Analysis Qualifying Examination 2 **August 18, 2015**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $f\colon X \to [0,\infty]$ be measurable. Define $\nu\colon \mathscr{A} \to [0,\infty]$ by $\nu(A) = \int_A f\,d\mu$. Then ν is a measure on $\mathscr{A}$. (You may take this for granted.) Suppose in addition that $\mu(X) < \infty$.

[5] **(a)** Prove that ν is expressible in the form $\nu = \sum_n \nu_n$, where (ν_n) is a sequence of finite measures on $\mathscr{A}$.

[10] **(b)** Prove that ν is σ-finite if and only if $f < \infty$ μ-a.e.

2. Let $(X, \mathscr{A}, \nu)$ be a measure space with $\nu(X) < \infty$.

[5] **(a)** Let $\mathscr{E}$ be a subset of $\mathscr{A}$ such that $\mathscr{E}$ is closed under countable unions. Prove that $\mathscr{E}$ has a ν-essentially largest element.[1]

[10] **(b)** Let μ be any measure on $\mathscr{A}$. Prove that ν has a Lebesgue decomposition with respect to μ.[2]

Notational Reminder. $C(\mathbf{T})$ denotes the space of continuous functions $f\colon \mathbf{R} \to \mathbf{C}$ such that $f(x+1) = f(x)$ for all $x \in \mathbf{R}$. $L^1(\mathbf{T})$ denotes the space of all Lebesgue-measurable functions $f\colon \mathbf{R} \to \mathbf{C}$ such that $f(x+1) = f(x)$ for all $x \in \mathbf{R}$ and $\int_0^1 |f(x)|\,dx < \infty$. For each $f \in L^1(\mathbf{T})$, $\|f\|_1 = \int_0^1 |f(x)|\,dx$ and $\hat{f}$ is the function on $\mathbf{Z}$ defined by $\hat{f}(k) = \int_0^1 e^{-2\pi ikx} f(x)\,dx$, where $i = \sqrt{-1}$. $L^2(\mathbf{T})$ denotes the space of all Lebesgue-measurable functions $f\colon \mathbf{R} \to \mathbf{C}$ such that $f(x+1) = f(x)$ for all $x \in \mathbf{R}$ and $\int_0^1 |f(x)|^2\,dx < \infty$. For each $f \in L^2(\mathbf{T})$, $\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}$.

[10] **3. (a)** Let $f, g\colon [0,1] \to \mathbf{C}$ be absolutely continuous. Prove that fg is absolutely continuous and

$$\int_0^1 f'(x)g(x)\,dx = f(1)g(1) - f(0)g(0) - \int_0^1 f(x)g'(x)\,dx.$$

[10] **(b)** Let $f \in C(\mathbf{T})$. Suppose f is absolutely continuous on $[0,1]$ and $f' \in L^2(\mathbf{T})$. Prove that $\sum_{k\in\mathbf{Z}} |\hat{f}(k)| < \infty$. (Warning: Please try to avoid dividing by zero.)

4. Recall that $\ell^\infty(\mathbf{Z})$ denotes the space of bounded functions $g\colon \mathbf{Z} \to \mathbf{C}$. For each $g \in \ell^\infty(\mathbf{Z})$, $\|g\|_\infty = \sup_{k\in\mathbf{Z}} |g(k)|$, by definition. For each $f \in L^1(\mathbf{T})$, observe that by the triangle inequality for integrals, $\hat{f} \in \ell^\infty(\mathbf{Z})$ and $\|\hat{f}\|_\infty \le \|f\|_1$, where $\hat{f}$ is as defined in the notational reminder above. Define $\mathscr{F}$ on $L^1(\mathbf{T})$ by $\mathscr{F}(f) = \hat{f}$. By what we just observed, $\mathscr{F}\colon L^1(\mathbf{T}) \to \ell^\infty(\mathbf{Z})$. Finally, recall that $c_0(\mathbf{Z})$ denotes the space of functions $g\colon \mathbf{Z} \to \mathbf{C}$ such that $g(k) \to 0$ as $k \to \pm\infty$.

[10] **(a)** Prove the Riemann-Lebesgue lemma, that for each $f \in L^1(\mathbf{T})$, we have $\hat{f} \in c_0(\mathbf{Z})$. Thus $\mathscr{F}$ actually maps $L^1(\mathbf{T})$ into $c_0(\mathbf{Z})$.

[10] **(b)** Let F be the range of $\mathscr{F}$. Prove that F is a proper subset of $c_0(\mathbf{Z})$. (You may use without proof the fact that $\|D_n\|_1 \to \infty$ as $n \to \infty$, where $D_n\colon \mathbf{R} \to \mathbf{C}$ is the n-th Dirichlet kernel, which is defined by $D_n(x) = \sum_{k=-n}^{n} e^{2\pi ikx}$.)

---

[1] To say that E *is a ν-essentially largest element of* $\mathscr{E}$ means that $E \in \mathscr{E}$ and for each $D \in \mathscr{E}$, we have $\nu(D \setminus E) = 0$.

[2] A *Lebesgue decomposition of ν with respect to μ* is a pair (α, β) of measures on $\mathscr{A}$ such that $\nu = \alpha + \beta$, α is absolutely continuous with respect to μ, and β is singular with respect to μ. To say that α *is absolutely continuous with respect to* μ (denoted $\alpha \ll \mu$) means that for each $A \in \mathscr{A}$, if $\mu(A) = 0$, then $\alpha(A) = 0$. To say that β *is singular with respect to* μ (denoted $\beta \perp \mu$) means that there exists $E \in \mathscr{A}$ such that $\mu(E) = 0$ and β lives on E. To say that β *lives on* E means that $\beta(X \setminus E) = 0$.

5. Let E be a normed linear space. Recall that the dual space E^* of E is the vector space of continuous linear functionals f on E, equipped with the norm defined by

$$\|f\| = \sup\{\, |f(x)| : x \in E \text{ and } \|x\| \leq 1\,\}.$$

You may take it for granted that this does define a norm on E^*. You may also take it for granted that under this norm, E^* is a Banach space.[3] Of course we may also consider E^{**}, the dual space of E^*, which is the vector space of all continuous linear functionals φ on E^*, equipped with the norm defined by

$$\|\varphi\| = \sup\{\, |\varphi(f)| : f \in E^* \text{ and } \|f\| \leq 1\,\}.$$

[10] **(a)** For each $x \in E$, define φ_x on E^* by $\varphi_x(f) = f(x)$ and prove that φ_x is a continuous linear functional on E^* with $\|\varphi_x\| = \|x\|$.

[10] **(b)** Let $B \subseteq E$ such that for each $f \in E^*$, we have $\sup\{\, |f(x)| : x \in B\,\} < \infty$. Prove that

$$\sup\{\, \|x\| : x \in B\,\} < \infty.$$

[10] **6.** Let H be a Hilbert space and let $B = \{\, x \in H : \|x\| \leq 1\,\}$. Prove that B is weakly compact.

[3] A *Banach space* is a normed linear space which is complete with respect to the metric induced by its norm.