mathbb{N}$*;*
(c) *every finite direct sum of copies of* $R^{(k)}$ *is a Rickart R-module for some* $k \in \mathbb{N}$*;*
(d) $\mathsf{Mat}_k(R)$ *is a right semihereditary ring for some* $k \in \mathbb{N}$*;*
(e) *R is a right semihereditary ring.*

Note that (d)⇔(e) in Theorem 6.2 was also proved by Small in a conceptual manner, using different arguments.

We recall that a module is said to be *torsionless* if it can be embedded in a direct product of copies of the base ring. In our next result we provide a characterization of rings R for which every finitely generated free right R-module is Baer.

Theorem 6.3. *(Theorem 3.5,* [47]*) The following are equivalent for a ring R:*

(a) *every finitely generated free (projective) right R-module is a Baer module;*
(b) *every finitely generated torsionless right R-module is projective;*
(c) *every finitely generated torsionless left R-module is projective;*
(d) *R is left semihereditary and right* Π*-coherent (i.e. every finitely generated torsionless right R-module is finitely presented);*
(e) *R is right semihereditary and left* Π*-coherent;*
(f) $M_n(R)$ *is Baer ring for all* $n \in \mathbb{N}$*.*

In particular, a ring R satisfying these equivalent conditions is right and left semihereditary.

Remark 6.4. Note that Theorem 6.3 generalizes Theorem 2.2 in [16], which states that, for a von Neumann regular ring R, every finitely generated torsionless right R-module embeds in a free right R-module (FGTF property) iff $M_n(R)$ is a Baer ring for every $n \in \mathbb{N}$. Our result in fact establishes that every finitely generated torsionless right module is projective iff $M_n(R)$ is Baer for all $n \in \mathbb{N}$, even in the absence of von Neumann regularity of R.

As a consequence of Theorems 6.2 and 6.3, we have the next three corollaries.

Corollary 6.5. *(Corollary 3.10,* [33]*) R is a left* Π*-coherent ring and every finitely generated free R-module is Rickart iff every finitely generated free R-module is Baer.*

We obtain a characterization of Prüfer domains in terms of the Rickart or Baer property for finitely generated free (projective) right R-modules.

Corollary 6.6. *(Corollary 3.7,* [33] *and Corollary 15,* [54]*) Let R be a commutative integral domain. Then the following conditions are equivalent:*

(a) *every finitely generated free (projective) right R-module is a Baer module;*
(b) *every finitely generated free (projective) right R-module is a Rickart module;*
(c) *the free R-module* $R^{(k)}$ *is a Rickart module for some* $k \geq 2$*;*
(d) *the free R-module* $R^{(2)}$ *is a Rickart module;*
(e) $\mathsf{Mat}_2(R)$ *is a right Rickart ring;*
(f) *R is a Prüfer domain.*

Note that in Part(c) of Corollary 6.6, $k \geq 2$ is required. For $k = 1$ we have the example of the commutative domain $\mathbb{Z}$ (obviously a Rickart $\mathbb{Z}$-module), which is not a Prüfer domain.

We also obtain the following characterization of a Prüfer domain R in terms of the summand intersection property for finitely generated free (projective) right R-modules.

Corollary 6.7. *(Corollary 3.8,* [33]*) Let R be a commutative integral domain. Then the following conditions are equivalent:*

(a) *every finitely generated free (projective) right R-module has the SIP;*
(b) *the free R-module $R^{(k)}$ has the SIP for some $k \geq 3$;*
(c) *the free R-module $R^{(3)}$ has the SIP;*
(d) *R is a Prüfer domain.*

The next example shows a commutative integral domain R for which the free module $R^{(2)}$ has the SIP yet R is not a Prüfer domain. Thus, by Corollary 6.7 $R^{(3)}$ does not have the SIP. In this case, $R^{(2)}$ is not a Rickart R-module as well.

