Analysis Qualifying Exams

OSU Math Grad Students

July 31, 2014

To pass, it suffices to solve four problems correctly. You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem just by quoting a theorem that reduces what you are asked to prove to a triviality. Justify the applicability of theorems you use. If you are not sure whether you may use a particular theorem, ask the proctor.

Please remember to write your code name at the top of each page. Note that a good code name can be the difference between passing and failing.

Many of the solutions contained in this document were provided in the summer 2013 head-start course led by Professor Aurel Stan and TA Donald Robertson.

Contents

2013 - Autumn

Exercise 1. *Let $a_1 = a \geq 0$ and $a_{n+1} = \sqrt{|a_n^2 - a_n^4|}$, $n \in \mathbb{N}$. Find all values of a for which the sequence (a_n) converges. (Justify your answer.)*

Proof. (A. Newman) The sequence (a_n) converges for $a \in [0, \sqrt{2}]$ and diverges otherwise. Consider the function given by the recurrence $f(x) = \sqrt{|x^2 - x^4|}$. On the interval $[0, 1]$, we see that $f(x)$ is increasing for $x < \frac{\sqrt{2}}{2}$ and decreasing for $x > \frac{\sqrt{2}}{2}$. So if we consider $a_1 \in (0, \frac{\sqrt{2}}{2})$ we see that $a_1 \geq a_2$, so it follows that $a_2 \geq a_3$, $a_3 \geq a_4$, and so on. Thus (a_n) is a decreasing sequence that is bounded below by zero, so it is convergent. For $a_1 = \frac{\sqrt{2}}{2}$ we have $a_2 = \frac{1}{2}$, so we are back in the interval $(0, \frac{\sqrt{2}}{2})$ and we have that (a_n) converges in this case. Likewise for $x \in (\frac{\sqrt{2}}{2}, 1)$ we have $f(\frac{\sqrt{2}}{2}) > f(x) > f(1)$, so $f(x) \in (0, \frac{1}{2})$. Thus if $a_1 \in (\frac{\sqrt{2}}{2}, 1)$, then $a_2 \in (0, \frac{1}{2}) \subseteq (0, \frac{\sqrt{2}}{2})$, so the sequence (a_n) converges. Obviously the sequence converges when $a = 1$. For all $x \in (1, \sqrt{2})$, $f(x)$ is increasing, and if $a = a_1 \in (1, \sqrt{2})$, then $a_2 \leq a_1$, so the sequence (a_n) is decreasing and bounded below by zero, so it is convergent. If $a = \sqrt{2}$, then the sequence is constant and therefore convergent. Finally if $a > \sqrt{2}$, we have that $a_2 > a_1$, and so the sequence is increasing. However it is not bounded. Since for $x > 2$ we have $x^4 - x^2 > \frac{3}{4}x^4$ and the sequence given by the recurrance $a_{n+1} = \frac{\sqrt{3}}{2}a_n^2$ diverges for $a > \sqrt{2}$. □

Exercise 2. *Let $a, b > 0$. Find the smallest possible constant C for which the inequality*

$$x^a y^b \leq C(x+y)^{a+b} \tag{1}$$

holds for all $x, y > 0$. (Justify your answer.)

Proof. (A. Newman) The smallest value of C that works is $C = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$. To verify this, observe that we wish to maximize:

$$f(x, y) = \left(\frac{x}{x+y}\right)^a \left(\frac{y}{x+y}\right)^b.$$

To maximize this function, we will set $z = \dfrac{x}{x+y}$ and $w = \dfrac{y}{x+y}$. And now we try to maximize $f(z, w) = z^a w^b$ subject to the constraint $z + w = 1$, using the method of Lagrange multipliers we must have λ so that $az^{a-1}w^b - \lambda = 0$ and $bz^a w^{b-1} - \lambda = 0$ simultaneously. Thus we must have global maxima at $z = \frac{aw}{b}$, that is global maxima at $y = \frac{bx}{a}$. Substituting this for y yields that for any (x, y) with $x > 0$ and $y = \frac{bx}{a}$ we have $f(x, y) = \left(\dfrac{a}{a+b}\right)^a \left(\dfrac{b}{a+b}\right)^b$. So this proves the claim.

(R. Ye) Sketch: Since $x, y > 0$, let $t = \frac{y}{x}$, then $y = tx$. Put it into the original inequality and after simplification, we get:

$$\frac{t^b}{(1+t)^{a+b}} \leq C.$$

Let $f(t) = \frac{t^b}{(1+t)^{a+b}}$. Now it's easy to maximize $f(t)$ on $t > 0$ to obtain C, by using derivative. $\square$

Exercise 3. *Prove that the function defined by* $f(x) = \dfrac{e^{\pi x} - 1}{e^x - 1}$ *for* $x \neq 0$ *and* $f(0) = \pi$ *is infinitely differentiable on* $\mathbb{R}$.

Proof. (H. Lyu) By quotient rule, f is certainly infinitely many differentiable for $x \neq 0$. So it suffices to show that f is so at $x = 0$. Using the power series expansion $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots$, $x \in \mathbb{R}$, we can write

$$f(x) = \frac{\frac{\pi x}{1} + \frac{\pi^2 x^2}{2!} + \cdots}{\frac{x}{1} + \frac{x^2}{2!} + \cdots} = \frac{\pi + \frac{\pi^2 x}{2!} + \cdots}{1 + \frac{x}{2!} + \cdots} \tag{$*$}$$

where $x \neq 0$. But since $f(0) = \pi$ as defined, $(*)$ holds for all x in $\mathbb{R}$. Note that the two power series after canceling out the common factor x in $(*)$ have the same radius of convergence, by Hadamard's formula since the coefficients do not change. Also, a power series is analytic, i.e., infinitely differentiable, in its radius of convergence. Thus $(*)$ tells us that $f(x)$ can be written as a fraction of two smooth functions where the denominator does not vanish at $x = 0$.

Now suppose we have a function $g(x) = p(x)/q(x)$ where $p, q \in C^\infty(\mathbb{R})$ and $q(0) \neq 0$. Then since q is continuous and $q(0) \neq 0$, $q(x) \neq 0$ near $x = 0$. By quotient rule, $g'(x) = \frac{p'(x)q(x) - q'(x)p(x)}{q(x)^2}$ near $x = 0$. Note that both the numerator and denominator are smooth and the denominator does not vanish at 0, so g' satisfies the same hypothesis as g. Hence, by induction, g is infinitely differentiable near $x = 0$, and in particular, at $x = 0$. This observation applies to $(*)$, and this shows the assertion. $\square$

Exercise 4. *Prove that* $\displaystyle\sum_{k=2}^{n} \frac{1}{k \log k} = \log\log n + C + a_n$, $n \in \mathbb{N}$, *where* $C \in \mathbb{R}$ *and* $a_n \to 0$ *as* $n \to \infty$.

Proof #1. (O. Khalil)

By the integral test, since the function $\frac{1}{x \log x}$ is decreasing, we have that for each $n \geq 2$,

$$\int_2^n \frac{1}{x \log x} dx \leq \sum_{k=2}^{n} \frac{1}{k \log k} \leq \frac{1}{2 \log 2} + \int_2^n \frac{1}{x \log x} dx$$

Let $b_n = \displaystyle\sum_{k=2}^{n} \frac{1}{k \log k} - \int_2^n \frac{1}{x \log x} dx$. So, $0 \leq b_n \leq \frac{1}{2 \log 2}$. Note that for each $n \in \mathbb{N}$, we have that

$$b_{n+1} - b_n = \frac{1}{(n+1)\log(n+1)} - \int_n^{n+1} \frac{1}{x \log x}$$

But, since the function $\frac{1}{x \log x}$ is decreasing on the interval $[n, n+1]$, then, we ge that $b_{n+1} - b_n \leq 0$. Hence, b_n is monotonically decreasing, so the sequence b_n is convergent with a finite limit.

