MODULES WHOSE ENDOMORPHISM RINGS ARE VON NEUMANN REGULAR

GANGYONG LEE, S. TARIQ RIZVI, AND COSMIN ROMAN

Abstract. Abelian groups whose endomorphism rings are von Neumann regular have been extensively investigated in the literature. In this paper, we study modules whose endomorphism rings are von Neumann regular, which we call endoregular modules. We provide characterizations of endoregular modules and investigate their properties. Some classes of rings R are characterized in terms of endoregular R-modules. It is shown that a direct summand of an endoregular module inherits the property, while a direct sum of endoregular modules does not. Necessary and sufficient conditions for a finite direct sum of endoregular modules to be an endoregular module are provided. We show that $End_R(M)$ is strongly regular precisely when a module M decomposes into a direct sum of the image and the kernel of any $\varphi \in End_R(M)$. As a special case, modules whose endomorphism rings are semisimple artinian are characterized. We provide a precise description of an indecomposable endoregular module over an arbitrary commutative ring. A structure theorem for extending endoregular abelian groups is also provided.

1. Introduction

The study of the class of von Neumann regular rings has been a topic of wide interest. Among other factors, the abundance of idempotent elements in such a ring makes this study interesting. It is well-known that a ring R is von Neumann regular iff $Im\varphi_a = aR$ is a direct summand of R_R for all $a \in R$ and for all R-homomorphisms $\varphi_a : R \to R$ given by the left multiplication by a. Fuchs (Problem 50, [5]) in 1958 raised the question of characterizing abelian groups whose endomorphism rings are von Neumann regular. This was answered in 1967 by Rangaswamy (Theorem 4, [19]). It is also known that Rangaswamy's result can be extended to any module as given below (see also Exercises 50, Page 48, in [25] and Corollary 3.2 in [27]).

Theorem 1.1. *Let M be a right R-module. Then $End_R(M)$ is a von Neumann regular ring iff $Ker\varphi$ and $Im\varphi$ are direct summands of M for all $\varphi \in End_R(M)$.*

While the notion of a regular module has been studied by Fieldhouse [4], Ware [27], and Zelmanowitz [30], our focus of study in this paper is on modules whose endomorphism rings are von Neumann regular. We call these modules *endoregular*. Other than a natural desire to extend the notion of a von Neumann regular ring to a general module theoretic setting, one of the motivations of our study is to investigate whether or not the endoregular property of modules is inherited by direct summands and direct sums of endoregular modules. More specifically, a natural question is: If R is a von Neumann regular ring and $e^2 = e \in R$, what *kind* of regularity will the right R-module eR have? From Example 2.10 we will see that a direct sum of two endoregular modules is not necessarily endoregular. Therefore, another obvious quest is: When are the direct sums of endoregular modules, endoregular? We investigate these questions and obtain several related results. Furthermore, for a right R-module M, in two recent studies we separately considered the conditions "$Ker\varphi$ is a direct summand of M for all $\varphi \in End_R(M)$" and "$Im\varphi$ is a direct summand of M for all $\varphi \in End_R(M)$" (see Definitions 1.5 and 1.7). In view of Theorem 1.1, our studies are closely related to that of endoregular modules. These make the study of endoregular modules a logical topic of our research.

After necessary background results and notations in this section, we provide several characterizations of endoregular modules and obtain their basic properties in Section 2. We show that every direct summand of an endoregular module is endoregular, while the direct sums of endoregular modules are not endoregular, in general. The classes of von Neumann regular rings, semisimple artinian rings, and right V-rings are characterized in terms of endoregular R-modules. We show that $End_R(M)$ is strongly regular precisely when a module M decomposes into a direct sum of the image and the kernel of any endomorphism.

In Examples 2.10 and 3.1 it is shown that a direct sum of endoregular modules is not always endoregular, Section 3 is devoted to the investigation of conditions required for a direct sum of two or more endoregular modules to be endoregular. We introduce the notion of the relative endoregular property between two modules and include a characterization. Then we use this notion to prove that $\bigoplus_{i=1}^{n} M_i$ is endoregular iff M_i is M_j-endoregular for all i, j where $1 \leq i, j \leq n$. As a consequence it follows that every finite direct sum of copies of an endoregular module is an endoregular module. In addition, we obtain a characterization for an arbitrary direct sum of endoregular modules to be endoregular, provided that each module is fully invariant in the direct sum.

The focus of our study in Section 4 is on indecomposable endoregular modules. We show that a module whose endomorphism ring is semisimple artinian decomposes into a finite direct sum of indecomposable endoregular modules. A precise description of indecomposable endoregular modules over an arbitrary commutative ring is provided. We also provide a number of examples which delimit and illustrate our results.

Throughout this paper, R is a ring with unity and M is a unital right R-module. For a right R-module M, $S = End_R(M)$ denotes the endomorphism ring of M; thus M can be viewed as a left S- right R-bimodule. For $\varphi \in S$, $Ker\varphi$ and $Im\varphi$ stand for the kernel and the image of φ, respectively. The notations $N \subseteq M$, $N \leq M$, $N \trianglelefteq M$, $N \leq^{ess} M$, or $N \leq^{\oplus} M$ mean that N is a subset, a submodule, a fully invariant submodule, an essential submodule, or a direct summand of M, respectively. $M^{(n)}$ denotes the direct sum of n copies of M and $\mathsf{Mat}_n(R)$ denotes an $n \times n$ matrix ring over R. By $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of complex, real, rational, integer, and natural numbers, respectively. $E(M)$ denotes the injective hull of M and $\mathbb{Z}_n$ denotes the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$.
We also denote $r_M(I) = \{m \in M \mid Im = 0\}$, $r_S(I) = \{\varphi \in S \mid I\varphi = 0\}$ for $\emptyset \neq I \subseteq S$; $r_R(N) = \{r \in R \mid Nr = 0\}$, $l_S(N) = \{\varphi \in S \mid \varphi N = 0\}$ for $N \leq M$.

While the notions of a (Zelmanowitz) regular module and an endoregular module (see Definition 2.1) coincide when the module $M = R_R$, the next examples show the independence of the two notions for the case of arbitrary modules: One shows a (Zelmanowitz) regular module which is not an endoregular module, and the other exhibits an endoregular module which is not a (Zelmanowitz) regular module. A module M is said to be (*Zelmanowitz*) *regular* if, given any $m \in M$, there exists $f \in Hom_R(M, R)$ such that $mf(m) = m$. Every free module over a von Neumann regular ring is a (Zelmanowitz) regular module (Theorem 2.8, [30]).

Example 1.2. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Take $M = R^{(R)}$. Then M is a (Zelmanowitz) regular R-module. However, M is not an endoregular module (see Example 2.36).

Remark 1.3. Every finitely generated (Zelmanowitz) regular module is an endoregular module. Conversely, every projective endoregular module over a commutative ring is a (Zelmanowitz) regular module. (See Theorems 3.6 and 3.9 in [27], respectively.)

Example 1.4. (i) The $\mathbb{Z}$-module $\mathbb{Z}_p$, where p is a prime number, is endoregular because $\mathbb{Z}_p$ is simple. However, $\mathbb{Z}_p$ is not a (Zelmanowitz) regular $\mathbb{Z}$-module since $Hom_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}) = 0$.
(ii) Let $\mathbb{F}$ be a field, $R = \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{smallmatrix}\right)$ a ring, and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ the idempotent. Note that R is not a von Neumann regular ring because there does not exist an $s \in R$ such that $r = rsr$ for $r = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in R$. Let $M = eR = \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{smallmatrix}\right)$. Since $End_R(M)$ is a field, M is an endoregular R-module. However, since $\left(\begin{smallmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{smallmatrix}\right)$ is a cyclic submodule of M which is not a direct summand, M is not a (Zelmanowitz) regular module (see Example 3.8 in [27] for details and see also [14]). For more examples, see Page 349 in [30].

Some of our arguments utilize techniques from our previous papers, [12], [13], and [14] and may be included for the convenience of the reader. We begin with some basic definitions and results.

Definition 1.5. Let M be a right R-module. Then M is said to be a *Rickart module* if $r_M(\varphi) = Ker\varphi$ is a direct summand of M for all $\varphi \in End_R(M)$ ([12], [22]).

Recall that a module M is said to satisfy C_2 *condition* if, $\forall N \leq M$ with $N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$. M is said to satisfy C_3 *condition* if, $\forall N_1, N_2 \leq^{\oplus} M$ with $N_1 \cap N_2 = 0$, we have $N_1 \oplus N_2 \leq^{\oplus} M$. M is said to satisfy D_2 *condition* if, $\forall N \leq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$. M is said to satisfy D_3 *condition* if, $\forall N_1, N_2 \leq^{\oplus} M$ with $N_1 + N_2 = M$, we have $N_1 \cap N_2 \leq^{\oplus} M$.

Theorem 1.6. *(Proposition 2.11 and Theorem 3.9,* [12]*) The following conditions are equivalent for a module M and $S = End_R(M)$:*
- (a) *M is a Rickart module;*
- (b) *M satisfies D_2 condition, and $Im\varphi$ is isomorphic to a direct summand of M for any $\varphi \in S$;*
- (c) *S is a right Rickart ring and M is k-local-retractable.*

Recall that a module M is said to be *k-local-retractable* if $r_M(\varphi) = r_S(\varphi)(M)$ for any $\varphi \in S = End_R(M)$ [12]. (This condition was called "P-flat over S" in [17].)

