Ph.D. Algebra Qualifying Exam
6111

August, 2014

Directions

1. Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

2. Answer each question on a separate sheet or sheets of paper, and write your *code name* and the *problem number* on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

3. Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

4. This is a closed book, closed notes exam.

1. Construct all groups of order 275 ($= 5^2 \times 11$).

2. (a) Prove that A_n is a normal subgroup of S_n.

   (b) If $n \geq 5$, show that A_n is the *only* non-trivial normal subgroup of S_n. (You may use the simplicity of A_n.)

   (c) Show that this is not the case when $n = 4$.

3. Find (with proof) the character table of $D_8 = \langle r, d \mid r^4 = d^2 = 1, dr = r^{-1}d \rangle$. You may use the fact that D_8 is the group of symmetries of the square.

4. Let A be a commutative ring with 1. A is said to be noetherian if it satisfies the ascending chain condition on ideals:

   (ACC) If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_k \subseteq \ldots$ is an ascending chain of ideals in A, then $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \mathfrak{a}_{k+2} \cdots$ for k sufficiently large.

   An equivalent condition is

   (FG) Every ideal of A is finitely generated.

   Show that $(ACC) \iff (FG)$.

5. Let A be a commutative ring with 1, $\mathfrak{p} \subseteq A$ a prime ideal, and $S = A \backslash \mathfrak{p}$ the complement of $\mathfrak{p}$ in A. Let $A_{\mathfrak{p}} = S^{-1}A$ be the localization of A at $\mathfrak{p}$. Let M be an A-module, and let $M_{\mathfrak{p}} = S^{-1}M$ be the localization of M at $\mathfrak{p}$.

   (a) Give a precise definition of $M_{\mathfrak{p}}$, and show that $M_{\mathfrak{p}}$ is naturally a module over $A_{\mathfrak{p}}$.

   (b) Let $\varphi_{\mathfrak{p}} : M \to M_{\mathfrak{p}}$ be the natural map. Show that if $m \in M$, $m \neq 0$, then there is a prime $\mathfrak{p}$ of A such that $\varphi_{\mathfrak{p}}(m) \neq 0$.