Example 6.8. Consider $R = \mathbb{Z}[x]$ which is not a Prüfer domain. Let $M = (R \oplus R)_R$. If $(g, h)R$ and $(g', h')R$ are two proper direct summands of $R \oplus R$ for $g, g', h, h' \in R$, then by simple calculations we can show that either $(g, h)R \cap (g', h')R = (0, 0)$ or $(g, h)R = (g', h')R$. Thus M has the SIP but $R \oplus R \oplus R$ can not satisfy the SIP as a $\mathbb{Z}[x]$-module by Corollary 6.7. Furthermore, let $N = R_R$. By Example 5.4 we know that M is not N-Rickart. Thus, by Theorem 5.1 M is not a Rickart $\mathbb{Z}[x]$-module.

Note that $\mathbb{Z}[x]$ and $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ are Rickart $\mathbb{Z}$-modules because $\mathbb{Z}[x] \oplus \mathbb{Z}[x] \cong_{\mathbb{Z}} \mathbb{Z}^{(\mathbb{N})} \oplus \mathbb{Z}^{(\mathbb{N})} \cong_{\mathbb{Z}} \mathbb{Z}^{(\mathbb{N})}$ (Remark 2.28, [32]).

Definition 6.9. A ring R is said to be *right n-hereditary* if every n-generated right ideal of R is projective.

Theorem 6.10. *(Proposition 3.13,* [33]*) The following conditions are equivalent for a ring R and a fixed $n \in \mathbb{N}$:*

(a) *every n-generated free (projective) right R-module is a Rickart module;*
(b) *the free R-module $R^{(n)}$ is a Rickart module;*
(c) $\mathsf{Mat}_n(R)$ *is a right Rickart ring;*
(d) *R is a right n-hereditary ring.*

For a fixed $n \in \mathbb{N}$, we obtain the following characterization for every n-generated free R-module to be Baer.

Theorem 6.11. *(Theorem 3.12,* [47]*) The following conditions are equivalent for a ring R and a fixed $n \in \mathbb{N}$:*

(a) *every n-generated free (projective) right R-module is a Baer module;*
(b) *every n-generated torsionless right R-module is projective.*

It is interesting to note that, as opposed to related notions (such as injectivity, quasi-injectivity, continuity and quasi-continuity), having $M \oplus M$ Baer does not imply that $M \oplus M \oplus M$ is also Baer.

We start with a lemma and by recalling the concept of an *n-fir*.

Definition 6.12. A ring R is said to be a *right n-fir* if any right ideal that can be generated with $\leq n$ elements is free of unique rank (i.e., for every $I \leq R_R$, $I \cong R^{(k)}$ for some $k \leq n$, and if $I \cong R^{(l)} \Rightarrow k = l$) (for alternate definitions see Theorem 1.1, [10]).

The definition of (right) n-firs is left-right symmetric, thus we will call such rings simply n-firs. For more information on n-firs, see [10].

Theorem 6.13. *(Theorem 3.16,* [47]*) Let R be a n-fir. Then $R^{(n)}$ is a Baer R-module. Consequently, $M_n(R)$ is a Baer ring.*

We remark that a right n-hereditary ring may not be a right $(n+1)$-hereditary ring. In Example 6.8, while $\mathbb{Z}[x]$ is a right 1-hereditary ring, $\mathbb{Z}[x]$ is not a right 2-hereditary ring. The next example due to Jøndrup [24] exhibits a module M such that $M^{(n)}$ is a Baer module, while $M^{(n+1)}$ is not a Rickart module.

Example 6.14. ([24]) Let n be any natural number, K be any commutative field, and let R be the K-algebra on the $2(n+1)$ generators X_i, Y_i $(i = 1, \cdots, n+1)$ with the defining relation

$$\sum_{i=1}^{n+1} X_i Y_i = 0.$$

Since R is an n-fir (Theorem 2.3, [24]), $R^{(n)}$ is a Baer R-module by Theorem 6.13. In particular, since R is not $(n+1)$-hereditary, $R^{(n+1)}$ is not a Rickart R-module.