Definition 1.7. Let M be a right R-module. Then M is said to be a *d-Rickart* (or *dual Rickart*) *module* if $Im\varphi$ is a direct summand of M for all $\varphi \in End_R(M)$ [14].

Theorem 1.8. *(Proposition 2.21 and Theorem 3.5,* [14]*) The following conditions are equivalent for a module M and $S = End_R(M)$:*
- (a) *M is a d-Rickart module;*
- (b) *M satisfies C_2 condition, and $Im\varphi$ is isomorphic to a direct summand of M for any $\varphi \in S$;*
- (c) *S is a left Rickart ring and $\varphi M = r_M(l_S(\varphi M))$ for any $\varphi \in S$.*

In view of Theorem 1.1, a right R-module M is endoregular iff M is Rickart and d-Rickart (see Proposition 2.3).

Recall that a module M is said to have the *summand intersection property*, SIP, if the intersection of any two direct summands is a direct summand of M. M is said to have the *summand sum property*, SSP, if the sum of any two direct summands is a direct summand of M.

Lemma 1.9. *(Lemma 2.1,* [6]*) A module M satisfies the SIP (resp., SSP) iff for every idempotent pair $e, f \in End_R(M)$, $Ker(ef) \leq^{\oplus} M$ (resp., $Im(ef) \leq^{\oplus} M$).*

Theorem 1.10. *(Theorem 3.17,* [12]*, and Theorem 3.8,* [14]*) A module M is a Rickart module with C_2 condition if and only if M is a d-Rickart module with D_2 condition if and only if $End_R(M)$ is a von Neumann regular ring.*

Proof. It follows from Theorems 1.1, 1.6, and 1.8. □

2. Endoregular modules

In this section we introduce the notion of an endoregular module and obtain several characterizations. In particular, an endoregular module is characterized in terms of the SIP and the SSP (Theorem 2.4). Basic properties of endoregular modules will also be studied. It is shown that every direct summand of an endoregular module inherits the property (Proposition 2.7), while a direct sum of endoregular modules may not be endoregular (Example 2.10). We characterize the class of rings R for which every finitely generated free R-module is endoregular as that of the von Neumann regular rings (Proposition 2.11). Rings R for which finitely cogenerated right R-module is endoregular, is precisely that of the right V-rings (Proposition 2.14). The class of rings for which every every (free) right R-module is endoregular, are shown to be exactly the semisimple artinian rings (Proposition 2.17). We show that an abelian endoregular module is precisely one which decomposes as a direct sum of $Ker\varphi$ and $Im\varphi$ for any endomorphism φ (Theorem 2.22).

Definition 2.1. A module M is called *endoregular* if $End_R(M)$ is a von Neumann regular ring.

Obviously, R_R (respectively, $_RR$) is an endoregular module iff R is a von Neumann regular ring.

Example 2.2. Every semisimple module is endoregular. Every finitely generated projective module over a von Neumann regular ring is an endoregular module (Proposition 2.11), thus every finitely generated right ideal of a von Neumann regular ring is an endoregular module. Every $\mathcal{K}$-nonsingular continuous module is endoregular, and dually every $\mathcal{T}$-noncosingular discrete module is endoregular (Proposition 2.31), for example, $\mathbb{Q}^{(\mathbb{R})}$ is an endoregular $\mathbb{Z}$-module.

In view of Theorem 1.1, it is easy to see that an endoregular module can be characterized as follows: (See also Theorem 2.4, Corollary 3.8, and Proposition 4.1).

Proposition 2.3. *The following conditions are equivalent for a module M:*

(a) *M is an endoregular module;*
(b) *M is a Rickart and a d-Rickart module;*
(c) *M satisfies C_2 and D_2 conditions, and $Im\varphi$ is isomorphic to a direct summand of M for all $\varphi \in End_R(M)$.*

Proof. This follows from Theorem 1.1, and Theorems 1.6 and 1.8. □

Next, we characterize an endoregular module in terms of the SIP and the SSP.

Theorem 2.4. *The following are equivalent for a module M and $S = End_R(M)$:*

(a) *M is an endoregular module;*
(b) *$\mathsf{Mat}_2(S)$ has the SSP;*
(c) *$M^{(2)}$ has the SIP and the SSP.*

Proof. (a)⇒(b) Since S is von Neumann regular, so is $\mathsf{Mat}_2(S)$. It follows from the fact that a von Neumann regular ring has the SSP.
(b)⇒(c) Set $H := \mathsf{Mat}_2(S) = End_R(M^{(2)})$. Suppose H has the SSP. Then for any idempotent pair $e, f \in H$, there exist idempotents $g, h \in H$ such that $efH = gH$ and $Hef = Hh$ from Lemma 1.9. Since $efM^{(2)} = efHM^{(2)} = gHM^{(2)} = gM^{(2)}$, $M^{(2)}$ has the SSP from Lemma 1.9. Also, since $Ker(ef) = r_M(Hef) = r_M(Hh) = (1-h)M$, $M^{(2)}$ has the SIP from Lemma 1.9. (See also Theorem 2.3 in [6].)
(c)⇒(a) Let $\varphi \in End_R(M)$. Consider $N = \{(m, \varphi m) \mid m \in M\}$. Then $N \leq^{\oplus} M^{(2)}$. Since $Ker\varphi \oplus 0 = (M \oplus 0) \cap N \leq^{\oplus} M^{(2)}$ as $M^{(2)}$ has the SIP, $Ker\varphi \leq^{\oplus} M$, so M is a Rickart module. Also, since $(M \oplus 0) + N = M \oplus \varphi M \leq^{\oplus} M^{(2)}$ as $M^{(2)}$ has the SSP, $\varphi M \leq^{\oplus} M$, so M is a d-Rickart module. Thus, M is an endoregular module from Proposition 2.3. □

Remark 2.5. (i) It is well-known that every ring with the SSP has the SIP. Thus, in Theorem 2.4(b) $\mathsf{Mat}_2(S)$ also has the SIP.
(ii) From Lemma 3.16 in [13] and Theorem 2.4 we observe that M is an endoregular module iff $M^{(2)}$ has the SIP with C_3 condition iff $M^{(2)}$ has the SSP with D_3 condition.

According to [7], a 2-sided ideal I in a ring R is said to be *regular* if, for each $r \in I$, there exists an $s \in I$ such that $rsr = r$. It is well-known that every two-sided ideal of a von Neumann regular ring is regular. From the preceding argument, we may conjecture that every fully invariant submodule of an endoregular module may possibly be endoregular. However, the next example from [7] shows that this is not true in general. Note that every ideal of a commutative von Neumann regular ring is an endoregular module (Theorem 2, [29]).

Example 2.6. (Example 1.8, [7]) Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$ be a ring. Consider $T = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant}\}$ and $I = \{(a_n)_{n=1}^{\infty} \in A \mid a_n = 0 \text{ eventually}\} = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Then T is a von Neumann regular ring and I is a regular ideal of T. Let the ring $R = \left(\begin{smallmatrix} T & I \\ I & T \end{smallmatrix}\right)$, which is a von Neumann regular ring.

Now set $N = \left(\begin{smallmatrix} I & I \\ I & T \end{smallmatrix}\right)$ a 2-sided ideal of the endoregular module $M = R_R$. However, N is not an endoregular module: For $\varphi = \begin{pmatrix} 0 & 0 \\ (1,1,\dots) & 0 \end{pmatrix} \in End_R(N)$, $\varphi N = \left(\begin{smallmatrix} 0 & 0 \\ I & I \end{smallmatrix}\right) \lneq^{ess} \left(\begin{smallmatrix} 0 & 0 \\ I & T \end{smallmatrix}\right) \leq^{\oplus} N$. In particular, $End_R(N) = \left(\begin{smallmatrix} T & I \\ T & T \end{smallmatrix}\right)$ is not von Neumann regular.

For a von Neumann regular ring R, one may wonder if $eR, e^2 = e \in R$, has some kind of regularity. This is answered in the affirmative by our next result. In contrast to Example 2.6 the endoregularity of a module is inherited by its direct summands. We provide the proof of the next result by utilizing submodules of a module.

Proposition 2.7. *Every direct summand of an endoregular module is endoregular.*

Proof. Let M be an endoregular module, $N = eM$ for some $e^2 = e \in End_R(M)$, and $\psi \in End_R(N)$ be arbitrary. Since $Ker\psi e = Ker\psi \bigoplus (1-e)M$ and $Ker\psi e \leq^{\oplus} M$, $Ker\psi \leq^{\oplus} M$. Thus, $Ker\psi \leq^{\oplus} N$. In addition, $\psi N = \psi e M \leq^{\oplus} M$ as $\psi e \in End_R(M)$ and M is d-Rickart. Thus, $Im\psi \leq^{\oplus} N$. Therefore N is endoregular. (For an alternative proof, see Lemma 3.3, [27].) □

Remark 2.8. If M is an endoregular module then so are $Ker\varphi$ and $Im\varphi$ for every $\varphi \in End_R(M)$.

Corollary 2.9. *If R is a von Neumann regular ring then eR is an endoregular R-module for every $e^2 = e \in R$.*

Corollary 2.9 also follows from the fact that if R is a von Neumann regular ring then so is eRe, $e^2 = e \in R$.

The next example shows that a direct sum of endoregular modules may not inherit the endoregular property (see also Example 3.1).