Let $l = \lim_{n\to\infty} b_n$ and let $C = -\log\log 2 + l$. Let $a_n = -l + b_n$. So, $a_n \to 0$ as $n \to \infty$. Now, we have that $\frac{d}{dx}(\log\log x) = \frac{1}{x\log x}$, and hence

$$\int_2^n \frac{1}{x\log x}dx = \log\log n - \log\log 2$$

So, we can write

$$\sum_{k=2}^{n} \frac{1}{k\log k} = \log\log n - \log\log 2 + b_n = \log\log n + C + a_n$$

as desired. □

Proof #2. (K. Nowland) Let $f(x) = 1/x\log x$. Let $k > 2$ be an integer. Note that

$$\int_{k-1}^{k} f(x)dx = \int_{k-1}^{k} \frac{d}{dx}\left(x - k + \frac{1}{2}\right) f(x)dx.$$

Integrating by parts,

$$\int_{k-1}^{k} f(x)dx = \frac{1}{2}f(k) + \frac{1}{2}f(k-1) - \int_{k-1}^{k}\left(x - k + \frac{1}{2}\right) f'(x)dx.$$

Rearranging,

$$f(k) = \int_{k-1}^{k} f(x)dx + \frac{1}{2}f(k) - \frac{1}{2}f(k-1) + \int_{k-1}^{k}\left(x - k + \frac{1}{2}\right) f'(x)dx.$$

Summing from $k = 3$ to n,

$$\sum_{k=3}^{n} f(k) = \int_2^n f(x)dx + \frac{1}{2}f(n) - \frac{1}{2}f(2) + \sum_{k=3}^{n}\int_{k-1}^{k}\left(x - k + \frac{1}{2}\right) f'(x)dx.$$

Since $f(x) = 1/x\log x$,

$$\int_2^n f(x)dx = \int_2^n \frac{dx}{x\log x} = \log\log n - \log\log 2.$$

Also, $f(n) \to 0$ as $n \to \infty$. Note that $|x - k + 1/2| \le 1/2$ in the sum of integrals. Therefore,

$$\begin{aligned}\left|\int_{k-1}^{k}\left(x - k + \frac{1}{2}\right) f'(x)dx\right| &\le \frac{1}{2}\int_{k-1}^{k} \frac{1}{x^2\log x} + \frac{1}{x^2\log^2 x}dx \\ &\le \frac{1}{2(k-1)^2\log(k-1)} + \frac{1}{2(k-1)^2\log^2(k-1)}.\end{aligned}$$

Since $\sum \frac{1}{n^2} < \infty$, the sum of the integrals converges absolutely as $n \to \infty$. We can therefore write the sum of integrals as $c - b_n$ where c is constant and b_n tends to zero as n tends to infinity. Let $a_n = 1/2n\log n - b_n$ and $C = -\log\log 2 + 1/4\log 2 + c$. □

Exercise 5. *Let $f \in C^1([0,1])$. Prove that* $\sum_{k=1}^{n} f(k/n) - n\int_0^1 f(x)dx \underset{n\to\infty}{\longrightarrow} \frac{f(1)-f(0)}{2}$.

Proof. (O. Khalil) Let $\varepsilon > 0$ be fixed. Since f' is continuous on $[0,1]$ which is compact, it is uniformly continuous. Let $N \in \mathbb{N}$ be such that $\forall x, y \in [0,1]$, whenever $|x-y| < 1/N$, we have that $|f'(x) - f'(y)| < \varepsilon$. Let $n > N$ be arbitrary. Now, for every $x \in [0,1]$, write

$$F(x) = \int_0^x f(t)dt$$

Since f is differentiable, F is twice differentiable. Let $a_k = k/n$ for each k. Now, using Taylor's expansion for F, write

$$F\left(\frac{a_k + a_{k-1}}{2}\right) = F(a_{k-1}) + \frac{F'(a_{k-1})}{2n} + \frac{F''(\theta_k)}{8n^2} \tag{2}$$

$$F\left(\frac{a_k + a_{k-1}}{2}\right) = F(a_k) - \frac{F'(a_k)}{2n} + \frac{F''(\omega_k)}{8n^2} \tag{3}$$

for some $\theta_k \in (a_{k-1}, \frac{a_k+a_{k-1}}{2})$ and $\omega_k \in (\frac{a_k+a_{k-1}}{2}, a_k)$.
Substracting 2 from 3

$$0 = F(a_k) - F(a_{k-1}) - \left(\frac{F'(a_k) + F'(a_{k-1})}{2n}\right) + \frac{F''(\omega_k) - F''(\theta_k)}{8n^2}$$

Substituting $F(a_k) - F(a_{k-1}) = \int_{a_{k-1}}^{a_k} f(x)dx$, $F'(x) = f(x)$ and $F''(x) = f'(x)$, we get

$$n\int_{a_{k-1}}^{a_k} f(x)dx = \frac{f(a_k) + f(a_{k-1})}{2} + \frac{f'(\theta_k) - f'(\omega_k)}{8n} \tag{4}$$

Now, write

$$\frac{f(1) - f(0)}{2} = \sum_{k=1}^{n} \frac{f(a_k) - f(a_{k-1})}{2}$$

Hence, we have that

$$\begin{aligned}
\left|\sum_{k=1}^{n} f(k/n) - n\int_0^1 f(x)dx - \frac{f(1)-f(0)}{2}\right| &= \left|\sum_{k=1}^{n}\left(\frac{f(a_k) + f(a_{k-1})}{2} - n\int_{a_{k-1}}^{a_k} f(x)dx\right)\right| \\
&= \left|\sum_{k=1}^{n} \frac{f'(\theta_k) - f'(\omega_k)}{8n}\right| \\
&\le \sum_{k=1}^{n}\left|\frac{f'(\theta_k) - f'(\omega_k)}{8n}\right| \\
&< \sum_{k=1}^{n} \frac{\varepsilon}{8n} \\
&< \varepsilon
\end{aligned}$$

where the second equality follows from 4, and the second inequality follows from the uniform continuity of f' and the choice of N since θ_k and $\omega_k \in (a_{k-1}, a_k)$ and so $|\theta_k - \omega_k| < 1/n < 1/N$. Hence, $\displaystyle\sum_{k=1}^{n} f(k/n) - n\int_0^1 f(x)dx \longrightarrow \frac{f(1)-f(0)}{2}$ as $n \to \infty$ as desired.

□

Exercise 6. *Let (a_n) be a sequence of nonzero real numbers such that $\sum_{n=1}^{\infty} |a_n|^{-1} < \infty$. Prove that the series $\sum_{n=1}^{\infty}(x - a_n)^{-1}$ converges uniformly on every bounded set $S \subset \mathbb{R}$ that does not contain the points a_n, $n \in \mathbb{N}$.*

Proof. (R. Ye) Since $S \subset \mathbb{R}$ is bounded, there exists a $M > 0$ such that $|x| < M$ for all $x \in S$. Since $\sum_{n=1}^{\infty} |a_n|^{-1} < \infty$, $|a_n|^{-1} \to 0$, or $|a_n| \to \infty$ as $n \to \infty$. Therefore, there exists an $N \in \mathbb{N}$ such that $|a_n| > 2M$ for all $n \geq N$. For $n \geq N$:

$$|\frac{1}{x - a_n}| = \frac{1}{|x - a_n|} \leq \frac{1}{||a_n| - |x||} \leq \frac{1}{|a_n| - M} = \frac{|a_n|}{|a_n| - M}\frac{1}{|a_n|} < \frac{2}{|a_n|}.$$

The first inequality derives from triangle inequality, the second one holds since $|a_n| > 2M$ and $|x| < M$, and the last one uses the fact that $|a_n| > 2M$. By Weierstrass-M test, $\sum_{n=N}^{\infty}(x - a_n)^{-1}$ converges uniformly on S. Note that $\sum_{n=1}^{N}(x - a_n)^{-1} < \infty$, since $a_n \notin S$ $\forall n \in \mathbb{N}$. So $\sum_{n=1}^{\infty}(x - a_n)^{-1}$ converges uniformly on S. □

2013 - Spring

Exercise 1. *Let $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and suppose that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that there exists a sequence $0 < b_1 < b_2 < \cdots$ of real numbers such that $b_n \to \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$.*

Proof. (A. Newman) Since $\sum_{n=1}^{\infty} a_n$ converges, we know that the sequence $\{R_n\}_{n=1}^{\infty}$ where $R_n = \sum_{k=n}^{\infty} a_k$ goes to zero. Thus, there is $n_1 \in \mathbb{N}$ with $n_1 > 4$ so that $R_{n_1} < \frac{1}{4}$, and there is $n_2 \in \mathbb{N}$, $n_2 \geq n_1$ so that $R_{n_2} < \frac{1}{4^2}$ and we can continue in this way to find n_i so that $R_{n_i} < \frac{1}{4^i}$ for each $i \in \mathbb{N}$. Now for $i \in \{1, .., n_1\}$, set $b_i = 2^0 + i/(n_1)^2$, for $i \in \{n_1+1, ..., n_2\}$, set $b_i = 2^1 + i/(n_2)^2$, for $i \in \{n_2+1, ..., n_3\}$, set $b_i = 2^2 + i/(n_3)^2$ and so on. Now $0 < b_1 < b_2 < ...$ and the b_i go to infinity and

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n b_n &= \sum_{n=1}^{n_1} a_n b_n + \sum_{n=n_1+1}^{n_2} a_n b_n + \sum_{n=n_2+1}^{n_3} a_n b_n + \cdots \\
&\leq 2\sum_{n=1}^{n_1} a_n + 3\sum_{n=n_1+1}^{n_2} a_n + 5\sum_{n=n_2+1}^{n_3} a_n + \cdots \\
&\leq 2R_1 + 3R_{n_1} + 5R_{n_2} + 9R_{n_3} + \cdots \\
&\leq 2R_1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{4} + \frac{1}{64} + \frac{1}{8}
\end{aligned}$$

This last infinite sum converges since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge absolutely. □

Exercise 2. *Let z be a complex number such that $|z| = 1$ but $z \neq 1$. Let $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and suppose $a_n \to 0$ as $n \to \infty$. Prove that the series*

$$\sum_{n=1}^{\infty} a_n z^n$$

converges. (Don't just deduce this from a more general theorem. Give a detailed proof.)

