Ph.D. Algebra Qualifying Exam
6112

August, 2014

Directions

1. Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

2. Answer each question on a separate sheet or sheets of paper, and write your *code name* and the *problem number* on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

3. Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

4. This is a closed book, closed notes exam.

1. (a) In the category of $\mathcal{C}$ of abelian groups, state the universal mapping property that characterizes the free abelian group F_S on a set S.

   (b) Show that F_S exists by giving a construction and showing your construction satisfies the universal mapping property given in (a).

2. Let R be a ring with 1. Prove that an R module Q is injective iff any homomorphism of a left ideal $\mathfrak{a}$ into Q can be extended to an R-module homomorphism of R into Q.

3. Let $\mathcal{C}$ be the category of abelian groups. Let p be a prime number, and define functors F and G from $\mathcal{C}$ to $\mathcal{C}$ by $F(A) = A[p]$ (the elements of order dividing p in A) and $G(A) = A/pA$. Suppose that $0 \to A \stackrel{u}{\to} B \stackrel{v}{\to} C \to 0$ is exact:

   (a) Show that $0 \to F(A) \stackrel{F(u)}{\to} F(B) \stackrel{F(v)}{\to} F(C) \to 0$ is exact on the left, and fully exact if $G(A) = 0$.

   (b) Show that $0 \to G(A) \stackrel{G(u)}{\to} G(B) \stackrel{G(v)}{\to} G(C) \to 0$ is exact on the right, and fully exact if $F(C) = 0$.

4. Let K/F be a cyclic Galois extension of degree n, and suppose that the n^{th} roots of unity lie in F. By Hilbert's Theorem 90, we can write K as $F(\sqrt[n]{a})$ for some $a \in F^{\times}$. Describe (with proof) the set of all elements $b \in F^{\times}$ such that $K = F(\sqrt[n]{b})$.

5. Prove the Primitive Element Theorem for finite Galois extensions: if K/k is a finite Galois extension, then there exists an $\alpha \in K$ such that $K = k(\alpha)$.