Example 2.10. Consider the ring $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and the right R-module $L = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Let $N = R_R$. Then $L \oplus N$ is not an endoregular R-module, while L and N are endoregular R-modules. For, define $\varphi \in End_R(L \oplus N)$ by $\varphi(l, n) = (0, l)$ where $l \in L$ and $n \in N$, then $\varphi(L \oplus N) = 0 \oplus L \leq^{ess} 0 \oplus N$, thus $L \oplus N$ is not an endoregular R-module.

We will investigate conditions under which a direct sum of endoregular modules will be endoregular in Section 3. Next, we characterize several classes of rings in terms of endoregular modules. The following proposition extends Theorem 1.7 in [7].

Proposition 2.11. *The following conditions are equivalent for a ring R:*

(a) *every finitely generated free (projective) right R-module is endoregular;*
(b) *the free module $R^{(n)}$ is an endoregular R-module for some $n \in \mathbb{N}$;*
(c) *R is a von Neumann regular ring.*

Proof. This follows from the well-known fact that R is von Neumann regular iff so is $\mathsf{Mat}_n(R)$. (See also Proposition 2.25, [14].) □

There exists a finitely generated module over an (even commutative) von Neumann regular ring which is not an endoregular module.

Corollary 2.12. *Let M be a projective module over a von Neumann regular ring. Then every finitely generated submodule of M is an endoregular module.*

Proof. Let N be a finitely generated submodule of M. It is well-known that every finitely generated submodule of a projective module over a von Neumann regular ring R is isomorphic to a direct summand of a finitely generated free R-module [7]. Hence $N \cong K \leq^{\oplus} R^{(n)}$. Therefore N is an endoregular module by Propositions 2.7 and 2.11(b). □

Remark 2.13. A projective module over a von Neumann regular ring may not be an endoregular module, in general (see Proposition 2.34).

A ring R is said to be a *right V-ring* if every simple right R-module is injective. R is said to be an *SSI-ring* if every semisimple R-module is injective.

Recall that a module M is said to be *finitely cogenerated* if, for every set $\mathcal{A}$ of submodules of M, $\cap\mathcal{A} = 0$ implies $\cap\mathcal{F} = 0$ for some finite $\mathcal{F} \subseteq \mathcal{A}$, while M is said to be *subdirectly irreducible* if the intersection of its nonzero submodules is nonzero. Note that every subdirectly irreducible module is finitely cogenerated.

Proposition 2.14. *The following conditions are equivalent for a ring R:*

(a) *every finitely cogenerated right R-module is endoregular;*
(b) *R is a right V-ring.*

Proof. Let N be any simple right R-module. Then $E(N)$ is subdirectly irreducible. Assume that there exists $n \in E(N)$ but $n \notin N$. Consider the family of submodules $\{K \mid N \leq^{ess} K \leq^{ess} E(N) \text{ and } n \notin K\}$. By Zorn's Lemma, there exists a maximal element L in the above family. Then $E(L) \oplus (E(L)/L)$ is an endoregular module.

Let $\varphi : E(L) \oplus (E(L)/L) \to E(L) \oplus (E(L)/L)$ be an endomorphism defined by $(x, y + L) \mapsto (0, x + L)$ where $x, y \in E(L)$. Thus, $Ker\varphi = L \oplus (E(L)/L) \leq^{\oplus} E(L) \oplus (E(L)/L) \Rightarrow L = E(L) = E(N)$ which contradicts to the choice of L. So $N = E(N)$ and, hence R is a right V-ring. The converse follows easily. (See also Theorem 3.25, [13].) □

Corollary 2.15. *R is an SSI-ring if and only if R is a right noetherian ring and every finitely cogenerated right R-module is endoregular.*

Next we reformulate and extend Theorem 3.5 in [27] in terms of d-Rickart modules.

Lemma 2.16. *Let R be a ring. A countably infinitely generated free R-module is d-Rickart if and only if R is a semisimple artinian ring.*

Proof. Consider a descending chain of principal left ideals $Ra_1 \supseteq Ra_2a_1 \supseteq Ra_3a_2a_1 \supseteq \cdots$. Let F_R be a countably infinitely generated free d-Rickart module with free basis, $x_1, x_2, \ldots$ and G be a submodule of F spanned by $y_i = x_i - x_{i+1}a_i$. Then $F \cong G$ given by $\varphi : x_i \to y_i$, and also $\varphi F = G \leq^{\oplus} F$ as F is d-Rickart. Thus, the chain stops after finite steps (Lemma 28.2, [1]). So R is a perfect ring. R is also von Neumann regular because R_R is d-Rickart (a direct summand of the d-Rickart module F_R). Therefore R is a semisimple artinian ring. The converse is obvious. □

Proposition 2.17. *The following conditions are equivalent for a ring R:*

(a) *every right R-module is endoregular;*
(b) *the free module $R^{(R)}$ is an endoregular R-module;*
(c) *a countably infinitely generated free R-module is endoregular;*
(d) *R is a semisimple artinian ring.*

Proof. (b)⇒(d) Let K be a right ideal of R. Then there exist a free module $F_R = R^{(\Lambda)}$ and an epimorphism φ such that $\varphi F_R = K$ where Λ is an index set. Since F_R is a direct summand of $R^{(R)}$, it is endoregular. So, $\varphi F_R = K \leq^{\oplus} F_R$. Thus $K \leq^{\oplus} R_R$. Therefore R is a semisimple artinian ring. (d)⇒(a)⇒(b) are easy to see. (c)⇔(d) Follows from Lemma 2.16. □

Remark 2.18. From Theorem 2.25 in [12], every injective (or extending) right R-module is endoregular iff R is a semisimple artinian ring.

Recall that a ring R is said to be *π-regular* if, for any $r \in R$, there exist $s \in R$ and $n \in \mathbb{N}$ such that $r^n s r^n = r^n$. Next, we characterize modules whose endomorphism rings are π-regular.

Proposition 2.19. *The following are equivalent for a module M and $S = End_R(M)$:*

(a) *For any $\varphi \in S$, there exists $n \in \mathbb{N}$ such that $Ker\varphi^n \leq^{\oplus} M$ and $Im\varphi^n \leq^{\oplus} M$;*
(b) *S is a π-regular ring.*

Proof. (a)⇒(b) Let $\varphi \in S$ be arbitrary. By hypothesis, there exists $n \in \mathbb{N}$ such that $Ker\varphi^n \leq^{\oplus} M$ and $Im\varphi^n \leq^{\oplus} M$. Set $M = Ker\varphi^n \oplus N$ for some $N \leq M$. Since $\varphi^n M = \varphi^n N \leq^{\oplus} M$ and $\varphi^n|_N$ is a monomorphism, there exists $\psi \in S$ such that $\psi\varphi^n|_N = 1|_N$. Then $(\varphi^n - \varphi^n\psi\varphi^n)(M) = 0$. Hence S is a π-regular ring.
(b)⇒(a) Let $\varphi \in S$ be arbitrary. Then there exist $\psi \in S$ and $n \in \mathbb{N}$ such that $\varphi^n = \varphi^n\psi\varphi^n$. Take $e = \varphi^n\psi$ and $f = \psi\varphi^n$. Then e and f are idempotents. Since $\varphi^n M = eM$, $Im\varphi^n \leq^{\oplus} M$. Also, since $S\varphi^n = Sf$, $Ker\varphi^n = r_M(S\varphi^n) \leq^{\oplus} M$. □

Corollary 2.20. *If M satisfies C_2 condition and for any $\varphi \in End_R(M)$ $Ker\varphi^n \leq^{\oplus} M$ with some $n \in \mathbb{N}$, then $End_R(M)$ is a π-regular ring.*

Proof. Since $M = Ker\varphi^n \oplus N$ for some $N \leq M$, $\varphi^n M = \varphi^n N \leq^{\oplus} M$ as M satisfies C_2 condition and $\varphi^n|_N$ is a monomorphism. From Proposition 2.19, $End_R(M)$ is a π-regular ring. □

Remark 2.21. If M satisfies D_2 condition and for any $\varphi \in End_R(M)$ $Im\varphi^n \leq^{\oplus} M$ with some $n \in \mathbb{N}$, then $End_R(M)$ is a π-regular ring.

In 1978, Armendariz, Fisher, and Snider showed that a right R-module M satisfies Fitting's Lemma (i.e., $\forall \varphi \in End_R(M)$, $M = Ker\varphi^n \oplus Im\varphi^n$ for some $n \in \mathbb{N}$) iff $End_R(M)$ is strongly π-regular (Proposition 2.3, [2]). We show that $End_R(M)$ is strongly regular precisely when M decomposes into a direct sum of the image and the kernel of any endomorphism (a ring is said to be *strongly regular* if it is a von Neumann regular ring with all idempotents central). A ring is said to be *abelian* if its every idempotent is central. A module M is said to be *abelian* if $End_R(M)$ is abelian.