Proof. (A. Newman) This can be proved using Dirichlet's test. However, we will need to prove this first as we are not allowed to deduce this result from a more general theorem. We must prove:

Dirichlet's Test: If $a_n \geq a_{n+1} \geq 0$ for all n, and if $a_n \to 0$ and if there exists M so that $\left|\sum_{n=0}^{N} b_n\right| < M$, then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Proof of claim: Let $s_n = \sum_{k=0}^{n} a_k b_k$ and let $B_n = \sum_{k=0}^{n} b_k$ by summation by parts $s_n = a_{n+1}B_n + \sum_{k=0}^{n} B_k(a_k - a_{k+1})$. Now $a_{n+1}B_n \to 0$ as $n \to \infty$ since $|B_n| < M$ and $a_n \to 0$, furthermore for every k, we have $|a_k - a_{k+1}| = a_k - a_{k+1}$. Thus $|\sum_{k=0}^{n} B_k(a_k - a_{k+1})| \leq M\sum_{k=0}^{n}(a_k - a_{k+1}) = M(a_0 - a_{n+1}) \to Ma_0$. So we have that $|\sum_{k=0}^{n} a_k b_k| \leq Ma_n + Ma_0 \to Ma_0$. Thus $\sum_{k=0}^{\infty} a_k b_k$ converges absolutely so in particular it converges. □

Now we just have to show that if $|z| = 1$, but $z \neq 1$ we have for some absolute constant M, $\left|\sum_{n=1}^{N} z^n\right| < M$ for all n. Observe that $\sum_{n=0}^{N} z^n = \frac{z^{N+1}-1}{z-1}$ for a fixed z with $|z| = 1$

and $z \neq 1$ we have that $|z-1|$ is some positive constant c and that $|z^{N+1}-1| \leq 2$, thus $\left|\sum_{n=0}^{N} z^n\right| < \frac{2}{c} + 1$ for any $N \in \mathbb{N}$. Now the result follows by Dirichlet's test.

$\square$

Exercise 3. *Suppose that $f : \mathbb{R} \to [0, \infty)$ is twice continuously differentiable. Let K be the support of f. In other words, let K be the closure of $\{x \in \mathbb{R} : f(x) \neq 0\}$. Suppose that K is compact. Prove that there is a constant C (depending on f), such that for each $x \in \mathbb{R}$, we have*

$$f'(x)^2 \leq Cf(x). \tag{5}$$

Proof. (A. Newman) On $\mathbb{R} \setminus K$, f is identically zero so all derivatives of f are zero. Furthermore f'' attains a maximum on K since K is compact. Thus for all $x \in \mathbb{R}$ one has $f''(x) \leq C$ for some constant C. Using Taylor's Theorem with Lagrange remainders we have for all $x \in R$ and for all $h > 0$,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2$$

for some $c \in (x, x+h)$. Now it follows that for all $h > 0$ and for any fixed x, $0 \leq f(x+h) \leq f(x) + f'(x)h + \frac{C}{2}h^2$. Now $\frac{C}{2}h^2 + f'(x)h + f(x)$ is a quadradic polynomial in h that is always nonnegative so it's discriminant $(f'(x))^2 - 2Cf(x) \leq 0$, so $(f'(x))^2 \leq 2Cf(x)$. Since x was arbitrary, this proves the claim. $\square$

Exercise 4. *Define a sequence (p_n) of polynomials $p_n : [0,1] \to \mathbb{R}$ recursively as follows: For each $x \in [0,1]$, let $p_0(x) = x$ and if $p_0(x), \ldots, p_n(x)$ have already been defined, let*

$$p_{n+1}(x) = p_n(x) + \frac{x - p_n(x)^2}{2}.$$

Prove that as $n \to \infty$, $p_n(x) \uparrow \sqrt{x}$ uniformly on $[0,1]$.

Proof. (O. Khalil) We begin by showing $p_n(x) \leq \sqrt{x}$ for each n and $\forall x \in [0,1]$. We proceed by induction. We have that $\forall x, p_0(x) = x \leq \sqrt{x}$. Now, assume it's true for $p_n(x)$. But, then, we have that $\forall x \in [0,1]$

$$\begin{aligned} p_{n+1}(x) - p_n(x) &= \frac{(\sqrt{x} - p_n(x))(\sqrt{x} + p_n(x))}{2} \\ &\leq (\sqrt{x} - p_n(x))\sqrt{x} \\ &\leq \sqrt{x} - p_n(x) \end{aligned}$$

where we used the fact that $\sqrt{x} \leq 1$ for the last inequality. Hence, $p_{n+1}(x) \leq \sqrt{x}$ as desired. Now, since $x \geq p_n(x)^2$, then $p_{n+1}(x) - p_n(x) \geq 0$. Hence, the sequence $(p_n(x))$ is monotonically increasing and bounded above by $\sqrt{x}$ for each x. Hence, the point-wise limit exists. For each x, let $l(x) = \lim_{n\to\infty} p_n(x)$. But, then, $l(x)$ satisfies

$$l(x) = l(x) + \frac{x - l(x)^2}{2}$$

Solving for $l(x)$, we get that $l(x) = \sqrt{x}$. Now, since $(p_n(x))$ is a sequence of continuous functions (being polynomials) converging monotonically to a continuous function on a compact set, then by Dini's theorem, the sequence converges uniformly. $\square$

Exercise 5. *Let $f : [0, \infty) \to \mathbb{R}$ be bounded and continuous. Prove that*

$$\limsup_{b\to\infty} \frac{1}{b}\int_0^b f(x)dx \leq \limsup_{x\to\infty} f(x). \tag{6}$$

Proof. (O. Khalil) Let $M < \infty$ be such that $f(x) < M$, $\forall x$. Let $L = \limsup_{x\to\infty} f(x)$. If $L = \infty$, then there is nothing to prove. If L is finite, let $t > L$ be arbitrary. Then, by definition of limit superior, $\exists x_o > 0$, such that $\forall x > x_o$, we have that $t > f(x)$. But, then, we have that $\forall b > x_o$,

$$\begin{aligned}
\frac{1}{b}\int_0^b f(x)dx &= \frac{1}{b}\left(\int_0^{x_o} f(x)dx + \int_{x_o}^b f(x)dx\right) \\
&< \frac{1}{b}\int_0^{x_o} f(x)dx + \frac{1}{b}\int_{x_o}^b tdx \\
&< \frac{1}{b}\int_0^{x_o} f(x)dx + \frac{1}{b-x_o}t(b-x_o) \\
&= \frac{1}{b}\int_0^{x_o} f(x)dx + t \\
&< \frac{Mx_o}{b} + t
\end{aligned}$$

Hence, by taking limsup as $b \to \infty$ on both sides, we get that

$$\limsup_{b\to\infty} \frac{1}{b}\int_0^b f(x)dx \leq t$$

since $\frac{Mx_o}{b} \to 0$ as $b \to \infty$. Now, t was arbitrary, so taking limit as $t \to L^-$, we get 6 as desired. $\square$

Exercise 6. *Let $\mathscr{R}$ be the real vector space of Riemann integrable functions $f : [0, 1] \to \mathbb{R}$. For each $f \in \mathscr{R}$, let $\|f\| = (\int_0^1 |f(x)|^2 dx)^{1/2}$. Let $f \in \mathscr{R}$ and let $\varepsilon > 0$. Prove that there is a continuous function $g : [0, 1] \to \mathbb{R}$ such that $\|f - g\| < \varepsilon$.*

Proof. (H. Lyu) Let $f \in \mathbb{R}$ and fix $\epsilon > 0$. We claim that there is a continuous function $g : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 |f(x) - g(x)|\, dx < \epsilon.$$

Since f is integrable on $[0, 1]$, it is bounded on $[0, 1]$ by some number, say $M > 0$, and there is a partition $P : 0 = x_0 < x_1 < \cdots < x_n = 1$ of $[0, 1]$ such that $U(f, P) - L(f, P) < \epsilon/2$. For each $i = 0, 1, \cdots, n-1$, let $M_i = \sup_{[x_{i-1},x_i]}(f(x))$ and $m_i = \inf_{[x_{i-1},x_i]}(f(x))$. Now we construct a continuous function $g \leq f$ that is "close enough to f" as follows. First let g_0 be the step function defined by

$$g_0 = \sum_{i=0}^{n-1} \chi_{[x_{i-1},x_i)} \cdot m_i,$$

where χ_I is the characteristic function on the interval $I \subset [0,1]$. Note that

$$\begin{aligned} \int_0^1 |f(x) - g_0(x)|\, dx &= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x) - g_0(x)|\, dx \\ &\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = U(f,P) - L(f,P) < \epsilon/2. \end{aligned}$$

Now, the idea is the following. This "infimum stepfunction" g_0 is already close enough to f in 1-norm from the definition of Riemann integral. But since there are only finitely many jumps in g_0, a slight modification on g_0 to make it continuous will suffice. We are going to use line segments to connect each steps as follows. Let $\delta = \frac{\epsilon}{4M(n-1)}$. For each $i \in \{1, \cdots, n-1\}$, define $g(x) = \frac{m_{i+1}-m_i}{\delta}(x - x_i) + m_i$ on $[x_i, x_i + \delta]$, and for x not contained in any of such intervals we define $g(x) = g_0(x)$. Obviously g is continuous from the construction. On the other hand, it differes from g_0 by at most M on $n-1$ "small" intervals of length δ. Hence

$$\int_0^1 |g_0(x) - g(x)|\, dx \leq \sum_{i=1}^{n-1} |m_{i+1} - m_i|\delta \leq \sum_{i=1}^{n-1} 2M\delta = \epsilon/2.$$

This yields

$$\int_0^1 |f(x) - g(x)|\, dx \leq \int_0^1 |f(x) - g_0(x)|\, dx + \int_0^1 |g_0(x) - g(x)|\, dx \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows the claim.