Next, we provide an alternate proof of an earlier result of Small using the theory of Rickart modules (see Theorem 7.62, [30]).

Theorem 6.15. *For any $k \in \mathbb{N}$, R is a right hereditary ring if and only if $\mathsf{Mat}_k(R)$ is a right hereditary ring.*

Theorem 6.16. *(Theorem 2.26,* [32] *and Proposition 3.20,* [33]*) The following conditions are equivalent for a ring R:*

(a) *every free (projective) right R-module is a Rickart module;*
(b) *every direct sum of copies of $R^{(k)}$ is a Rickart R-module for some $k \in \mathbb{N}$;*
(c) *every column finite matrix ring over R, $CFM(R)$, is a right Rickart ring;*
(d) *the free R-module $R^{(R)}$ is a Rickart module;*
(e) *$CFM_{\Gamma_0}(R)$ is a right Rickart ring for $|\Gamma_0| = |R|$;*
(f) *$\mathsf{Mat}_k(R)$ is a right hereditary ring for some $k \in \mathbb{N}$;*
(g) *R is a right hereditary ring.*

In the following we characterize the class of rings R for which every projective R-module is a Baer module. A ring R is said to be a *semiprimary ring* if the Jacobson radical, $\text{Rad}(R)$, is nilpotent and $R/\text{Rad}(R)$ is semisimple.

Theorem 6.17. *(Theorem 3.3,* [47] *and Corollary 3.23,* [33]*) The following conditions are equivalent for a ring R:*

(a) *every free (projective) right R-module is a Rickart module and R is a semiprimary ring;*
(b) *every free (projective) right R-module is a Baer module;*
(c) *every free (projective) right R-module has the SSIP;*
(d) *R is a right hereditary, semiprimary ring.*

We remark that in the preceding result,'projective' can be replaced by 'flat'. The semiprimary condition in Theorem 6.17(d) is not superfluous as next example shows.

Example 6.18. $\mathbb{Z}$ is a non-semiprimary right hereditary ring. $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart $\mathbb{Z}$-module which is not a Baer $\mathbb{Z}$-module (Remark 2.28, [32]).

From Theorem 2.20 in [46] (see Theorem 6.21) we showed that a ring is semisimple artinian if and only if every R-module is Baer. For the commutative rings one can restrict the requirement of "every R-module" to "every free R-module" to obtain the same conclusion.

Proposition 6.19. *(Theorem 6,* [55]*) Let R be a commutative ring. Every free R-module is Baer if and only if R is semisimple artinian. In particular, every R-module is Baer if every free R-module is so.*

Theorem 6.20. *(Theorem 3.18,* [33]*) The following are equivalent for a ring R:*

(a) *every finitely generated free (projective) right R-module is a Rickart module with C_2 condition;*
(b) *every finitely generated free (projective) right R-module is a Rickart module with C_3 condition;*
(c) *the free module $R^{(k)}$ is a Rickart module with C_2 condition for some $k \in \mathbb{N}$;*
(d) *the free module $R^{(k)}$ is a Rickart module with C_3 condition for some $k \geq 2$;*
(e) *the free module $R^{(2)}$ is a Rickart module with C_3 condition;*
(f) *R is a von Neumann regular ring.*

We remark that in Part(d) of Theorem 6.20, $k \geq 2$ is required. For $k = 1$, even though R_R may be a Rickart module with C_3 condition, R may not be a von Neumann regular ring. In Example 6.8, $\mathbb{Z}[x]$ is a Rickart $\mathbb{Z}[x]$-module with C_3 condition while $\mathbb{Z}[x]$ is not a von Neumann regular ring.

We now characterize the semisimple artinian rings in terms of Rickart and Baer modules.