Theorem 2.22. *The following conditions are equivalent for a module M:*

(a) *M is an abelian endoregular module (i.e., $End_R(M)$ is strongly regular);*
(b) *$M = Ker\varphi \oplus Im\varphi$ for all $\varphi \in End_R(M)$.*

Proof. (a)⇒(b) Let $\varphi \in S = End_R(M)$ be arbitrary. Then there exists $\psi \in S$ such that $\varphi = \varphi\psi\varphi$ as M is endoregular. It is not difficult to observe that $M = Ker(\varphi - \varphi\psi\varphi) = Ker\varphi \oplus Ker(1 - \psi\varphi)$. Say $e = \psi\varphi$ and $f = \varphi\psi$. Then e and f are idempotents. We claim that $Im\varphi = Ker(1 - \psi\varphi) = eM$: Since M is abelian, $\varphi M = \varphi e M = e\varphi M \subseteq eM$ and $eM = \psi\varphi M = \psi\varphi\psi\varphi M = \psi f \varphi M = f\psi\varphi M \subseteq \varphi M$, which proves the claim. Thus, $M = Ker\varphi \oplus Im\varphi$.
(b)⇒(a) From Theorem 1.1 and Proposition 2.3, M is an endoregular module. To show the abelian property, let $e \in S$ be an arbitrary idempotent and $\varphi \in S$ be arbitrary. Consider $\alpha = e\varphi(1 - e)$. Then since $M = Ker\alpha \oplus Im\alpha$ and $\alpha^2 = 0$, $\alpha M = 0 \Rightarrow \alpha = 0 \Rightarrow e\varphi = e\varphi e$. Similarly, $(1 - e)\varphi e = 0$. Thus $e\varphi = e\varphi e = \varphi e$. Therefore M is abelian. □

Remark 2.23. (i) M is an endoregular module iff for each $\varphi \in End_R(M)$ there exists $\psi \in End_R(M)$ such that $M = Im\varphi \oplus Im(1 - \varphi\psi)$. Also, if M is an endoregular module, then for any $\varphi \in End_R(M)$, there exists $\psi \in End_R(M)$ such that $M = Ker\varphi \oplus Im\psi\varphi = Ker\varphi\psi \oplus Im\varphi$.
(ii) Lemma 3.3 in [30] directly follows from Theorem 2.22.
(iii) Every abelian endoregular module is a morphic module (a module M is said to be *morphic* if $Ker\varphi \cong Coker\varphi = M/Im\varphi$ for all $\varphi \in End_R(M)$ [18]).
(iv) In view of Theorem 2.22(b), since each surjective and each injective endomorphism is an isomorphism, every abelian endoregular module is hopfian and cohopfian (see also Corollary 2.4, [2]).
(v) Every cyclic module over a strongly regular ring is cohopfian: Let R be a strongly regular ring and M be a cyclic R-module. Then $M \cong R/I$ for a right ideal I of R. Let $\varphi \in End_R(R/I)$ be any monomorphism of R/I and $\varphi(1 + I) = x + I$. Since $x = x^2 y$ for some $y \in R$, $\varphi(1 + I) = \varphi^2(y + I)$. So, $\varphi(R/I) \subseteq \varphi^2(R/I)$, thus $\varphi(R/I) = \varphi^2(R/I)$. Since φ is a monomorphism, $\varphi(R/I) = R/I$. Therefore φ is an isomorphism.

Corollary 2.24. *Let M be a right R-module and $e^2 = e \in End_R(M)$:*

(a) *eM is an abelian endoregular module;*
(b) *$M = Ker(e\varphi e) \oplus Im(e\varphi e)$ for any $\varphi \in End_R(M)$.*

Proof. In view of Theorem 2.22 and the fact that $(1-e)M \subseteq Ker(e\varphi e)$, the proof follows directly. □

The following example illustrates Theorem 2.22.

Example 2.25. Let $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the ring. Consider the ring $R = \mathsf{Mat}_2(H)$ and the idempotent $e = \begin{pmatrix} (1,1) & 0 \\ 0 & 0 \end{pmatrix}$. Take $M = eR = \begin{pmatrix} \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, hence $End_R(M) = eRe = \begin{pmatrix} \mathbb{Z}_2\times\mathbb{Z}_2 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, M is abelian endoregular. Let $\varphi = \begin{pmatrix} (1,0) & 0 \\ 0 & 0 \end{pmatrix} \in End_R(M)$. Then $Ker\varphi = \begin{pmatrix} (0,\mathbb{Z}_2) & (0,\mathbb{Z}_2) \\ 0 & 0 \end{pmatrix}$, $Im\varphi = \begin{pmatrix} (\mathbb{Z}_2,0) & (\mathbb{Z}_2,0) \\ 0 & 0 \end{pmatrix}$ and $M = Ker\varphi \oplus Im\varphi$.

The 'abelian' condition in Theorem 2.22(a) is not superfluous.

Example 2.26. Let $V = \mathbb{R}^2$ be a vector space over a field $\mathbb{R}$. Take $\varphi = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \in End_{\mathbb{R}}(V) = \mathbb{M}_2(\mathbb{R})$. Then $Ker\varphi = (2,1)\mathbb{R} = Im\varphi \neq V$. In fact, $End_{\mathbb{R}}(V)$ is not abelian.

Proposition 2.27. *M is a projective Rickart module if and only if $Im\varphi$ is projective for each $\varphi \in End_R(M)$.*

Proof. Suppose M is projective and Rickart. Let $\varphi \in End_R(M)$ be arbitrary. Since $Ker\varphi \leq^{\oplus} M$, there exists a projective submodule N of M such that $M = Ker\varphi \oplus N$ and $N \cong M/Ker\varphi \cong Im\varphi$. Therefore $Im\varphi$ is projective.
Conversely, suppose $Im\varphi$ is projective for all $\varphi \in End_R(M)$. Then $Ker\varphi \leq^{\oplus} M$, i.e., M is a Rickart module. Set $\varphi = 1_M$. Then $Im\varphi = M$ is projective. □

Remark 2.28. Every projective d-Rickart module is projective endoregular (Theorem 1.10). However, a projective Rickart module may not be d-Rickart (hence it may not be endoregular). For example, the $\mathbb{Z}$-module $\mathbb{Z}^{(n)}$ is projective Rickart but is not d-Rickart for any $n \in \mathbb{N}$.

Since every von Neumann regular ring is nonsingular, we expect that an endoregular module will also have some *kind* of nonsingularity. Rizvi and Roman introduced the notion of $\mathcal{K}$-nonsingularity and showed that every Baer module is $\mathcal{K}$-nonsingular (Lemma 2.15, [21]). A module M is said to be *$\mathcal{K}$-nonsingular* if, for all $0 \neq \varphi \in End_R(M)$, $Ker\varphi$ is not essential in M. The endoregular $\mathbb{Z}$-module $\mathbb{Z}_p$ is $\mathcal{K}$-nonsingular but not nonsingular. Endoregular modules also satisfy the dual property of $\mathcal{K}$-nonsingularity, which is called $\mathcal{T}$-noncosingular property. A module M is said to be *$\mathcal{T}$-noncosingular* if, for all $0 \neq \varphi \in End_R(M)$, $Im\varphi$ is not small in M [26].

Proposition 2.29. *The following statements hold true:*
(i) *Every endoregular module is $\mathcal{K}$-nonsingular and $\mathcal{T}$-noncosingular.*
(ii) *Every endoregular module satisfies the SIP and the SSP.*

Proof. Since every endoregular module is a Rickart and a d-Rickart module, this follows from Propositions 2.12 and 2.16 in [12] and Proposition 2.11 in [14]. □

Remark 2.30. From the SIP and SSP for endoregular modules in Proposition 2.29, $\bigcap_{i=1}^{n} Ker\varphi_i$ and $\sum_{i=1}^{n} Im\varphi_i$ are direct summands of M for every finite set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ in $End_R(M)$ if and only if M is an endoregular module.

Proposition 2.31. *The following statements hold true:*
(i) *Every $\mathcal{K}$-nonsingular continuous module is an endoregular module.*

(ii) *Every $\mathcal{T}$-noncosingular discrete module is an endoregular module.*

Proof. For the proof of (i), let M be a $\mathcal{K}$-nonsingular continuous module. Then $End_R(M)/J$ is a von Neumann regular ring where J is Jacobson radical from Theorem 3.11 in [15]. Since M is also $\mathcal{K}$-nonsingular, $J = 0$. Thus, $End_R(M)$ is a von Neumann regular ring, i.e., M is an endoregular module. (Using Theorem 2.12 and Proposition 2.22 in [21], and Theorem 1.10, we have an alternating proof.)
For (ii), since every $\mathcal{T}$-noncosingular lifting module is a d-Rickart module from Theorems 2.1 and 2.14 in [26], by Theorem 1.10 M is an endoregular module. □

Example 2.32. Let $M = \mathbb{Q}^{(\mathbb{N})}$ or $\mathbb{Q}^{(\mathbb{R})}$ be a right $\mathbb{Z}$-module. Since M is a nonsingular injective module, M is an endoregular Baer module. In addition, M is a dual Baer module.

Remark 2.33. Every $\mathcal{K}$-nonsingular extending module is a Baer module (Lemma 2.14, [21]) and every $\mathcal{T}$-noncosingular lifting module is a dual Baer module (Theorem 2.14, [26]). From Proposition 2.31, every nonsingular injective R-module is an endoregular Baer module. In addition, if R is right hereditary right noetherian then every nonsingular injective R-module is also dual Baer from Corollary 2.30 in [14].

Recall that a module M is said to be *Baer* if $r_M(I) \leq^{\oplus} M$ for every $\emptyset \neq I \subseteq End_R(M)$. M is said to be a *dual Baer module* if $\sum_{\varphi \in I} Im\varphi \leq^{\oplus} M$ for every $I \subseteq End_R(M)$. In reference to Propositions 2.11 and 2.17, we have the following results.