Now let $f \in \mathscr{R}$, fix $\epsilon > 0$, and choose a continuous function $g : [0,1] \to \mathbb{R}$ such that $\int_0^1 |h(x) - g(x)|\, dx < \frac{\epsilon^2}{2M}$. Notice that from the construction of g, we get

$$\inf_{x\in[0,1]} (f(x)) \leq g \leq \sup_{x\in[0,1]} (f(x))$$

In particular, this yields $|f - g| \leq 2M$ on $[0,1]$. Now g is such a function that is close to f in 2-norm, i.e., $||f - g||_2 < \epsilon$, since

$$\int_0^1 |f(x) - g(x)|^2\, dx \leq \int_0^1 2M|f(x) - g(x)|\, dx = 2M \int_0^1 |f(x) - g(x)|\, dx < 2M \cdot \frac{\epsilon^2}{2M} = \epsilon^2.$$

This shows the assertion. $\square$

2012 - Autumn

Exercise 1. *Prove that the sequence* $(1+\frac{1}{n})^{n^2}e^{-n}$, $n \in \mathbb{N}$, *converges and find its limit.*

Proof. (K. Nowland) The sequences converges to $1/\sqrt{e}$. Let x_n be the nth term of the sequence. Since $x_n > 0$ for all n, it makes sense to talk of the logarithm $\log x_n$. Since the logarithm is continuous for $x_n > 0$, if $x_n \to x$, then $\log x_n \to \log x$, where if $x = 0$, then the the sequence $\log x_n$ diverges to $-\infty$. We see that

$$\log x_n = n^2 \log\left(1 + \frac{1}{n}\right) - n.$$

$\log(1+x) = x - x^2/2 + x^3/3 - \cdots$ is a Taylor expansion for $\log(1+x)$ about $x = 0$ and is valid for $|x| < 1$. Using this expansion, we see that

$$\log x_n = -\frac{1}{2} + \frac{1}{3n} - \frac{1}{4n^2} + \cdots .$$

This is valid since $0 < 1/n \leq 1$ for all n. Thus $\log x_n \to -\frac{1}{2}$ as $n \to \infty$. Therefore $x_n \to 1/\sqrt{e}$, as claimed. □

Exercise 2. *Prove or disprove that the function* $f(x) = \sin(x^3)/x$, $x > 0$, *is uniformly continuous on* $(0, \infty)$.

Proof. (O. Khalil) We wish to show that $f(x)$ is uniformly continuous on $(0,\infty)$. First, define $g(x)$ on $[0,\infty)$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Now, by l'Hospital's rule, we find that

$$\lim_{x \to 0^+} f(x) = 0$$

Thus, since $f(x)$ is continuous on $(0,\infty)$ and $\lim_{x\to 0^+} g(x) = g(0)$, then $g(x)$ is continuous on $[0,\infty)$. Therefore, $g(x)$ is uniformly continuous on any compact subset of $[0,\infty)$. Moreover, for all $x > y > 0$, observe that

$$|g(x) - g(y)| \leq |g(x)| + |g(y)| \leq \frac{1}{x} + \frac{1}{y} \leq \frac{2}{y}$$

Now, let $\varepsilon > 0$ be fixed. Let $x_o > 0$ be so that $\frac{2}{x_o} < \varepsilon$. Hence, we have that $g(x)$ is uniformly continuous on $[0, x_o]$ and that for all $x, y > x_o$, we have

$$|g(x) - g(y)| = |f(x) - f(y)| \leq \frac{2}{x_o} < \varepsilon$$

Thus, $g(x)$ is uniformly continuous on $[0,\infty)$ and so f is uniformly continuous on $(0,\infty)$ as desired. □

Exercise 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function with $f''(x) > 0$ for all $x \in [0, 1]$. Assume that $f(0) > 0$ and $f(1) = 1$. Prove that there exists $x_0 \in (0, 1)$ such that $f(x_0) = x_0$ if and only if $f'(1) > 1$.*

Proof. (O. Khalil) "$\Rightarrow$" Suppose that there exists $x_0 \in (0, 1)$ such that $f(x_0) = x_0$. Hence, using Taylor's expansion with Lagrange remainder, we get

$$x_o = f(x_o) = f(1) + (x_o - 1)f'(1) + \frac{(x_o - 1)^2 f''(\theta)}{2}$$

for some $\theta \in (x_o, 1)$. Rearranging

$$f'(1) = \frac{x_o - f(1)}{x_o - 1} - \frac{(x_o - 1)f''(\theta)}{2}$$

Now, since $f(1) = 1$, $x_o < 1$ and $f''(\theta) > 0$, then we get that $f'(1) > 1$ as desired.
"$\Leftarrow$" Now, assume that $f'(1) > 1$. Suppose by way of contradiction that $\nexists x \in (0, 1)$ such that $f(x) = x$. Hence, since, f is continuous, then, on $(0, 1)$, f lies on one side of the line $y = x$ (by applying the intermediate value theorem to the function $f(x) - x$). But, since $f(0) > 0$ by assumption, then, $f(x) > x$, $\forall x \in [0, 1)$. But, then, we have the following: $\forall x \in [0, 1)$

$$1 = \frac{f(1) - x}{1 - x} > \frac{f(1) - f(x)}{1 - x}$$

Hence, taking the limit as $x \to 1^-$ and using the continuity of f', we get that

$$1 \geq f'(1)$$

contrary to our assumption. Hence, $\exists x_0 \in (0, 1)$ such that $f(x_0) = x_0$. □

Exercise 4. *Prove that* $\sup_{x\geq 0} xe^{x^2} \int_x^\infty e^{-t^2} dt = \frac{1}{2}$.

Proof #1. (O. Khalil) Let $f(x) = xe^{x^2} \int_x^\infty e^{-t^2} dt$. First, observe that

$$0 < \int_1^\infty e^{-t^2} dt < \int_1^\infty e^{-t} dt = \frac{1}{e} < \infty$$

also

$$0 < \int_0^1 e^{-t^2} dt < e^0(1 - 0) = 1$$

Moreover, since $e^{-t^2} > 0$ for all t, then the function $g(y) = \int_x^y e^{-t^2} dt$ is positive, bounded and monotonically increasing for all $y > x$. Hence, for all $x \geq 0$, we have that

$$\int_x^\infty e^{-t^2} dt < \infty$$

Hence, we find that $f(0) = 0$ and that $f(x) \geq 0$ for all $x \geq 0$. Hence, $\sup_{x\geq 0} f(x) \geq 0$ and so we may assume $x > 0$ (strictly). Now, for a given $x > 0$, we have that on $[x, \infty)$, $t \geq x$ and so $\frac{te^{-t^2}}{x} > e^{-t^2}$. Hence, we have that

$$\begin{aligned}\int_x^\infty e^{-t^2} dt &< \int_x^\infty \frac{te^{-t^2}}{x} dt \\ &= \frac{e^{-x^2}}{2x}\end{aligned}$$

Hence, we have that for all $x \geq 0$

$$f(x) \leq \frac{1}{2} \tag{7}$$

Moreover, we have that on $[x, x+1]$, $t \leq x+1$ and so we find that on such interval

$$e^{-t^2} \geq \frac{te^{-t^2}}{x+1}$$

Hence, we get that

$$\begin{aligned}\int_x^\infty e^{-t^2} dt &\geq \int_x^{x+1} e^{-t^2} dt \\ &\geq \int_x^{x+1} \frac{te^{-t^2}}{x+1} dt \\ &= \frac{-e^{-(x+1)^2 + e^{-x^2}}}{2(x+1)}\end{aligned}$$

And, thus, we get that

$$f(x) \geq \frac{1}{2}(1 - e^{-2x-1})\frac{x}{x+1}$$

But, we have the following

$$\sup_{x\geq 0}(1 - e^{-2x-1})\frac{x}{x+1} \geq \limsup_{x\to\infty}(1 - e^{-2x-1})\frac{x}{x+1} = \lim_{x\to\infty}(1 - e^{-2x-1})\frac{x}{x+1}$$

for all $x \geq 0$ and $\lim_{x\to\infty}(1 - e^{-2x-1})\frac{x}{x+1} = 1$, we find that

$$\frac{1}{2} \geq \sup_{x\geq 0} f(x) \geq \sup_{x\geq 0} \frac{1}{2}(1 - e^{-x-1})\frac{x}{x+1} \geq \frac{1}{2} \tag{8}$$

as desired. □

Proof #2. (H. Lyu) (similar but little bit simpler) Let $f(x) = xe^{x^2} \int_x^\infty e^{-t^2} dt$ as before. One can show $f(0) = 0$ following Osama's argument. So it suffices to show that $\sup_{x>0} f(x) = 1/2$. The key is to use the substitution $t^2 \mapsto u$, which would yield

$$\int_x^y e^{-t^2}\, dt = \int_{x^2}^{y^2} \frac{e^{-u}}{2\sqrt{u}}\, du. \tag{1}$$

Suppose $x \leq y$. Then (1) yields the following estimation

$$\frac{e^{-x^2} - e^{-y^2}}{2y} \leq \int_x^y e^{-t^2}\, dt \leq \frac{e^{-x^2} - e^{-y^2}}{2x}. \tag{2}$$

Now the second inequality immediately yields $f(x) \leq 1/2$, so that $\sup_{x>0} f(x) \leq 1/2$. On the other hand, from the first inequality we get

$$\frac{e^{-x^2} - e^{-(x+1)^2}}{2(x+1)} \leq \int_x^{x+1} e^{-t^2}\, dt \leq \int_x^{\infty} e^{-t^2}\, dt$$

so that

$$\frac{x - xe^{-2x-1}}{2(x+1)} \leq f(x).$$

From this, letting $x \to \infty$, we get $\limsup_{x\to\infty} f(x) \geq 1/2$. Thus

$$\frac{1}{2} = \limsup_{x\to\infty} f(x) \leq \sup_{x>0} f(x) \leq 1/2.$$

This shows the assertion. □

Exercise 5. *If f is a Riemannn integrable function on a closed bounded interval $[a, b]$, prove that* $\lim_{n\to\infty} \int_a^b f(x) \cos^n x dx = 0$.

