You may submit solutions for at most 5 out of the following 7 problems. Each question will be graded out of 10 points.

(1) Suppose $X$ is a Banach space and $Y \subseteq X$ is a closed subspace. Show that

$$\|x + Y\|_{X/Y} := \inf \{\|x + y\|_X \mid y \in Y\}$$

is a well-defined norm on $X/Y$ under which $X/Y$ is a Banach space.

(2) Suppose $X$ is a Banach space and $F$ is a finite dimensional vector space. Show that a linear map $\varphi : X \to F$ is bounded if and only if $\ker(\varphi)$ is closed.

(3) State and prove the Baire Category Theorem for complete metric spaces.

(4) Suppose $X$ and $Y$ are Banach spaces and $T : X \to Y$ is a continuous linear map. Show that the following are equivalent.
   
   (a) There exists a constant $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.
   
   (b) $T$ is injective and has closed range.

(5) (a) Find a Banach space whose dual space is (isometrically isomorphic to) $\ell^1$.
   
   (b) Show that $L^1[0,1]$ with respect to Lebesgue measure is not a dual space.

(6) Let $X$ be a Banach space. Prove that every weakly convergent sequence in $X$ and every weak* convergent sequence in $X^*$ are bounded (with respect to the respective norms).

(7) Let $X$ be a Banach space and $B$ its closed unit ball. Let $i : X \to X^{**}$ be the canonical inclusion. Prove that $i(B)$ is weak*-dense in $B^{**}$, the closed unit ball of $X^{**}$. 