Proposition 2.34. *Let R be a von Neumann regular ring which is not Baer. Then every finitely generated free R-module is an endoregular module (which is neither a Baer nor a dual Baer module). However, any infinitely generated free R-module is not an endoregular module.*

Proof. For a von Neumann regular ring R, every finitely generated free R-module is endoregular by Proposition 2.11. Also, any finitely generated free R-module can not be Baer as R_R is a direct summand which is not Baer and it is not dual Baer since R is not semisimple artinian. From Proposition 2.17, it is easy to see that any infinitely generated free R-module is not an endoregular module. □

Example 2.35. Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Consider $R = \{(a_n)_{n=1}^{\infty} \in A \mid a_n$ is eventually constant$\}$. Then R is a von Neumann regular ring which is not Baer. Thus, every finitely generated free R-module is an endoregular module which is neither Baer nor dual Baer.

Example 2.36. Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. A is a self-injective von Neumann regular ring which is not semisimple artinian. Then every finitely generated free A-module is an endoregular Baer A-module (see Proposition 2.31), which is not dual Baer (see Example 2.28, [14]). In view of Proposition 2.17, any infinitely generated free A-module is not an endoregular module.

3. Direct sums of endoregular modules

It is of interest to investigate whether or not an algebraic property is inherited by direct summands and direct sums (see e.g., [24]). Proposition 2.7 shows that endoregularity is inherited by direct summands. However, Examples 2.10 and 3.1 exhibit that direct sums of endoregular modules need not be endoregular. In this section, we investigate when a finite direct sum of endoregular modules is also endoregular. First we introduce the notion of the relative endoregular property between two modules. Next, using this notion we fully characterize when a finite direct sum of endoregular modules is endoregular. More specifically, we show that $\bigoplus_{i=1}^{n} M_i$ is endoregular iff M_i is M_j-endoregular for all i, j where $1 \leq i, j \leq n$ (Theorem 3.14). In particular, every finite direct sum of copies of an endoregular module is an endoregular module (Corollary 3.15). A characterization for an arbitrary direct sum of endoregular modules to be endoregular under certain assumptions, is provided (Proposition 3.20).

Example 3.1. Let $\mathbb{F}$ be a field, $R = \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{smallmatrix}\right)$ a ring, and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ the idempotent. Let $L = eR = \left(\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{smallmatrix}\right)$ and $N = (1-e)R = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{F} \end{smallmatrix}\right)$. Then $L \oplus N = R_R$ is not an endoregular module (see Example 1.4(ii)), while L and N are endoregular modules.

The next result shows how a general example can be constructed.

Proposition 3.2. *If M is an indecomposable endoregular module which has a proper socle N, then $M \oplus N$ is not an endoregular module, while M and N are endoregular modules.*

Proof. Let $\varphi \in End_R(M \oplus N)$ defined by $\varphi(m, n) = (n, 0)$ for $m \in M, n \in N$. Since $Im\varphi = N \oplus 0$ is not a direct summand of $M \oplus N$, $M \oplus N$ is not endoregular. □

To investigate when direct sums of endoregular modules are endoregular, we now introduce the notion of the relative endoregular property between two modules.

Definition 3.3. Let M and N be R-modules. Then an element $\varphi \in Hom_R(M, N)$ is said to be *regular* if there exists $\psi \in Hom_R(N, M)$ such that $\varphi\psi\varphi = \varphi$ [9]. A subset H of $Hom_R(M, N)$ is said to be *regular* if each element of H is regular. A module M is called *N-endoregular* (or *relatively endoregular to a module N*) if $Hom_R(M, N)$ is regular.

In view of the above definition, a right R-module M is endoregular iff M is M-endoregular.

Kasch and Mader showed that $\varphi \in Hom_R(M, N)$ is a regular element iff $Ker\varphi \leq^{\oplus} M$ and $Im\varphi \leq^{\oplus} N$ (Theorem 2.1, [9]). Next proposition is a reformulation of Theorem 2.1 in [9].

Proposition 3.4. *Let M and N be right R-modules. Then M is N-endoregular if and only if, $\forall \varphi \in Hom_R(M, N)$, $Ker\varphi \leq^{\oplus} M$ and $Im\varphi \leq^{\oplus} N$.*

Proof. We include a proof for the convenience of the reader: Let $\varphi \in Hom_R(M, N)$ and $\psi \in Hom_R(N, M)$ such that $\varphi = \varphi\psi\varphi$. Take $e = \psi\varphi \in End_R(M)$ and $f = \varphi\psi \in End_R(N)$. Since $\varphi|_{eM} : eM \cong fN$ is an isomorphism, $Im\varphi = fN \leq^{\oplus} N$ and $Ker\varphi = (1-e)M \leq^{\oplus} M$.

Conversely, for an arbitrary $\varphi \in Hom_R(M, N)$, suppose $Ker\varphi = eM$, $e^2 = e \in End_R(M)$, and $Im\varphi = fN$, $f^2 = f \in End_R(N)$. This induces $(1-e)M \cong fN$, thus there exists $\psi \in Hom_R(N, M)$ such that $\psi\varphi|_{(1-e)M} = 1_{(1-e)M}$. Then $\varphi\psi\varphi m = \varphi\psi\varphi(1-e)m = \varphi(1-e)m = \varphi m$ for all $m \in M$. Therefore $\varphi = \varphi\psi\varphi$. □

Recall that a module M is said to be *N-Rickart* if, $\forall \varphi \in Hom_R(M, N)$, $Ker\varphi \leq^{\oplus} M$ for a module N. M is said to be *N-d-Rickart* if, $\forall \varphi \in Hom_R(M, N)$, $Im\varphi \leq^{\oplus} N$. Thus, a module M is N-endoregular iff M is N-Rickart and N-d-Rickart.

Proposition 3.5. *Let M and N be indecomposable modules. If M is N-endoregular then either $Hom_R(M, N) = 0$ or $M \cong N$.*

Proof. Assume that $Hom_R(M,N) \neq 0$. Let $0 \neq \varphi \in Hom_R(M,N)$. Since $Ker\varphi \leq^{\oplus} M$ and $Im\varphi \leq^{\oplus} N$, $Ker\varphi = 0$ and $Im\varphi = N$ as M and N are indecomposable. Thus, $M \cong N$. □

Theorem 3.6. *Let M and N be right R-modules. Then M is N-endoregular iff, for any direct summand $M' \leq^{\oplus} M$ and any submodule $N' \leq N$, M' is N'-endoregular.*

Proof. The sufficient condition is trivial. Conversely, let $M' = eM$ for some $e^2 = e \in End_R(M)$, $N' \leq N$, and $\psi \in Hom_R(M', N')$ be arbitrary. Since $\psi eM = \psi M' \subseteq N$, $Ker\psi e \leq^{\oplus} M$ and $Im\psi e \leq^{\oplus} N$ as M is N-endoregular. Thus, $\psi M' \leq^{\oplus} N'$, i.e., M' is N'-d-Rickart. Also, since $Ker\psi e = (1-e)M \oplus Ker\psi$, $Ker\psi \leq^{\oplus} M \Rightarrow Ker\psi \leq^{\oplus} M'$. Hence M' is N'-Rickart. □

Remark 3.7. Consider $L \leq M$. If M is N-endoregular then M/L is N-endoregular (Corollary 5.2, [9]). Thus, if M is N-endoregular and there exists an epimorphism $\varphi \in Hom_R(M,N)$ then N is an endoregular module.

Corollary 3.8. *The following conditions are equivalent for a module M:*

(a) *M is an endoregular module;*
(b) *for any submodule N of M, every direct summand L of M is N-endoregular;*
(c) *for every pair of direct summands L, N of M and for any $\varphi \in Hom_R(M,N)$, the kernel and the image of the restricted map $\varphi|_L$ are direct summands of L and N, respectively.*

Proof. Implications (a)⇒(b)⇒(c)⇒(a) are easy to check from Theorem 3.6. (See also Proposition 2.24 in [12] and Corollary 2.20 in [14].) □

Lemma 3.9. *(Lemma 4.2,* [9]*) Let M, N be right R-modules, $\varphi \in Hom_R(M,N)$, and $\psi \in Hom_R(N,M)$. If $\varphi - \varphi\psi\varphi$ is regular then φ is regular.*

Theorem 3.10. *Let M_i and N be right R-modules for $i \in \mathcal{I} = \{1, 2, \ldots, n\}$. Then the following statements hold true:*

(i) *$\bigoplus_{i\in\mathcal{I}} M_i$ is N-endoregular if and only if M_i is N-endoregular for all $i \in \mathcal{I}$.*
(ii) *N is $\bigoplus_{i\in\mathcal{I}} M_i$-endoregular if and only if N is M_i-endoregular for all $i \in \mathcal{I}$.*

Proof. Let $M = \bigoplus_{i\in\mathcal{I}} M_i$, $H = Hom_R(M,N)$, $T = Hom_R(N,M)$, $M_i = e_iM$ for $e_i^2 = e_i \in End_R(M)$, $He_i = Hom_R(e_iM, N)$, and $e_iT = Hom_R(N, e_iM)$. For the proof of (i), the necessary condition follows from Theorem 3.6. Conversely, suppose M_j is N-endoregular for all $j \in \mathcal{I}$. Let $\varphi \in H$ be arbitrary. We proceed by induction on n. If $n = 1$, the proof is trivial. If $n = 2$, then $M = M_1 \oplus M_2$: Suppose M_j is N-endoregular for $j = 1, 2$. Since φe_1 is a regular element in He_1, there exists $\psi \in e_1T$ such that $\varphi e_1\psi\varphi e_1 = \varphi e_1$. Thus, $(\varphi\psi\varphi - \varphi)e_1 = 0$. Set $x := \varphi\psi\varphi - \varphi \in H$. Since xe_2 is regular in He_2, there exists $y \in e_2T$ such that $xe_2yxe_2 = xe_2$. Thus, $(xyx - x)e_2 = 0$. Since $(xyx - x)e_1 = (xy - 1)xe_1 = 0$, $xyx - x = (xyx - x)(e_1 + e_2) = 0$. So, x is regular in H. Therefore φ is regular in H from Lemma 3.9. Assume that the theorem holds true for the case $n - 1$. Take $(1 - e_n)M = \bigoplus_{i=1}^{n-1} M_i$. Then $(1 - e_n)M$ is N-endoregular. From a similar way of the case for $n = 2$, M is N-endoregular.