Proof. (H. Lyu) Fix $\epsilon > 0$. Let $x_1, \cdots, x_r \in [a, b]$ be an enumeration of all numbers of the form $k + \frac{\pi}{2}$, $k \in \mathbb{Z}$, so that $x_1, \cdots, x_r$ is an enumeration of the solutions of the equation $|\cos x| = 1$ in $[a, b]$. Since f is Riemann integrable on $[a, b]$, it is bounded by certain number, say, $M > 0$. Let $\delta > 0$, $N \in \mathbb{N}$, to be determined. Write $[a, b] = A \sqcup B$ where $B = \bigcup_{k=1}^{r} (x_k - \delta, x_k + \delta) \cap [a, b]$ and $A = [a, b] \setminus B$. Then we get

$$\left| \int_a^b f(x) \cos^n(x)\, dx \right| \leq \int_A |f(x) \cos^n(x)|\, dx + \int_B |f(x) \cos^n(x)|\, dx. \tag{1}$$

First note that $|f(x) \cos^n(x)| \leq M$ on B so the trivial estimation yields

$$\int_B |f(x) \cos^n(x)|\, dx \leq M \sum_{k=1}^{r} l(x_k - \delta, x_k + \delta) = 2Mr\delta,$$

so we may choose $\delta < \frac{\epsilon}{4Mr}$ so that $\int_B |f(x) \cos^n(x)|\, dx < \epsilon/2$ for all $n \in \mathbb{N}$.

On the other hand, for this fixed $\delta > 0$, we will choose large $N = N(\epsilon, \delta) \in \mathbb{N}$ so that the integral over A is $< \epsilon/2$. To this end, notice that A is a closed subset of the compact interval $[a, b]$, so A is compact. Since the function $x \mapsto |\cos x|$ is continuous on $\mathbb{R}$, it has absolute maximum on A, say, $R > 0$. But since $|\cos x| < 1$ on A, we must have $0 < R < 1$. Thus there is a natural number N such that $R^N < \frac{\epsilon}{2(b-a)M}$. Now for any $n > N$, we have

$$\int_A |f(x) \cos^n(x)|\, dx \leq \int_A MR^n\, dx \leq M(b - a)R^N < \epsilon/2.$$

Thus it is possible to choose $\delta > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, the estimation (1) yields

$$\left| \int_a^b f(x) \cos^n(x)\, dx \right| < \epsilon, \tag{2}$$

which is independent of $\delta > 0$. This shows the assertion. $\square$

Exercise 6. *Prove that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, $x \in [0,1]$, *does not converge uniformly on* $[0,1]$.

Proof. (H. Lyu) This solution is due to Prof. Stan. The series does converge pointwise by summation by parts. To show that the convergence is not uniform, let us estimate the Cauchy segment and try to find a lower bound. Let $f(x) = \sin x$. Then $f''(x) = -\sin x < 0$ on $[0,\pi]$. so f is concave down on $[0,\pi]$. Hence the graph of f is above any secant line. In particular, $\sin(x) \geq \frac{2}{\pi}x$ on $[0,\pi/2]$. Let $n, m \in \mathbb{N}$ with $n < m$. Then

$$\frac{\sin nx}{n} + \cdots + \frac{\sin mx}{m} \geq \frac{2nx}{n\pi} + \cdots + \frac{2mx}{m\pi} = \frac{2x(m-n+1)}{\pi}$$

provided $nx, \cdots, mx \in [0,\pi/2]$, which holds if $0 \leq x \leq \frac{\pi}{2m}$. Let $m = 2n$. Then we have

$$\frac{\sin nx}{n} + \cdots + \frac{\sin 2nx}{2n} \geq \frac{2x(n+1)}{\pi}$$

if $x \in [0, \frac{\pi}{4n}]$. Let $x = \frac{\pi}{4(n+1)}$. Then the above inequality holds and for such x, we have

$$\frac{\sin nx}{n} + \cdots + \frac{\sin 2nx}{2n} \geq \frac{1}{2}.$$

Since n is arbitrary, this shows that sequence of the partial sums is not uniformly Cauchy. Thus the series does not converge uniformly.

$\square$

Proof. (O. Khalil) Let $S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$ be the n^{th} partial sum of this series and let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$ be the pointwise limit. Suppose for contradiction that $S_n \to f$ uniformly on $[0,1]$.

First, we use the mean value theorem and the fact that the function $\sin x$ is differentiable on $[0,1]$ to rewrite S_n as follows

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin kx - \sin 0}{k} = \sum_{k=1}^{n} x \cos\theta_k$$

for some $\theta_k \in (0, kx)$. Moreover, the function $\cos x$ is continuous at 0. Hence, for fixed ε_o such that $0 < \varepsilon_o < 1$, $\exists \delta > 0$ such that whenever $|x - 0| = |x| < \delta$, we have that

$$1 - \varepsilon_o < |\cos x| < 1 + \varepsilon_o$$

Since $0 < \varepsilon_0 < 1$, $\delta < \pi/2$. Now, let ε be such that $0 < \varepsilon < \frac{\delta(1-\varepsilon_o)}{4}$. Since $S_n \to f$ uniformly on $[0,1]$ by assumption, then $(S_n(x))$ is uniformly Cauchy on $[0,1]$. Hence, $\exists N \in \mathbb{N}$ such that $\forall n > m > N$ and $\forall x \in [0,1]$, we have that

$$|S_n(x) - S_m(x)| = \left|\sum_{k=m}^{n} x \cos\theta_k\right| < \varepsilon$$

Now, let $n > 2N$ be some even integer. Now, let $x = \frac{\delta}{2n}$. Since, $x < \frac{\pi}{2n}$, then $0 < \theta_k < kx < \pi/2$. Hence, $\cos\theta_k > 0$ for each k in the above sum and the absolute values can be dropped. Moreover, $0 < \theta_k < kx < \delta$ and so we have that $\cos\theta_k > 1 - \varepsilon_o$.
Hence, we have

$$\begin{aligned}
\varepsilon > |S_n(\delta/2n) - S_{n/2}(\delta/2n)| &= \frac{\delta}{2n}\sum_{k=n/2}^{n}\cos\theta_k \\
&> \frac{\delta n(1-\varepsilon_o)}{4n} = \frac{\delta(1-\varepsilon_o)}{4} \\
&> \varepsilon
\end{aligned}$$

Thus, we reached a contradiction as desired. $\square$

2012 - Spring

Exercise 1. *Study the convergence of the sequence:*

$$\sqrt{2},\ \sqrt{1+\sqrt{2}},\ \sqrt{1+\sqrt{1+\sqrt{2}}},\ \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}},\dots$$

More precisely, decide if it is divergent (in this case, does it have an infinite limit?), or convergent (in which case find the limit if possible, otherwise estimate it).

Proof. (H. Lyu) Let $f(x)=\sqrt{1+x}$. Then the above sequence can be represented as the recurrence $x_{n+1}=f(x_n)$, $x_0=\sqrt{2}$. We claim that the sequence (x_n) is convergent and $\lim_{n\to\infty} x_n=\frac{1+\sqrt{5}}{2}$. Define $g(x)=x-f(x)$. Then $g(x)=0$ has the unique solution $\alpha=\frac{1+\sqrt{5}}{2}$, and g is increasing on $[0,\infty)$ since $g'(x)=1-\frac{1}{2\sqrt{1+x}}>0$ for $x\geq 0$. Thus $g<0$ on $[0,\alpha)$ and $g\geq 0$ on $[\alpha,\infty)$. Now for $t\in[0,\alpha)$, $g(t)<0$ means $t\leq f(t)$, so if $x_n\in[0,\alpha)$ then $x_n<f(x_n)=x_{n+1}$. On the other hand, as f is increasing and $x_n<\alpha$, we get $x_{n+1}<\alpha$. Combining these observations, we conclude that if $x_n\in[0,\alpha)$, then $x_n<x_{n+1}<\alpha$. Since $x_0=\sqrt{2}$ is in this range, (x_n) is an increasing sequence which is bounded above by α. To find the limit, let $\beta\geq 0$ be this limit and solve the equation $\beta=\sqrt{1+\beta}$, which would yield $\beta=\alpha$. This proves the claim. $\square$

Exercise 2. *Let $f: I\to\mathbb{R}$ be continuous and satisfy the inequality*

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}(f(x)+f(y))$$

for all $x,y\in I$, where I is an interval in $\mathbb{R}$. Prove that f is convex. In other words, prove that

$$f((1-t)x+ty)\leq(1-t)f(x)+tf(y) \tag{9}$$

for all $x,y\in I$ and for all $t\in[0,1]$.