Theorem 6.21. *(Theorem 2.20,* [46] *and Theorem 2.25,* [32]*) The following conditions are equivalent for a ring R:*

(a) *every right R-module is a Baer module;*
(b) *every right R-module is a Rickart module;*
(c) *every extending right R-module is a Rickart module;*
(d) *every injective right R-module is a Rickart module;*
(e) *every injective right R-module is a Baer module;*
(f) *R is a semisimple artinian ring.*

We extend Theorem 6.1 to a module theoretic setting using Lemma 3.6 (every direct sum of copies of an arbitrary retractable module is retractable).

Proposition 6.22. *(Proposition 3.2,* [33]*) Let M be a right R-module. If every finite direct sum of copies of M is a Rickart module then $End_R(M)$ is a right semihereditary ring. Conversely, if M is a retractable module and if $End_R(M)$ is a right semihereditary ring, then every finite direct sum of copies of M is a Rickart module.*

The next example illustrates the necessary direction in Proposition 6.22.

Example 6.23. Consider $M = \mathbb{Q} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. Note that $M^{(n)} = \mathbb{Q}^{(n)} \oplus \mathbb{Z}^{(n)}$ is a Rickart $\mathbb{Z}$-module for any $n \in \mathbb{N}$ (see Example 4.12). It is easy to see that $End_{\mathbb{Z}}(M) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ is a right semihereditary ring.

The following example shows that the condition "M is a retractable module" in the hypothesis of the converse in Proposition 6.22, is not superfluous.

Example 6.24. Consider $M = \mathbb{Z}_{p^\infty}$ as a right $\mathbb{Z}$-module. Then it is well-known that M is not retractable. Note that $End_{\mathbb{Z}}(M)$ is the ring of p-adic integers which is a Dedekind domain and hence is a (semi)hereditary ring. However, M is not a Rickart $\mathbb{Z}$-module, (and neither are direct sums of copies of M).

As a consequence of Theorem 6.3, we can obtain the following result for finite direct sums of copies of an arbitrary retractable Baer module.

Corollary 6.25. *(Corollary 3.7,* [47]*) Let M be a retractable module. Then every finite direct sum of copies of M is a Baer module iff $End(M)$ is left semihereditary and right Π-coherent.*

Corollary 6.26. *(Corollary 3.14,* [33]*) Let M be a retractable module. Then $M^{(n)}$ is a Rickart module iff $End_R(M)$ is a right n-hereditary ring for a fixed $n \in \mathbb{N}$.*

Our next result provides a rich source of more examples of when the concepts of Rickart and Baer modules differ.

Proposition 6.27. *(Proposition 3.11,* [33]*) Let R be a right semihereditary ring which is not a Baer ring. Then every finitely generated free R-module is Rickart, but is not Baer.*

In Example 4.6, the ring R exhibits right semihereditary which is not Baer.

Proposition 6.28. *(Proposition 3.19,* [33]*) Let M be an indecomposable artinian Rickart module. Then any finite direct sum of copies of M is a Rickart module and satisfies C_2 condition.*

The next proposition extends Theorem 6.16 to endomorphism rings of finitely generated retractable modules.

Proposition 6.29. *(Corollary 3.21,* [33]*) Let M be a finitely generated retractable module. Then every direct sum of copies of M is a Rickart module iff $End_R(M)$ is a right hereditary ring.*

Proposition 6.30. *(Proposition 2.29,* [32]*) Let R be a right hereditary ring which is not a Baer ring. Then every free right R-module is Rickart, but is not Baer.*

Proposition 6.31. *Let R be a right hereditary ring which is not a semiprimary ring. Then there exist an index set $\mathcal{I}$ such that $M^{(\mathcal{I})}$ is a Rickart R-module, but is not a Baer R-module.*

Example 6.32. From Example 6.18, since $\mathbb{Z}$ is a non-semiprimary right hereditary ring, there exists an index set $\mathbb{R}$ such that $\mathbb{Z}^{(\mathbb{R})}$ is a Rickart $\mathbb{Z}$-module which is not a Baer $\mathbb{Z}$-module.