For (ii), the necessary condition follows from Theorem 3.6. Conversely, suppose N is M_i-endoregular for all $i \in \mathcal{I}$. Let $\varphi \in T$ be arbitrary. We proceed by induction on n. If $n = 1$, the proof follows obviously. If $n = 2$, then $M = M_1 \oplus M_2$: Suppose N is M_i-endoregular for $i = 1, 2$. Since $e_1\varphi$ is a regular element in e_1T, there exists $\psi \in He_1$ such that $e_1\varphi\psi e_1\varphi = e_1\varphi$. Thus, $e_1(\varphi\psi\varphi - \varphi) = 0$. Set $x := \varphi\psi\varphi - \varphi \in T$. Since e_2x is regular in e_2T, there exists $y \in He_2$ such that

$e_2xye_2x = e_2x$. Thus, $e_2(xyx - x) = 0$. Since $e_1(xyx - x) = e_1x(yx - 1) = 0$, $xyx - x = (e_1 + e_2)(xyx - x) = 0$. So, x is regular in T. Therefore φ is regular in T from Lemma 3.9. Assume that the theorem holds true for the case $n - 1$. Then N is $(1 - e_n)M$-endoregular. Similar to the case for $n = 2$, we obtain that N is M-endoregular. □

Corollary 3.11. *Let $\{M_i\}_{1\leq i\leq m}$ and $\{N_j\}_{1\leq j\leq n}$ be classes of right R-modules where $m, n \in \mathbb{N}$. Then $\bigoplus_{i=1}^m M_i$ is $\bigoplus_{j=1}^n N_j$-endoregular iff M_i is N_j-endoregular for all i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. Since the necessary condition follows from Theorem 3.6, it remains to prove the sufficient condition. Suppose M_i is N_j-endoregular for all i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$. From Theorem 3.10(i), for all fixed j, $\bigoplus_{i=1}^m M_i$ is N_j-endoregular. Also, from Theorem 3.10(ii), $\bigoplus_{i=1}^m M_i$ is $\bigoplus_{j=1}^n N_j$-endoregular. □

Remark 3.12. $Hom_R(\oplus_{i=1}^m M_i, \oplus_{j=1}^n N_j)$ is regular iff $Hom_R(M_i, N_j)$ is regular for all i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Next corollary extends Lemma 1.6 in [7].

Corollary 3.13. *Let M and N be right R-modules. Let $e_1, e_2, \ldots, e_m$ be orthogonal idempotents in $End_R(M)$ such that $e_1 + e_2 + \cdots + e_m = 1$ and let $f_1, f_2, \ldots, f_n$ be orthogonal idempotents in $End_R(N)$ such that $f_1 + f_2 + \cdots + f_n = 1$ where $m, n \in \mathbb{N}$. Then M is N-endoregular if and only if e_iM is f_jN-endoregular for all i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Theorem 3.14. *Let $\{M_i\}_{1\leq i\leq n}$ be a class of right R-modules where $n \in \mathbb{N}$. Then $\bigoplus_{i=1}^n M_i$ is an endoregular module if and only if M_i is M_j-endoregular for all i, j where $1 \leq i, j \leq n$.*

Proof. The proof follows in view of Corollary 3.11. □

Corollary 3.15. *Every finite direct sum of copies of an endoregular module is also an endoregular module.*

Corollary 3.16. *(Theorem 2.14, Part II, [16]) A ring R is von Neumann regular if and only if $\mathsf{Mat}_n(R)$ is also a von Neumann regular ring for any $n \in \mathbb{N}$.*

Proposition 3.17. *Let M be an endoregular module which is not Baer. Then every finite direct sum of copies of M is an endoregular module which is not a Baer module.*

Proof. The proof follows similar to that of Proposition 2.34. □

Example 3.18. Let T and I be as in Example 2.6. Consider the ring $R = \begin{pmatrix} T & T/I \\ 0 & T/I \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} (1,1,\ldots) & 0+I \\ 0 & 0+I \end{pmatrix} \in R$. Let $M = eR = \begin{pmatrix} T & T/I \\ 0 & 0 \end{pmatrix}$ be a right R-module. Then since $S = End_R(M) = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ is a von Neumann regular ring, M is an endoregular R-module. On the other hand, it is not a dual Baer module: For $U = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq S_S$, $\sum_{\varphi\in U} Im\varphi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. So, there is no idempotent $e \in S$ such that $\sum_{\varphi\in U} Im\varphi = eM$ (see also Example 4.1, [14]). Note that M is not a Baer module, either (see Example 2.18, [12]). In addition, M is not a (Zelmanowitz) regular module because a cyclic submodule $\begin{pmatrix} 0 & T/I \\ 0 & 0 \end{pmatrix}$ is not a direct summand of M. Therefore, every finite direct sum of copies of M is an endoregular module which is not a Baer module. In addition, $M^{(n)}$ is neither a (Zelmanowitz) regular nor a dual Baer module for any $n \in \mathbb{N}$.

The following example shows that an infinite direct sum of copies of an endoregular module is not an endoregular module, in general.

Example 3.19. (i) Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $M = R_R$. Note that R is a self-injective regular ring which is not semisimple artinian. So, M is an endoregular R-module. However, $M^{(R)}$ is not an endoregular R-module (Example 2.36).
(ii) Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$, $T = \mathsf{Mat}_2(R)$, and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Take $M = eT$. Since $End_T(M)$ is a von Neumann regular ring which is not semisimple artinian, M is an endoregular T-module. However, $End_T(M^{(\mathbb{N})})$ is not a von Neumann regular ring by Proposition 2.17. Thus $M^{(\mathbb{N})}$ is not an endoregular T-module.

We conclude this section with the following characterization.

Proposition 3.20. *Let $M_i \trianglelefteq \bigoplus_{i\in\mathcal{I}} M_i$, $\forall\ i \in \mathcal{I}$, $\mathcal{I}$ is an arbitrary index set. Then $\bigoplus_{i\in\mathcal{I}} M_i$ is an endoregular module iff M_i is an endoregular module for all $i \in \mathcal{I}$.*

Proof. The necessary condition follows from Theorem 3.6. Conversely, let $M = \bigoplus_{i\in\mathcal{I}} M_i$, $S = End_R(M)$, and $\varphi = (\varphi_{ij}) \in S$ be arbitrary with $\varphi_{ij} \in Hom_R(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \mathcal{I}$, $Ker\varphi = \bigoplus_{i\in\mathcal{I}} Ker\varphi_{ii} \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i$ and $Im\varphi = \bigoplus_{i\in\mathcal{I}} Im\varphi_{ii} \leq^{\oplus} \bigoplus_{i\in\mathcal{I}} M_i$ because $\varphi_{ii} \in End_R(M_i)$ and M_i is an endoregular module for all $i \in \mathcal{I}$. □

4. Indecomposable endoregular modules

In this section, we show that an indecomposable endoregular module has precisely a division ring as its endomorphism ring (Proposition 4.4). Modules whose endomorphism rings are semisimple artinian are characterized as those which are direct sums of fully invariant submodules, each of which is a direct sum of copies of an indecomposable endoregular submodule (Theorem 4.7). In view of our earlier results, we observe that every endoregular module which is noetherian or artinian over a commutative ring, is semisimple (Proposition 4.18). Using retractability, we reformulate Corollary in [28] in terms of indecomposable endoregular modules (Theorem 4.24). Finally, we provide a structure theorem for extending endoregular abelian groups (Theorem 4.27).

Proposition 4.1. *The following are equivalent for a module M and $S = End_R(M)$:*

(a) *M is an endoregular module;*
(b) *S is a left and right Rickart ring, M is k-local-retractable and $\varphi M = r_M(l_S(\varphi M))$ for all $\varphi \in S$;*
(c) *S is a von Neumann regular ring.*

Proof. This follows from Theorems 1.6 and 1.8, and Proposition 2.3. □

Recall that a ring R is said to be *unit regular* if, for any $a \in R$, there exists an invertible element $u \in R$ such that $aua = a$.

Proposition 4.2. *The following are equivalent for a module M and $S = End_R(M)$:*

(a) *M is an endoregular module and a morphic module;*
(b) *S is a left and right Rickart ring, M is morphic and k-local-retractable, and $\varphi M = r_M(l_S(\varphi M))$ for all $\varphi \in S$;*
(c) *S is a unit regular ring.*

Proof. This follows from Proposition 4.1, and Theorem 4.1 in [7]. □

Proposition 4.3. *The following are equivalent for a module M and $S = End_R(M)$:*

(a) *M is an abelian endoregular module;*
(b) *S is an abelian Rickart ring, M is k-local-retractable and $\varphi M = r_M(l_S(\varphi M))$ for all $\varphi \in S$;*
(c) *S is a strongly regular ring.*

Proposition 4.4. *The following are equivalent for a module M and $S = End_R(M)$:*
(a) *M is an indecomposable endoregular module;*
(b) *S is a domain, M is k-local-retractable and $\varphi M = r_M(l_S(\varphi M))$ for all $\varphi \in S$;*
(c) *S is a division ring.*

Note that every indecomposable hopfian d-Rickart (or cohopfian Rickart) module is an indecomposable endoregular module (see Corollary 4.8, [14]).