Proof. (E. Nash, though I'm not proud about it) Let $x,y\in I$ be arbitrary with $x<y$. Define a function $g:[0,1]\to\mathbb{R}$ by $g(t)=f((1-t)x+ty)-(1-t)f(x)-tf(y)$ and note that g is continuous as f is continuous. Note first that $g(0)=g(1)=0$ and that $g\left(\frac{1}{2}\right)\leq 0$ by assumption. Proving the claim is now equivalent to showing that $g(t)\leq 0$ for all $t\in[0,1]$.

Suppose to the contrary that $g(t)>0$ for some $t\in[0,1]$. Then because g is continuous and $g(0)=g(1)=0\geq g\left(\frac{1}{2}\right)$, there exists some interval $[a,b]\subseteq[0,1]$ such that $g(t)>0$ for all $t\in(a,b)$ and $g(a)=g(b)=0$. By assumption $g\left(\frac{a+b}{2}\right)>0$, so we have the following, after simplification:

$$\begin{aligned}
0 <& g\left(\frac{a+b}{2}\right) - \frac{1}{2}g(a) - \frac{1}{2}g(b) \\
=& f\left(\left(1-\frac{a+b}{2}\right)x + \left(\frac{a+b}{2}\right)y\right) - \left(1-\frac{a+b}{2}\right)f(x) - \left(\frac{a+b}{2}\right)f(y) \\
&- \frac{1}{2}f((1-a)x+ay) + \frac{1}{2}(1-a)f(x) + \frac{a}{2}f(y) - \frac{1}{2}f((1-b)x+by) + \frac{1}{2}(1-b)f(x) + \frac{b}{2}f(y) \\
=& f\left(\left(1-\frac{a+b}{2}\right)x + \left(\frac{a+b}{2}\right)y\right) - \frac{1}{2}f((1-a)x+ay) - \frac{1}{2}f((1-b)x+by)
\end{aligned}$$

But now setting $(1-a)x+ay = c$ and $(1-b)x+by = d$, we have $f\left(\frac{c+d}{2}\right) > \frac{1}{2}f(c) + \frac{1}{2}f(d)$, which contradicts the assumption that $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$ for all $x, y \in I$. Thus, $g(t) \leq 0$ for all t and the claim is proven. □

Exercise 3. *Let $f : (1,\infty) \to \mathbb{R}$ be differentiable and define $g, h : (1,\infty) \to \mathbb{R}$ by*

$$g(x) = \frac{f'(x)}{x} \qquad \text{and} \qquad h(x) = \frac{f(x)}{x}.$$

Suppose g is bounded. Prove that h is uniformly continuous.

Proof. (A. Newman) It suffices to show that h' is bounded on $(1,\infty)$, for if a differentiable function is has bounded derivative then it is uniformly continuous by the Mean Value Theorem. Toward that end observe that by the quotient rule,

$$h'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2}.$$

We will show that $\frac{f(x)}{x^2}$ is bounded. We first check that it is bounded on the interval $(1,2]$ by showing $f(x)$ is bounded on $(1,2]$. By our assumption on $(1,2]$, we have that $f'(x)$ is bounded by a constant. If we suppose for contradiction that $f(x)$ is unbounded on $(1,2]$ then we may construct a decreasing sequence $\{x_n\}$, with $2 > x_1$ and $x_n \to 1$ and $f(x_n) \to \infty$ (increasing sequence). Now by the mean value theorem for any x_n we have that there is an $l_n \in (x_n, 2)$ so that $\frac{f(x_n)-f(2)}{x_n-2} = f'(l_n)$. Now as n goes to infinty, we have $|f'(l_n)|$ goes to infinity as well since $x_n - 2$ goes to -1, but this contradicts $f'(x)$ being bounded on $(1,2]$. We next verfiy that $\frac{f(x)}{x^2}$ is bounded on $[2,\infty)$. By the Mean Value Theorem for any $x \in (2,\infty)$ one has

$$\left|\frac{f(x)-f(2)}{x-2}\right| = |f'(y_x)|$$

for some $y_x \in (2,x)$. By our assumption $|f'(y_x)| \leq cy_x$ for some constant c and since $y_x < x$ we have

$$\left|\frac{f(x)-f(2)}{x-2}\right| \leq cx$$

Thus,

$$\frac{|f(x)-f(2)|}{x^2} \leq c$$

And it follows that $\left|\frac{f(x)}{x^2}\right| \le c + \frac{|f(2)|}{x^2} \le c + \frac{|f(2)|}{4}$. It now follows that $h'(x)$ is bounded by a constant and therefore h is uniformly continuous. $\square$

Exercise 4. *Let $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$ be subintervals of $[a, b]$. Assume that each point x in $[a, b]$ lies in at least q of these subsets. Prove that there exists $k \in \{1, \ldots, n\}$ such that $(b_k - a_k) \ge (b - a)\frac{q}{n}$.*

Proof. (A. Newman) The sum $\sum_{i=1}^{n}(b_i - a_i)$ must be at least $q(b-a)$ since the intervals cover the whole interval $[a, b]$ at least q many times. Thus $\sum_{i=1}^{n}(b_i - a_i) \ge q(b-a)$ from which it follows that there is a k so that $(b_k - a_k) \ge (b-a)\frac{q}{n}$. $\square$

Exercise 5. *Let $f(x) = \sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$ for all $x > 0$ for which the series converges. Prove that f is defined and is differentiable on $(0, \infty)$.*

Proof. (E. Nash) We first show that f is defined on $(0, \infty)$, i.e. that $\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$ converges for $x > 0$. Note that $\sin y \le y$ for $y > 0$. In particular, $\sin \frac{1}{3^n x} \le \frac{1}{3^n x}$. Now fix $x > 0$ and choose N sufficiently large so that $\frac{1}{3^N x} < \pi$. Thus, $\sin \frac{1}{3^k x} > 0$ for all $k \ge N$. Now set $\sum_{n=0}^{N} 2^n \sin \frac{1}{3^n x} = M$ and observe the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x} &= \sum_{n=0}^{N} 2^n \sin \frac{1}{3^n x} + \sum_{n=N+1}^{\infty} 2^n \sin \frac{1}{3^n x} \\
&\le M + \frac{1}{x} \sum_{n=N+1}^{\infty} \left(\frac{2}{3}\right)^n \\
&< M + \frac{1}{x}\left(\frac{1}{1 - 2/3}\right) = M + \frac{3}{x}
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$ is bounded above. As $\sin \frac{1}{3^k x} > 0$ for all $k \ge N$, we know that the sequence of partial sums is strictly increasing beyond a point. Thus, the monotone convergence theorem implies that $\lim_{k\to\infty} \sum_{n=0}^{k} 2^n \sin \frac{1}{3^n x} = \sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}$ exists and is finite. Then because x was arbitrary, the function $f(x)$ is defined for all $x > 0$.

Now we turn to differentiability. Set $f_k(x) = \sum_{n=0}^{k} 2^n \sin \frac{1}{3^n x}$ and note that f_k is differentiable for all k and that $f_k'(x) = \sum_{n=0}^{k} -\frac{2^n}{3^n x^2} \cos \frac{1}{3^n x}$. Now let $\varepsilon > 0$ be arbitrary. We will show that $f(x)$ is differentiable at $x = \varepsilon$. Consider the interval $\left[\frac{\varepsilon}{2}, 2\varepsilon\right]$. As shown above, the sequence $(f_n(\varepsilon))$ converges, so to show $f(x)$ is differentiable at $x = \varepsilon$, it will be sufficient to show that (f_k') converges uniformly on $\left[\frac{\varepsilon}{2}, 2\varepsilon\right]$ as this will imply that (f_k) converges uniformly to a differentiable function on $\left[\frac{\varepsilon}{2}, 2\varepsilon\right]$. We observe that $\left|-\frac{2^n}{3^n x^2} \cos \frac{1}{3^n x}\right| \le \frac{4}{\varepsilon^2} \cdot \frac{2^n}{3^n}$. We know that $\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2} \cdot \frac{2^n}{3^n}$ is a convergent sequence, so the Weierstrass M-test implies that (f_k') converges uniformly on $\left[\frac{\varepsilon}{2}, 2\varepsilon\right]$. Thus, (f_k) converges to a function differentiable at $x = \varepsilon$ for arbitrary ε, completing the proof. $\square$

Exercise 6. *Prove that* $\sup_{x>0} x \int_0^{\infty} \frac{e^{-px}}{p+1} dp = 1.$