In the next result we provide a characterization for an arbitrary direct sum of copies of a Baer module to be Baer, for the case when M is finitely generated and retractable. In contrast to Corollary 6.25, we require the modules to be finitely generated.

Theorem 6.33. *(Theorem 3.4,* [47]*) Let M be a finitely generated retractable module. Then every direct sum of copies of M is a Baer module iff $End_R(M)$ is semiprimary and (right) hereditary.*

Given the connection provided by Theorem 3.4 between extending modules and Baer modules, we obtain the following result concerning Σ-extending (respectively, n-Σ-extending) modules, i.e., modules M with the property that direct sums of arbitrary (respectively, n) copies of M are extending. We generalize in this the results of Lemma 2.4 on polyform modules in [11] (recall that every polyform module is $\mathcal{K}$-nonsingular).

Theorem 6.34. *(Theorem 3.18,* [47]*) Let M be a $\mathcal{K}$-nonsingular module, with $S = End_R(M)$.*

(1) *If $M^{(n)}$ is extending, then every n-generated right torsionless S-module is projective; it follows that S is a right n-hereditary ring.*

(2) *If $M^{(n)}$ is extending for every $n \in \mathbb{N}$, then S is right a semihereditary and left Π-coherent ring.*
(3) *If $M^{(\mathcal{I})}$ is extending for every index set $\mathcal{I}$, and M is finitely generated, then S is a semiprimary hereditary ring.*

A more detailed discussion on these necessary conditions, as well as completing sufficient conditions for a module to be Σ-extending will appear in a sequel to this paper.

Proposition 6.35. *(Proposition 2.14,* [47]*) If a Baer module M can be decomposed into a finite direct sum of indecomposable summands, then every arbitrary direct sum decomposition of M is finite.*

If the endomorphism ring of a module is a PID (principal ideal domain), we obtain the following result, due to Wilson, which has been reformulated to our setting (Lemma 4, [52]).

Proposition 6.36. *(Proposition 3.11,* [47]*) Let M be a finite direct sum of copies of some finite rank, torsion-free module whose endomorphism ring is a PID. Then M is Baer module.*

We conclude this paper with information on some references for further results on the topics we have discussed. The list of these references is only suggestive and is not complete by any means. For results on Baer, quasi-Baer, and Rickart rings, see for example, [1], [2], [3], [4], [5], [6], [12], [14], [15], [20], [23], [24], [26], [35], [42], [54]. Results on Baer, quasi-Baer, and Rickart modules and related notions can be found in [21], [27], [28], [29], [31], [32], [33], [34], [44], [45], [46], [47], [49], [50], [51], [52], [55]. For results on (FI-)extending and (quasi-)continuous modules, see for example, [7], [9], [11], [13], [19], [22], [25], [36], [37], [38], [39], [43], [48].

OPEN PROBLEMS:
1. Obtain a characterization for a finite (infinite) direct sum of Baer modules to be Baer.
2. Obtain a characterization for a finite (infinite) direct sum of quasi-Baer modules to be quasi-Baer.
3. Obtain a characterization for a finite (infinite) direct sum of Rickart modules to be Rickart.

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References

[1] Berberian, S.K., *Baer ∗-Rings*, Springer-Verlag: Berlin-Heidelberg-New York, 1972
[2] G.F. Birkenmeier; H.E. Heatherly; J.Y. Kim; J.K. Park, Triangular matrix representations, J. Algebra, **2000** *230(2)*, 558–595
[3] G.F. Birkenmeier; J.Y. Kim; J.K. Park, Quasi-Baer ring extensions and biregular rings, Bull. Austral. Math. Soc., **2000** *61*, 39–52
[4] G.F. Birkenmeier; J.Y. Kim; J.K. Park, On quasi-Baer rings, Contemp. Math., Amer. Math. Soc., Providence, RI, **2000** *259*, 67–92
[5] G.F. Birkenmeier; J.Y. Kim; J.K. Park, A sheaf representation of quasi-Baer rings, J. Pure Appl. Algebra, **2000** *146*, 209–223