Proposition 4.5. *The following are equivalent for a module M and $S = End_R(M)$:*
(a) *M is an endoregular module and S has no infinite set of nonzero orthogonal idempotents;*
(b) *S is a semisimple artinian ring.*

Lemma 4.6. *(Theorem 17.9, [11]) Let M be a right R-module and $S = End_R(M)$. Then $S \cong \mathsf{Mat}_n(T)$ for some T iff $M \cong N^{(n)}$ for some right R-module N and some $n \in \mathbb{N}$. Moreover, if S is a simple artinian ring then $End_R(N)$ is a division ring.*

Theorem 4.7. *The following are equivalent for a module M and $S = End_R(M)$:*
(a) *M has a finite direct sum decomposition $M \cong \bigoplus_{i=1}^{k} M_i^{(n_i)}$ where $M_i^{(n_i)}$ is fully invariant with $n_i \in \mathbb{N}$ and M_i is an indecomposable endoregular module for all $1 \le i \le k$;*
(b) *S is a semisimple artinian ring.*

Proof. (a)⇒(b) Let $N_i = M_i^{(n_i)}$ for each $1 \le i \le k$. Then $End_R(N_i) = End_R(M_i^{(n_i)})$ $= \mathsf{Mat}_{n_i}(End_R(M_i))$. Since $End_R(M_i)$ is a division ring by Proposition 4.4, $End_R(N_i)$ is a simple artinian ring. Since N_i is a fully invariant submodule of M for all $1 \le i \le k$, S is a $k \times k$ matrix ring with elements of $End_R(N_i)$ in each (i,i)-position for $1 \le i \le k$ and 0 elsewhere as $Hom_R(N_i, N_j) = 0$ for all $1 \le i \ne j \le k$. Therefore S is a semisimple artinian ring.
(b)⇒(a) From Wedderburn-Artin Theorem and Lemma 4.6, M has a finite direct sum decomposition $M \cong \bigoplus_{i=1}^{k} M_i^{(n_i)}$ where $M_i^{(n_i)}$ is fully invariant with $n_i \in \mathbb{N}$ and $End_R(M_i)$ is a division ring for all $1 \le i \le k$. Thus M_i is indecomposable endoregular by Proposition 4.4. □

Proposition 4.8. *Every extending endoregular module is precisely a $\mathcal{K}$-nonsingular continuous module.*

Proof. Since an endoregular module satisfies C_2 condition and is $\mathcal{K}$-nonsingular from Propositions 2.3 and 2.29, an extending endoregular module is $\mathcal{K}$-nonsingular continuous. The converse follows from Theorem 2.12 in [21]. □

Proposition 4.9. *Let M be an extending endoregular module and $S = End_R(M)$. Then S is a right continuous von Neumann regular ring. In addition, $S = S_1 \times S_2$ where S_1 is a right self-injective von Neumann regular ring and S_2 is a right continuous strongly regular ring. In particular, M is a morphic module.*

Proof. By Proposition 4.8 M is a $\mathcal{K}$-nonsingular continuous module, from Proposition 3.1 in [23] S is a right continuous von Neumann regular ring. In addition, $S = S_1 \times S_2$ where S_1 is a right self-injective von Neumann regular ring and S_2 is a right continuous strongly regular ring from Theorem 13.17 in [7]. In particular, since every right continuous von Neumann regular ring is a unit regular ring by Corollary 13.23 in [7], M is a morphic module from Proposition 4.2. □

Next example shows that the converse of Proposition 4.9 is not true, in general.

Example 4.10. Let the ring $R = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{R} \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $M = eR = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$ and $End_R(M) = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{pmatrix}$. While $End_R(M)$ is a division ring (thus, it is a right continuous von Neumann regular ring), M is not extending: Let $N = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ be a submodule of M. There is no element $\alpha \in R$ such that $0 \neq \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \alpha \in N$. Note that M is an indecomposable endoregular module.

Definition 4.11. A module M is said to be *retractable* if, for every $0 \neq N \leq M$, $\exists 0 \neq \varphi \in End_R(M)$ with $\varphi M \subseteq N$, i.e., if $Hom_R(M, N) \neq 0$ for every $0 \neq N \leq M$.

Examples of retractable modules include free modules, generators, and semisimple modules (see [10]).

Proposition 4.12. *A module M is simple if and only if M is retractable and $End_R(M)$ is a division ring (i.e., M is indecomposable endoregular).*

Proof. The necessary condition follows easily. Conversely, for a proper submodule N, there is a nonzero endomorphism φ such that $\varphi M \subsetneq N$ as M is retractable. This contradicts that $End_R(M)$ is a division ring. Thus, M is a simple module. □

The following example shows that a retractable endoregular module may not be semisimple even though R is a commutative ring.

Example 4.13. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $M = R^{(n)}$ for some $n \in \mathbb{N}$. Then M is a retractable endoregular R-module from Corollary 3.15, but it is not a semisimple module.

We conclude this paper with some results on endoregular modules over a commutative ring. We begin with a reformulation of a result of [8] in terms of endoregular modules.

Lemma 4.14. *(Proposition 5,* [8]*) Every finitely generated indecomposable endoregular module over a commutative ring is a simple module.*

Remark 4.15. (i) Every indecomposable module over a commutative von Neumann regular ring is a simple module (Theorem 2.13, [3]). However, for the ring $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$, the right module $M_R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ is an indecomposable module over a von Neumann regular ring, but it is not simple.
(ii) Every indecomposable retractable endoregular module is a simple module (Proposition 4.12). Actually, every indecomposable retractable d-Rickart module is simple (Corollary 4.9, [14]).

Proposition 4.16. *Let M be a finitely generated module over a commutative ring. If an infinite direct sum of copies of M is endoregular then M is a semisimple module.*

Proof. Since M is finitely generated, $End_R(M^{(\mathcal{I})}) = End_R(S^{(\mathcal{I})})$ where $S = End_R(M)$ and $\mathcal{I}$ is an infinite index set. From Proposition 2.17, S is a semisimple artinian ring. Thus, M is a finite direct sum of indecomposable endoregular modules M_i from Theorem 4.7. Since each M_i is also finitely generated, from Lemma 4.14 M_i is simple. Therefore M is a semisimple module. □

Note that for a finitely generated module M over a commutative ring, if M is an endoregular module but is not semisimple then any infinite direct sum of copies of M is not endoregular by Proposition 4.16.

Example 4.17. Let M be as in Example 4.13. Then M is a finitely generated endoregular module over a commutative ring but it is not a semisimple module. Thus, from Proposition 4.16 any *infinite direct sum* of copies of M is not an endoregular module, while every *finite direct sum* of copies of M is an endoregular module from Corollary 3.15.

Next proposition follows from Proposition 4.14 in [12] or Proposition 4.12 in [14].

Proposition 4.18. *Let M be an artinian or a noetherian module over a commutative ring. Then M is an endoregular module iff M is a semisimple module.*

Remark 4.19. A structure theorem for a finitely generated endoregular module over a commutative noetherian ring is from Theorem 4.14 in [14]. Every finitely generated endoregular module M over a commutative noetherian ring R is a semisimple module, i.e., $M \cong R/\mathbf{m}_1 \oplus R/\mathbf{m}_2 \oplus \cdots \oplus R/\mathbf{m}_n$ where $\mathbf{m}_i$ are maximal ideals of R with $1 \leq i \leq n$ for some $n \in \mathbb{N}$.

Next example exhibits that the 'finitely generated' condition in Proposition 4.16 and the 'noetherian or artinian' condition in Proposition 4.18 are not superfluous:

Example 4.20. Let $M = \mathbb{Q}_{\mathbb{Z}}$. Then M is a non-artinian and non-noetherian indecomposable injective endoregular module over a commutative ring $\mathbb{Z}$, which is not finitely generated. While every infinite direct sum of copies of M is an endoregular module (see Example 2.32), M is not a simple module even though $End_{\mathbb{Z}}(M) = \mathbb{Q}$ is a division ring.

Lemma 4.21. *(Theorem,* [28]*) Let M be a torsion-free module over a commutative domain R. Then M is an endoregular module if and only if M is R-isomorphic to a direct sum of copies of the field of fractions Q of R.*

Lemma 4.22. *Let M be an indecomposable d-Rickart module over a commutative ring R. Then $P = r_R(M)$ is a prime ideal of R and M is an indecomposable torsion-free d-Rickart module over a commutative domain R/P.*

Proof. Suppose $ab \in P$ and $a \notin P$. Let $\varphi_a \in End_R(M)$ be defined by $\varphi_a(m) = ma$ for any $m \in M$. Then $0 \neq Im\varphi_a = Ma \leq^{\oplus} M$ and hence $Ma = M$ as M is indecomposable d-Rickart. Thus, $0 = Mab = Mb \Rightarrow b \in r_R(M) = P$. Therefore P is a prime ideal of R.