Proof. (H. Lyu) Let $f(x) = x\int_0^\infty \frac{e^{-px}}{p+1}\,dp$. By the change of variable $px \mapsto u$, we get

$$\int_s^t \frac{e^{-px}}{p+1}\,dp = \int_{sx}^{tx} \frac{e^{-u}}{u+x}\,du.$$

Note that for $0 \le s \le t$, we have

$$\int_{sx}^{tx} \frac{e^{-u}}{u+x}\,du \le \int_{sx}^{tx} \frac{e^{-u}}{sx+x}\,du = \frac{e^{-sx}-e^{-tx}}{sx+x}$$

and similarly

$$\int_{sx}^{tx} \frac{e^{-u}}{u+x}\,du \ge \int_{sx}^{tx} \frac{e^{-u}}{tx+x}\,du = \frac{e^{-sx}-e^{-tx}}{tx+x}.$$

Thus letting $s = 0$ and multiplying by $x > 0$, we have the following estimation

$$\frac{1-e^{-tx}}{1+t} \le x\int_0^t \frac{e^{-px}}{p+1}\,dp \le f(x) \le 1 - e^{-xt}. \tag{1}$$

Note that the last inequality of (1) gives $\sup_{x>0} f(x) \le 1$. On the other hand, let $t > 0$ be arbitrary. Then letting $x \to \infty$, we have

$$\frac{1}{1+t} = \limsup_{x\to\infty} \frac{1-e^{-tx}}{1+t} \le \limsup_{x\to\infty} f(x) \le \sup_{x>0} f(x).$$

Therefore we obtain

$$\frac{1}{1+t} \le \sup_{x>0} f(x) \le 1.$$

Since $t > 0$ was arbitrary, we get the desired result. □

2011 - Autumn

Exercise 1. *Let f, g, and h be real-valued functions which are continuous on $[a,b]$ and differentiable on (a,b), where $a,b \in \mathbb{R}$ with $a < b$. Define F on $[a,b]$ by*

$$F(x) = \det \begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}.$$

Prove that there exists $c \in (a,b)$ such that $F'(c) = 0$.

Proof. (E. Nash) First, note that $f(a)$, $f(b)$, $g(a)$, $g(b)$, $h(a)$, and $h(b)$ are all constants. Thus, we have that $F(x) = c_1 f(x) + c_2 g(x) + c_3 h(x)$ where c_1, c_2, and c_3 are the constants given by taking a cofactor expansion along the first row of the given matrix. Then F is a linear combination of f, g, and h, so F is continuous on $[a,b]$ and differentiable on (a,b) as each of f, g, and h satisfy these properties.

Now observe that plugging in a for x in the given matrix creates two identical rows. If two rows of a matrix are linearly dependent, then the determinant of the matrix is zero, so $F(a) = 0$. Similarly, $F(b) = 0$. Now we may apply Rolle's Theorem, so there exists some $c \in (a,b)$ with $F'(c) = 0$, as desired. $\square$

Exercise 2. *Prove that $e^\pi > \pi^e$.*

Proof. (E. Nash) Consider the function $f(x) = \frac{x}{\ln(x)}$ defined on $(0,\infty)$. By the quotient rule, we have that $f'(x) = \frac{\ln x - 1}{(\ln x)^2}$. Setting this derivative equal to zero delivers $x = e$. Further, we observe that $f'(x) < 0$ for all $x \in (0,e)$ and $f'(x) > 0$ for all $x \in (e,\infty)$. This implies that f attains a minimum value at $f(e) = e$ and that this minimum value is only met when $x = e$. In particular, this implies that $f(\pi) > e$. Thus, we have the following, which confirms the claim:

$$\frac{\pi}{\ln \pi} > e \Rightarrow \pi > e \ln \pi \Rightarrow \pi > \ln \pi^e \Rightarrow e^\pi > \pi^e$$

$\square$

Exercise 3. *Let $f : [1,\infty) \to \mathbb{R}$ be bounded and continuous. Prove that*

$$\lim_{n\to\infty} \int_1^\infty f(t) n t^{-n-1} dt = f(1). \tag{10}$$

Proof. (H. Lyu) Let $g(x) = f(x) - f(1)$, so g is continuous on $[1,\infty)$ and $g(1) = 0$. For each $b > 1$, $|g|$ has an absolute minimum $M_b \geq 0$ on the compact interval $[1,b]$ be the extreme value theorem. Notice that $M_b \to 0 = g(1)$ as $b \searrow 1$ by continuity. Since f is bounded, so is g. Let $M \geq 0$ be a bound for g.

Observe that $\int_1^\infty \frac{n}{t^{n+1}}\, dt = [-t^{-n}]_1^\infty = 1$. Hence

$$\int_1^\infty \frac{n}{t^{n+1}} f(t)\, dt - f(1) = \int_1^\infty \frac{n}{t^{n+1}} (f(t) - f(1))\, dt = \int_1^\infty \frac{n}{t^{n+1}} g(t)\, dt.$$

Thus it suffices to show that $\left|\int_1^\infty \frac{n}{t^{n+1}} g(t)\,dt\right| \to 0$ as $n \to \infty$. Indeed, for any $b > 1$ we have

$$\begin{aligned}
\left|\int_1^\infty \frac{n}{t^{n+1}} g(t)\,dt.\right| &\leq \left|\int_1^b \frac{n}{t^{n+1}} g(t)\,dt\right| + \left|\int_b^\infty \frac{n}{t^{n+1}} g(t)\,dt\right| \\
&\leq M_b \int_1^\infty \frac{n}{t^{n+1}}\,dt + M \int_b^\infty \frac{n}{t^{n+1}}\,dt \\
&= M_b + \frac{M}{b^n},
\end{aligned}$$

and since $b > 1$, this upper bound goes to zero as $n \to \infty$. This shows the assertion. □

Exercise 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be monotone and satisfy $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all x_1 and x_2 in $\mathbb{R}$. Prove that $f(x) = ax$ for all real numbers x, where $a = f(1)$.*

Proof. (E. Nash) First note that $f(1) = f(1+0) = f(1)+f(0)$, which implies that $f(0) = 0$, so the claim holds for $x = 0$. We now prove that $f\left(\frac{1}{s}\right) = \frac{a}{s}$ for all $s \in \mathbb{N}$. To see this, observe that $a = f(1) = \sum_{i=1}^{s} f\left(\frac{1}{s}\right)$ by iterating the identity $f(x_1 + x_2) = f(x_1) + f(x_2)$. Thus, $a = sf\left(\frac{1}{s}\right)$, so $f\left(\frac{1}{s}\right) = \frac{a}{s}$ and the claim holds for all $\frac{1}{s}$. Now for arbitrary $r \in \mathbb{N}$, we have similarly that $f\left(\frac{r}{s}\right) = \sum_{i=1}^{r} f\left(\frac{1}{s}\right)$, so $f\left(\frac{r}{s}\right) = rf\left(\frac{1}{s}\right) = \frac{ar}{s}$ whenever $\frac{r}{s} > 0$. We also observe that $f(-1) = f(-1) + f(-1) + f(1)$, which implies that $f(-1) = -a$ as we have already shown $f(1) = a$. A similar argument now allows us to conclude that $f\left(\frac{r}{s}\right) = \frac{ar}{s}$ when $\frac{r}{s} < 0$. Thus, we have that $f(q) = aq$ for all $q \in \mathbb{Q}$.

Now let $x \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$ be given. Because the rationals are dense in the reals, there exist $q_1, q_2 \in \mathbb{Q}$ such that $q_1 < x < q_2$ and $|q_2 - q_1| < \frac{\varepsilon}{2|a|}$. Further, because f is monotone, we know that $|f(x) - f(q_1)| \leq |f(q_2) - f(q_1)|$. Now we observe the following algebra:

$$\begin{aligned}
|f(x) - ax| &= |f(x) - f(q_1) + f(q_1) - ax| \\
&\leq |f(q_2) - f(q_1)| + |f(q_1) - ax| \\
&= |aq_2 - aq_1| + |aq_1 - ax| \\
&= |a|\left(|q_2 - q_1| + |q_1 - x|\right) \\
&< |a|\left(\frac{\varepsilon}{2|a|} + |q_1 - q_2|\right) \\
&< |a|\left(\frac{\varepsilon}{2|a|} + \frac{\varepsilon}{2|a|}\right) = \varepsilon
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary and $|f(x) - ax| < \varepsilon$, we have that $f(x) = ax$ for all $x \in \mathbb{R}$ and the claim is confirmed. □

Exercise 5. *Let $\{r_k\}_{k=1}^\infty$ be the set of rational numbers of the interval $[0, 1]$. Define $f : [0, 1] \to \mathbb{R}$ by*

$$f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|}{3^k}.$$

Then f is continuous on $[0,1]$. (You may take this for granted.) Prove that f is differentiable at every irrational point in $(0,1)$.