[6] G.F. Birkenmeier; J.Y. Kim; J.K. Park, Polynomial extensions of Baer and quasi-Baer rings, J. Pure Appl. Algebra, **2001** *159*, 25–42
[7] G.F. Birkenmeier; B.J. Müller; S.T. Rizvi, Modules in which every fully invariant submodule is essential in a direct summand, Comm. Algebra, **2002** *30(3)*, 1395–1415
[8] S.U. Chase, Direct products of modules, Trans. Amer. Math. Soc., **1960** *97(3)*, 457–473
[9] A.W. Chatters; S.M. Khuri, Endomorphism rings of modules over nonsingular CS rings, J. London Math. Soc., **1980** *21(2)*, 434–444
[10] P.M. Cohn, *Free Rings And Their Relations*, Academic Press: London & New York (1971)
[11] J. Clark; R. Wisbauer, Polyform and projective Σ-extending modules, Alg. Colloq., **1998** *5(4)*, 391–408
[12] W.E. Clark, Twisted matrix units semigroup algebras, Duke Math. J., **1967** *34*, 417–423
[13] N.V. Dung; D.V. Huynh; P.F. Smith; R. Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics Series, 313; Longman Scientific & Technical: Harlow (1994)
[14] S. Endo, Note on p.p. rings, Nagoya Math. J., **1960** *17*, 167–170
[15] M.W. Evans, On commutative P.P. rings, Pacific J. Math., **1972** *41(3)*, 687–697
[16] C. Faith, Embedding torsionless modules in projectives, Publ. Mat., **1990** *34(2)*, 379–387
[17] C. Faith, *Algebra: Rings, modules and categories I*, Springer-Verlag, New York-Heidelberg (1973)
[18] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London (1979); 2nd edn, Krieger (1991)
[19] K. Hanada; Y. Kuratomi; K. Oshiro, On direct sums of extending modules and internal exchange property, J. Algebra, **2002** *250(1)*, 115–133
[20] A. Hattori, A foundation of torsion theory for modules over general rings, Nagoya Math. J., **1960** *17*, 147–158
[21] J. Hausen, Modules with the summand intersection property, Comm. Algebra, **1989** *17(1)*, 135–148
[22] A. Harmancı; P.F. Smith, Finite direct sums of CS-modules, Houston J. Math., **1993** *19(4)*, 523–532
[23] H.L. Jin; J. Doh; J.K. Park, Group actions on quasi-Baer rings, Canad. Math. Bull., **2009** *52(4)*, 564–582
[24] S. Jøndrup, p.p. rings and finitely generated flat ideals, Proc. Amer. Math. Soc., **1971** *28(2)*, 431–435
[25] J. Kado; Y. Kuratomi; K. Oshiro, CS-property of direct sums of uniform modules, International Symposium on Ring Theory, Trends in Math., Birkäuser (2001) 149–159
[26] I. Kaplansky, *Rings of operators*, W. A. Benjamin Inc., New York-Amsterdam (1968)
[27] S.M. Khuri, Endomorphism rings and lattice isomorphisms, J. Algebra, **1979** *56(2)*, 401–408
[28] S.M. Khuri, Endomorphism rings of nonsingular modules, Ann. S. Math. Qu*é*bec, **1980** *IV(2)*, 145–152
[29] S.M. Khuri, Nonsingular retractable modules and their endomorphism rings, Bull. Austral. Math. Soc., **1991** *43(1)*, 63–71
[30] T.Y. Lam, *Lectures on Modules and Rings*, GTM 189, Springer-Verlag, Berlin-Heidelberg-New York (1999)
[31] Liu, Qiong; Ouyang, Bai Yu, Rickart Modules, Nanjing Daxue Xuebao Shuxue Bannian Kan, **2003** *23(1)*, 157–166
[32] G. Lee; S.T. Rizvi; C.S. Roman, Rickart modules, Comm. Algebra, **2010** *38(11)*, 4005–4027
[33] G. Lee; S.T. Rizvi; C.S. Roman, Direct sums of Rickart modules, manuscript, submitted
[34] G. Lee; S.T. Rizvi; C.S. Roman, Dual Rickart modules, Comm. Algebra, **2011** *39(12)*
[35] S. Maeda, On a ring whose principal right ideals generated by idempotents form a lattice, J. Sci. Hiroshima Univ. Ser. A, **1960** *24*, 509–525
[36] S.H. Mohamed; B.J. Müller, *Continuous and Discrete Modules*, Lecture Note Ser. No. 147, London Math. Soc., Cambridge Univ. Press (1990)
[37] S.H. Mohamed; B.J. Müller, Ojective modules, Comm. Algebra, **2002** *30(4)*, 1817–1827
[38] B.J. Müller; S.T. Rizvi, On injective and quasi-continuous modules, J. Pure Appl. Algebra, **1983** *28(2)*, 197–210
[39] B.J. Müller; S.T. Rizvi, Direct sums of indecomposable modules, Osaka J. Math., **1984** *21(2)*, 365–374
[40] C. Mucke, *Zerlegungseigenschaften von stetigen und quasi-stetigen Moduln*, Algebra Berichte 57, Verlag Reinhard Fischer, Munich (1988)
[41] J. von Neumann, *Continuous geometry*, With a foreword by Israel Halperin, Princeton Landmarks in Mathematics, Princeton University Press (1960)