For the second statement, since $End_R(M) \cong End_{R/P}(M)$, M is an indecomposable faithful d-Rickart module over a commutative domain R/P. Assume that M is not torsion-free. Then there exists $0 \neq \overline{r} \in R/P$ such that $Im\varphi_{\overline{r}} = M\overline{r} \lneq^{\oplus} M$ as M is d-Rickart. Thus, $M\overline{r} = 0$ because M is indecomposable. This contradicts that M is faithful over a commutative domain R/P. □

In Remark 4.15(ii), $r_R(M)$ is also a prime ideal of R.

Lemma 4.23. *Let R be a commutative domain and Q be the field of fractions of R. Then Q_R is retractable if and only if $R = Q$.*

Proof. The sufficient condition follows easily. Conversely, for every nonzero endomorphism $\varphi \in End_R(Q_R) \cong Q$, $\varphi Q_R = Q_R$. Also, for any nonzero $a \in Q$, there exists $\psi \in End_R(Q_R)$ such that $\psi Q_R \subseteq aR$ as Q_R is retractable. Thus $R = Q$. □

Using retractability, we reformulate Corollary in [28] in terms of an indecomposable endoregular module.

Theorem 4.24. *Let R be a commutative ring and M be a right R-module. Then M is an indecomposable endoregular module if and only if either (1) M is a simple (hence, retractable) right R-module; or (2) M is not retractable and M is R-isomorphic to the field of fractions Q of R/P where $P = r_R(M)$.*

Proof. Suppose M is an indecomposable endoregular module. Then from Lemma 4.22 M is an indecomposable torsion-free endoregular module over a commutative domain R/P because $End_{R/P}(M) \cong End_R(M)$. By Lemma 4.21 M is R-isomorphic to the field of fractions Q of R/P. In addition, if M is retractable then $Q = R/P$ by Lemma 4.23. So M is a simple R-module. The converse follows easily. □

Remark 4.25. (i) From Corollary in [28] and Theorem 4.24, if M is an indecomposable endoregular module over a commutative ring R then either (1) M is a retractable module $\Leftrightarrow r_R(M)$ is a maximal ideal of R; or (2) M is not a retractable module $\Leftrightarrow r_R(M)$ is a nonmaximal prime ideal of R. In Example 4.20, $\mathbb{Q}$ is not a retractable $\mathbb{Z}$-module and $r_{\mathbb{Z}}(\mathbb{Q}) = 0$ is a nonmaximal prime ideal of $\mathbb{Z}$.
(ii) If, in addition, the ring R is commutative in Theorem 4.7, then $End_R(M)$ is a semisimple artinian ring $\Leftrightarrow M = \bigoplus_{i=1}^{k} M_i^{(n_i)}$ where M_i is R-isomorphic to the field of fractions Q_i of R/P_i where $P_i = r_R(M_i)$ for $1 \leq i \leq k$ from the proof of Theorem 4.24. (See also Theorem in [28], Page 987.)

Theorem 4.24 is not valid over a noncommutative ring R (even, over a prime PI-ring R) as shown in the following.

Example 4.26. (i) Let R and M be as in Example 1.4(ii). Then M is an indecomposable endoregular module, while M is neither a simple module nor a retractable module, and $r_R(M) = 0$ is not a prime ideal of R.
(ii) Let the ring $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Consider $M = \mathbb{Q} \oplus \mathbb{Q}$ as a right R-module. Then $End_R(M) = \mathbb{Q}$. Thus, M is an indecomposable endoregular module, even though M is neither a simple module nor a retractable module, and $r_R(M) = 0$ is a prime ideal of R.

The next result provides a structure theorem for extending endoregular abelian groups.

Theorem 4.27. *Let M be an extending endoregular module over $\mathbb{Z}$. Then M is a direct sum of copies of $\mathbb{Q}$ or $\mathbb{Z}_{p_i}$ where p_i is prime in $\mathbb{N}$.*

Proof. Since M is extending and $\mathbb{Z}$ is noetherian, M is a direct sum of indecomposable modules. Since each indecomposable endoregular module is $\mathbb{Q}$ or $\mathbb{Z}_p$ from Theorem 4.24, M is a direct sum of copies of $\mathbb{Q}$ or $\mathbb{Z}_{p_i}$ where p_i is prime. □

Remark 4.28. The previous result also follows directly from Corollary 5 in [20]. This, since an extending endoregular module is continuous by Proposition 4.8, every continuous module over a commutative noetherian ring is quasi-injective (Corollary 5, [20]). It is easy to see that $\mathbb{Q}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^n}$ for $n > 1$, and $\mathbb{Z}_p$ are the indecomposable quasi-injective $\mathbb{Z}$-modules [15]. However, $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_{p^n}$ for $n > 1$ are not endoregular $\mathbb{Z}$-modules.

Acknowledgments

The authors are thankful to the Math Research Institute, the Ohio State University, Columbus and OSU-Lima, for the support of this research work.

References

[1] F.W. Anderson; K.R. Fuller, *Rings and Categories of Modules*, GTM 13, Springer Verlag, Berlin-Heidelberg-New York (1973)

[2] E.P. Armendariz; J.W. Fisher; R.L. Snider, On injective and surjective endomorphisms of finitely generated modules, Comm. Algebra, **1978** *6(7)*, 659–672

[3] F. Couchot, Modules with RD-composition series over a commutative ring, Comm. Algebra, **2003** *31(7)*, 3171–3194

[4] D.J. Fieldhouse, Pure theories, Math. Ann., **1969** *184*, 1–18.

[5] L. Fuchs, *Abelian groups*, Publishing House of the Hungarian Academy of Sciences, Budapest (1958)

[6] J.L. García, Properties of direct summands of modules, Comm. Algebra, **1989** *17(1)*, 73–92

[7] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London (1979); 2nd edn, Krieger (1991)

[8] Y. Hirano; J. Park, Rings for which the converse of Schurs lemma holds, Math. J. Okayama Univ., **1991** *33*, 121–131

[9] F. Kasch; A. Mader, Regularity and substructures of Hom, Comm. Algebra, **2006** *34(4)*, 1459–1478

[10] S.M. Khuri, Endomorphism rings and lattice isomorphisms, J. Algebra, **1979** *56(2)*, 401–408

[11] T.Y. Lam, *Lectures on Modules and Rings*, GTM 189, Springer Verlag, Berlin-Heidelberg-New York (1999)

[12] G. Lee; S.T. Rizvi; C.S. Roman, Rickart modules, Comm. Algebra, **2010** *38(11)*, 4005–4027

[13] G. Lee; S.T. Rizvi; C.S. Roman, Direct sums of Rickart modules, manuscript, submitted

[14] G. Lee; S.T. Rizvi; C.S. Roman, Dual Rickart modules, Comm. Algebra, **2011** *39(12)*

[15] S.H. Mohamed; B.J. Müller, *Continuous and Discrete Modules*, Lecture Note Ser. No. 147, London Math. Soc., Cambridge Univ. Press (1990)

[16] J. von Neumann, *Continuous geometry*, With a foreword by Israel Halperin, Princeton Landmarks in Mathematics, Princeton University Press (1960)

[17] W.K. Nicholson, On PP-endomorphism rings, Canad. Math. Bull., **1993** *36(2)*, 227–230

[18] W.K. Nicholson; E. Sánchez Campos, Morphic modules, Comm. Algebra, **2005** *33(8)*, 2629–2647

[19] K.M. Rangaswamy, Abelian groups with endomorphic images of special types, J. Algebra, **1967** *6*, 271–280

[20] S.T. Rizvi, Commutative rings for which every continuous module is quasi-injective, Arch. Math. (Basel), **1988** *50(5)*, 435–442

[21] S.T. Rizvi; C.S. Roman, Baer and quasi-Baer modules, Comm. Algebra, **2004** *32(1)*, 103–123

[22] S.T. Rizvi; C.S. Roman, Baer property of modules and applications, Advances in Ring Theory, **2005**, 225–241

[23] S.T. Rizvi; C.S. Roman, On $\mathcal{K}$-nonsingular modules and applications, Comm. Algebra, **2007** *35*, 2960–2982

[24] S.T. Rizvi; C.S. Roman, On direct sums of Baer modules, J. Algebra, **2009** *321(2)*, 682–696

[25] B. Stenström, *Rings of Quotients*, Band 217, Springer Verlag, Berlin-Heidelberg-New York (1975)

[26] D.K. Tütüncü; R. Tribak, On dual Baer module, Glasg. Math. J., **2010** *52(2)*, 261–269

[27] R. Ware, Endomorphism rings of projective modules, Trans. Amer. Math. Soc., **1971** *155*, 233–256

[28] R. Ware; J. Zelmanowitz, Simple endomorphism rings, Amer. Math. Monthly, **1970** *77(9)*, 987–989

[29] R. Wiegand, Endomorphism rings of ideals in a commutative regular ring, Proc. Amer. Math. Soc., **1969** *23*, 442–449

[30] J. Zelmanowitz, Regular modules, Trans. Amer. Math. Soc., **1972** *163*, 340–355

Gangyong Lee, Department of Mathematics, The Ohio State University

Columbus, OH 43210, USA

e-mail: lgy999@math.ohio-state.edu

S. Tariq Rizvi, Department of Mathematics, The Ohio State University

Lima, OH 45804, USA

e-mail: rizvi.1@osu.edu

Cosmin S. Roman, Department of Mathematics, The Ohio State University

Lima, OH 45804, USA

e-mail: cosmin@math.ohio-state.edu