1. Let \( \mathcal{F} \) be a field of subsets of a set \( X \). Let \( \nu: \mathcal{F} \to \mathbb{C} \) be \( \sigma \)-additive.\(^1\) Let \( |\nu| \) be the variation\(^2\) of \( \nu \). Prove that \( |\nu| \) is \( \sigma \)-additive.

2. Let \( A \) be the set of all functions \( f: [0, \infty) \to \mathbb{C} \) such that for each \( b \in (0, \infty) \), \( f \) is absolutely continuous on \([0, b] \). For each \( p \in (0, \infty) \), let \( L^p \) denote the set of all Lebesgue-measurable functions \( f: [0, \infty) \to \mathbb{C} \) such that

\[
\int_0^\infty |f(x)|^p \, dx < \infty.
\]

(a) Give an example of a real-valued function \( f \in \bigcap_{0 < p < \infty} L^p \cap A \) such that

\[
\limsup_{b \to \infty} f(b) = \infty.
\]

(Don’t work hard. There is a simple example. It would not be hard to give a formula for \( f \) but you don’t have to give a formula. A suitably labelled sketch of the graph and/or a suitable description in words would be fine.)

(b) Let \( p \in (0, \infty) \), let \( f \in L^p \cap A \), and suppose \( f' \in L^1 \). Prove that

\[
f(b) \to 0 \quad \text{as } b \to \infty.
\]

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\(^1\) To say that \( \nu \) is \( \sigma \)-additive means that for each finite or countable disjoint sequence \((F_n)\) in \( \mathcal{F} \), if \( \bigcup_n F_n \in \mathcal{F} \), then \( \nu(\bigcup_n F_n) = \sum_n \nu(F_n) \). Incidentally, this definition implies that if \( \nu \) is \( \sigma \)-additive, then \( \nu(\emptyset) = 0 \), because \( \emptyset \) is the union of the empty sequence of elements of \( \mathcal{F} \) and a sum with no terms has the value 0.

\(^2\) Reminder: The variation of \( \nu \) is the function \( |\nu|: \mathcal{F} \to [0, \infty] \) defined by

\[
|\nu|(E) = \sup \sum_m |\nu(E_m)|,
\]

where \((E_m)\) ranges over all finite disjoint sequences of elements of \( \mathcal{F} \) such that \( \bigcup_m E_m \subseteq E \). (We would get the same result if we required \( \bigcup_m E_m = E \). This is easy to see and you may take it for granted.)
3. Let $X$ be a topological space. By definition, $\text{Baire}(X)$ is the $\sigma$-field on $X$ generated by $\mathcal{H}$, where $\mathcal{H} = \{ f^{-1}[B] : f \in C(X, \mathbb{R}) \text{ and } B \in \text{Borel}(\mathbb{R}) \}$. The elements of $\text{Baire}(X)$ are called Baire subsets of $X$.

(a) Let $A \in \text{Baire}(X)$. Prove that there is a continuous function $f: X \to \mathbb{R}^N$ such that $A \in \mathcal{E}$, where $\mathcal{E} = \{ f^{-1}[E] : E \in \text{Borel}(\mathbb{R}^N) \}$. (Reminder: $\mathbb{R}^N$ denotes the space of all infinite sequences of real numbers, with its usual product topology. In other words, $\mathbb{R}^N$ denotes the Cartesian product of a countably infinite number of copies of the real line $\mathbb{R}$.)

(b) Let $K$ be a compact Baire subset of $X$. Prove that $K$ is closed in $X$ and that $K$ is a countable intersection of open subsets of $X$. (Warning: $X$ need not be Hausdorff.)

4. Let $E$ be a vector space and let $M$ and $N$ be linear subspaces of $E$ such that $M \cap N = \{0\}$. Define $P$ and $Q$ on $M + N$ by $P(x+y) = x$ and $Q(x+y) = y$ for all $x \in M$ and all $y \in N$. Then $P$ and $Q$ are well-defined linear operators on $M + N$. (You need not prove this. It is elementary.)

(a) Suppose in addition that $E$ is a normed linear space. Prove that $P$ is continuous if and only if $Q$ is continuous.

(b) Now suppose in addition that $E$ is a Banach space and that $M$ and $N$ are closed. Prove that $P$ and $Q$ are continuous if and only if $M + N$ is closed.

5. Let $K$ be $\mathbb{R}$ or $\mathbb{C}$. Let $E$ be a normed linear space over $K$, let $E^*$ be the Banach space of continuous linear functionals on $E$, and let $E^{**}$ be the Banach space of continuous linear functionals on $E^*$.

(a) For each $x \in E$, define $L_x: E^* \to K$ by $L_x(\varphi) = \varphi(x)$ and prove that $L_x \in E^{**}$ and $\|L_x\| = \|x\|$.

(b) Let $A \subseteq E$. Suppose that for each $\varphi \in E^*$, the set $\varphi[A]$ is bounded in $K$. Prove that $A$ is norm-bounded in $E$.

6. Let $X$ be a locally compact Hausdorff space and let $C_0(X)$ be the space of complex-valued continuous functions $f$ on $X$ such that $f$ tends to zero at infinity. Let $f$ be an element of $C_0(X)$ and let $(f_n)$ be a sequence in $C_0(X)$. Prove that $f_n \to f$ weakly in $C_0(X)$ if and only if $(f_n)$ is uniformly bounded and $f_n \to f$ pointwise. (For part of the forward implication, you may use the result of one of the parts of an earlier problem.)

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3 The fact that $E^*$ is a Banach space and not just a normed linear space follows from the fact that the scalar field $K$ is complete as a metric space. You may use the fact that $E^*$ is a Banach space without proof.