[42] W.K. Nicholson, On PP-endomorphism rings, Canad. Math. Bull., **1993** *36(2)*, 227–230
[43] S.T. Rizvi, Commutative rings for which every continuous module is quasi-injective, Arch. Math. (Basel), **1988** *50(5)*, 435–442
[44] S.T. Rizvi; C.S. Roman, Baer and quasi-Baer modules, Comm. Algebra, **2004** *32(1)*, 103–123
[45] S.T. Rizvi; C.S. Roman, Baer property of modules and applications, Advances in Ring Theory, **2005**, 225–241
[46] S.T. Rizvi; C.S. Roman, On $\mathcal{K}$-nonsingular modules and applications, Comm. Algebra, **2007** *35(9)*, 2960–2982
[47] S.T. Rizvi; C.S. Roman, On direct sums of Baer modules, J. Algebra, **2009** *321(2)*, 682–696
[48] C.S. Roman, A generalized relative injectivity for extending modules, Proceedings of the International Conference on Modules and Representation Theory, Presa Univ. Clujeana, Cluj-Napoca, **2009**, 171–181
[49] C.S. Roman, Baer and quasi-Baer modules, Doctoral Dissertation, The Ohio State University **2004**
[50] V. Stephenson; G.M. Tsukerman, Rings of endomorphisms of projective modules, Sibirsk. Mat. Ž., **1970** *11*, 228–232
[51] R. Wiegand, Endomorphism rings of ideals in a commutative regular ring, Proc. Amer. Math. Soc., **1969** *23*, 442–449
[52] G.V. Wilson, Modules with the summand intersection property, Comm. Algebra, **1986** *14(1)*, 21–38
[53] R. Wisbauer, *Modules and Algebras: bimodule structure and group actions on algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics 81, Longman, Harlow (1996)
[54] K.G. Wolfson, Baer rings of endomorphisms, Math. Ann., **1961** *143*, 19–28
[55] G.M. Tsukerman, Rings of endomorphisms of free modules, Sibirsk. Mat. Ž., **1966** *7*, 1161–1167
[56] J. Zelmanowitz, Regular modules, Trans. Amer. Math. Soc., **1972** *163*, 340–355

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