Ph. D. Analysis Qualifying Exam 6212 (August 2018)

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Solutions to five problems constitute a complete exam.

Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a two hour, closed book, closed notes exam.

1. Let $f : [0,1] \to \mathbb{C}$ be square-integrable. Show that the minimum of $\displaystyle\int_0^1 |f(s) - \sum_{k=1}^n c_k e^{2\pi iks}|^2 ds$ over all tuples $c_1, ..., c_n$ in $\mathbb{C}$ is attained if and only if $c_k = \displaystyle\int_0^1 f(s)e^{-2\pi iks} ds$ for all $1 \leqslant k \leqslant n$.

2. Let $\mathcal{S}$ be the space of Schwarz functions on $\mathbb{R}$. Let ϕ be defined on $\mathcal{S}$ by $\phi(f) = \sum_{k\in\mathbb{Z}} f(k)$. Show that $\phi \in \mathcal{S}'$.

3. Let $\mathcal{B} = \ell^\infty(\mathbb{N})$ be the Banach space of complex sequences $(x_n)_{n\in\mathbb{N}}$ of finite sup norm. Let L be defined on $\mathcal{B}$ by $L\left((x_n)_{n\in\mathbb{N}}\right) = \left(\frac{x_n}{n}\right)_{n\in\mathbb{N}}$.

(a) Show that the graph of L, $\Gamma(L) = \{(a, La) : a \in \mathcal{B}\}$ is a closed subset of $\mathcal{B} \times \mathcal{B}$.

(b) Is the range of L, $\{La : a \in \mathcal{B}\}$, closed in $\mathcal{B}$?

(c) For what values of $\lambda \in \mathbb{R}$ does $(L - \lambda I)^{-1}$ (where I is the identity) exist as a continuous map on $\mathcal{B}$?

4. Let P and Q be two polynomials; assume that Q has no real roots and define f by

$$f(x) = \frac{P(\cos 2\pi x)}{Q(\cos 2\pi x)}$$

Show that the symmetric Fourier sums $S_m f$ (the Fourier sums from $-m$ to m) converge pointwise to f for any $x \in [0,1]$.

(b) Show that the convergence above is uniform.

5. Let E and G be Banach spaces, let F be a normed linear space, and let $S : E \to F$ and $T : F \to G$ be linear, continuous, one-to-one, and onto. Prove that F is a Banach space.

6. Let E be a normed linear space, let $A \subseteq E$, and suppose $f[A]$ is bounded in the scalar field for each continuous linear functional f on E. Prove that A is bounded in E.

7. Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the vector space of real-valued functions on X which tend to zero at infinity. As usual, give $C_0(X)$ the uniform norm, defined by

$$\|f\|_u = \sup\{|f(x)| : x \in X\}$$

for each $f \in C_0(X)$. Let L be a linear functional on $C_0(X)$ such that for each $f \in C_0(X)$, if $f \geqslant 0$, then $L(f) \geqslant 0$. Prove that L is continuous.