Algebra Qualifying Exam I
2018

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer **any 5 out of the 6 given problems**. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Given a group G, define the following normal subgroups inductively: $Z_0(G) = 1$, $Z_1(G) = Z(G)$ is the center of G, and $Z_{i+1}(G) \trianglerighteq Z_i(G)$ is the subgroup of G such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. The chain $Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \cdots$ is called the upper central series of G. A group is called *nilpotent* if $Z_c(G) = G$ for some integer c, and the smallest such c (if it exists) is called the *nilpotence class* of G.

   Let p be a prime and suppose that G a group of order p^a where $a \geq 2$. Prove that the nilpotence class of G is $\leq a - 1$.

2. Let G be a finite group which acts transitively on a set S with $|S| \geq 2$. Show that there there exists an element $g \in G$ which has no fixed points, i.e., for all $s \in S$, $g \cdot s \neq s$ (i.e., $S^{\langle g \rangle} = \emptyset$).

3. Assume that A is a commutative integral domain. Prove that

$$\bigcap_{\substack{\mathfrak{m} \subsetneq A \\ \text{maximal ideal}}} A_{\mathfrak{m}} = A$$

   where all the rings above are to be viewed as subrings of K, the field of fractions of A. (Here $A_{\mathfrak{m}}$ denotes the localization of A at the multiplicative set $S = A \setminus \mathfrak{m}$.)

4. Let R be a commutative ring with 1. R is said to be Noetherian if it satisfies the ascending chain condition on ideals:

   (ACC) If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_k \subseteq \ldots$ is an ascending chain of ideals in R, then $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \mathfrak{a}_{k+2} \cdots$ for k sufficiently large.

   An equivalent condition is

   (FG) Every ideal of R is finitely generated.

   Show that $(ACC) \iff (FG)$.

5. Compute the irreducible decomposition of the representation of S_3 induced from the sign representation of S_2.

6. Let V be a vector space over $\mathbb{C}$. Suppose $v_1, \ldots, v_r$ are linearly independent vectors of V and let $w \in \bigwedge^p(V)$. Prove that w is expressible as $w = \sum_{i=1}^{r} v_i \wedge \psi_i$ for some $\psi_1, \ldots, \psi_r \in \bigwedge^{p-1}(V)$ if and only if $v_1 \wedge \cdots \wedge v_r \wedge w = 0$.