Proof. (E. Nash) Let $c \in [0,1]$ be irrational. We will show that $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ exists and is equal to $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$. Observe the following:

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0^+} \frac{1}{h}\left(\sum_{k=1}^{\infty} \frac{|c+h-r_k|-|c-r_k|}{3^k}\right) \\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\sum_{r_k\in[0,c)} \frac{h}{3^k} + \sum_{r_k\in(c,c+h]} \frac{2c-2r_k+h}{3^k} + \sum_{r_k\in(c+h,1]} \frac{-h}{3^k}\right).
\end{aligned}$$

Note now that

$$\lim_{h\to 0^+} \frac{1}{h} \sum_{r_k\in[0,c)} \frac{h}{3^k} = \sum_{r_k\in[0,c)} \frac{1}{3^k} \le \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{2}$$

and

$$\lim_{h\to 0^+} \frac{1}{h} \sum_{r_k\in(c+h,1]} \frac{-h}{3^k} = \sum_{r_k\in(c,1]} \frac{-1}{3^k} \ge \sum_{k=1}^{\infty} \frac{-1}{3^k} = -\frac{1}{2}.$$

Each of these summations is monotonic and bounded, so both must converge to some value. We claim that $\lim_{h\to 0^+} \frac{1}{h} \sum_{r_k\in(c,c+h]} \frac{2c-2r_k+h}{3^k} = 0$, which will show that $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \sum_{r_k\in[0,c)} \frac{1}{3^k} - \sum_{r_k\in(c,1]} \frac{1}{3^k}$. Observe the following:

$$\begin{aligned}
\lim_{h\to 0^+} \left| \frac{1}{h} \sum_{r_k\in(c,c+h]} \frac{2c-2r_k+h}{3^k} \right| &\le \lim_{h\to 0^+} \frac{1}{h} \sum_{r_k\in(c,c+h]} \frac{|2c-2r_k|+h}{3^k} \\
&\le \lim_{h\to 0^+} \frac{1}{h} \sum_{r_k\in(c,c+h]} \frac{3h}{3^k} \\
&\le \lim_{h\to 0^+} \sum_{r_k\in(c,c+h]} \frac{3}{3^k} = 0.
\end{aligned}$$

It may be similarly calculated that $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} = \sum_{r_k\in[0,c)} \frac{1}{3^k} - \sum_{r_k\in(c,1]} \frac{1}{3^k}$, so f is differentiable at c and $f'(c) = \sum_{r_k\in[0,c)} \frac{1}{3^k} - \sum_{r_k\in(c,1]} \frac{1}{3^k}$. □

Exercise 6. *Consider a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ defined by a power series with radius of convergence $R \in (0,\infty)$. Suppose the series converges at $x = R$. Prove that f is left-continuous at $x = R$. (Of course this is commonly known as Abel's theorem on endpoint behaviour of power series. You are being asked to prove that theorem, not just quote it. Warning: The convergence at $x = R$ may be only conditional. Indeed, the result is almost trivial when the convergence there is absolute.)*

Proof. (O. Khalil) First, we rescale the problem so that $R = 1$ by observing that $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n (\frac{x}{R})^n = \sum_{n=0}^{\infty} b_n t^n = f(t)$ which has radius of convergence $= 1$. We need to show that $\lim_{t \to 1^-} f(t) = f(1)$. Fix $\varepsilon > 0$. Then, $\exists N \in \mathbb{N}$ such that $\forall m \geq N$, $\left| \sum_{k=N}^{k=m} b_k \right| < \frac{\varepsilon}{3}$. Hence, by taking the limit, we have that $\left| \sum_{k=N}^{\infty} b_k \right| \leq \frac{\varepsilon}{3}$. Now, for a given $t \in (0, 1)$, we have have that

$$\left| \sum_{k=1}^{\infty} b_k (1 - t^k) \right| \leqslant (1 - t^{N-1}) \sum_{k=1}^{N-1} |b_k| + \left| \sum_{k=N}^{\infty} b_k \right| + \left| \sum_{k=N}^{\infty} b_k t^k \right|$$

Now, using Abel's summation by parts formula, letting $B_k = \sum_{j=N}^{j=k} b_j$ and $B_{N-1} = 0$, we get

$$|f(1) - f(t)| \leqslant (1 - t^{N-1}) \sum_{k=1}^{N-1} |b_k| + \frac{\varepsilon}{3} + \left| \sum_{k=N}^{\infty} (B_k - B_{k-1}) t^k \right|, \qquad (*)$$

where we have that $\forall k > N, b_k = B_k - B_{k-1}$. Now, observe that for each $t \in (0, 1)$

$$\left| \sum_{k=N}^{\infty} B_k t^k \right| \leq \sum_{k=N} |B_k| t^k \leq \varepsilon/3 \sum_{k=N} t^k < \infty$$

where we used the fact that the geometric series $\sum_1^\infty t^k$ is convergent for $\in (0, 1)$. Hence, we get the following

$$\begin{aligned} \left| \sum_{k=N}^{\infty} (B_k - B_{k-1}) t^k \right| &= \left| \sum_{k=N}^{\infty} B_k t^k - \sum_{k=N}^{\infty} B_{k-1} t^k \right| \\ &= \left| \sum_{k=N}^{\infty} B_k t^k - \sum_{k=N}^{\infty} B_k t^{k+1} \right| \\ &\leq \sum_{k=N}^{\infty} |B_k| (t^k - t^{k+1}) < \frac{\varepsilon}{3} \sum_{k=N}^{\infty} (t^k - t^{k+1}) \end{aligned}$$

The second equality follows from the fact that $B_{N-1} = 0$ and that both series converge by the above argument.

Now, notice that for any $m > N$, we have that $\sum_{k=N}^{m} (t^k - t^{k+1}) = t^N - t^{m+1} \to t^N$ as $m \to \infty$. Combining this fact with the above estimate and the fact that $t^N < 1$ for each $t \in (0, 1)$, we get that

$$\left| \sum_{k=N}^{\infty} (B_k - B_{k-1}) t^k \right| < \frac{\varepsilon}{3} t^N < \varepsilon/3$$

Now, since $lim_{t\to 1^-} 1 - t^{N-1} = 0$, $\exists\delta \in (0,1)$ such that $\forall t \in (1-\delta, 1)$, we have that

$$|1 - t^{N-1}| < \frac{\varepsilon}{3(\sum_{k=1}^{k=N-1} |b_k| + 1)}$$

Hence, $\forall t \in (1-\delta,1), (1-t^{N-1})\sum_{k=1}^{N-1} |b_k| \leq \dfrac{\varepsilon \sum_{k=1}^{k=N-1} |b_k|}{3(\sum_{k=1}^{k=N-1} |b_k| + 1)} < \dfrac{\varepsilon}{3}$.

Hence, plugging these estimates in (*), we get that $\forall t \in (1-\delta, 1)$,

$$|f(1) - f(t)| < \varepsilon$$

So, f is left continuous at 1. $\square$

2011 - Spring

Exercise 1. *Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the consecutive strictly positive solutions of the equation $x = \tan x$. Does the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ converge? Justify your answer. (As part of justifying your answer, you should justify any estimates on λ_n that you use.)*

Proof. (A. Newman) The series does converge. To prove this we will look at the solutions of $\tan x - x$ on intervals of the form $I_n = (\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2})$. We first check that there is always one solution in such an interval. Since $\tan x$ is surjective as a function on I_n for each n, we have that there is $y_n \in I_n$ so that $\tan y_n > \frac{(2n+3)\pi}{2}$ and so $\tan y_n - y_n > 0$. And also $\tan((n+1)\pi) - (n+1)\pi = -(n+1)\pi < 0$ for $n > 1$. So for $n \geq 2$ there is always at least one λ_k in I_n. Suppose that for $n \geq 2$ there are two solutions to $\tan x - x$ in I_n, call them a and b. Since $\tan x \leq 0$ on $(\frac{(2n+1)\pi}{2}, (n+1)\pi]$,we know that $a, b > (n+1)\pi$. By Rolle's theorem, there is a $d \in (a, b)$ so that $\sec^2 d - 1 = 0$. But the only places where $\sec^2 x = 1$ is on integer multiples of π, but there are no integer multiples of π between a and b so this is a contradiction. It follows from this that after reindexing and deleting a finite number of terms we have $\lambda_n > n\pi$. So it follows that $\sum_{n=1}^{\infty} \lambda_n^{-2}$ converges by comparison to $\sum_{n=1}^{\infty} n^{-2}$. □

Exercise 2. *Let $f : (0, \infty) \to \mathbb{R}$ be twice differentiable and suppose that $A, C \in [0, \infty)$ such that for each $x > 0$, we have $|f(x)| \leq A$ and $|f''(x)| \leq C$. Prove that for each $x > 0$ and each $h > 0$ we have*

$$|f'(x)| \leq \frac{A}{h} + Ch. \tag{11}$$

Proof. (A. Newman) Using Taylor's theorem, we have for any $x > 0$ and any $h > 0$ an ε in $(x, x+2h)$ so that $f(x+2h) = f(x) + f'(x)2h + \frac{f''(\varepsilon)}{2}4h^2$. It follows that $f'(x) = \frac{f(x+2h)-f(x)}{2h} - f''(\varepsilon)h$. And so by the triangle inequality $|f'(x)| \leq \frac{|A|}{h} + |C|h$ □

Exercise 3. *Find the least constant c such that*

$$(x_1 + x_2 + \cdots + x_{2011})^2 \leq c(x_1^2 + x_2^2 + \cdots + x_{2011}^2) \tag{12}$$

for all real values of $x_1, x_2, \ldots, x_{2011}$. (For emphasis, let's repeat that you are asked to find the least such c, not just some c.)

Proof. (A. Newman) The least constant c that works is $c = 2011$. By the Cauchy-Schwarz inequality we have $(1x_1 + 1x_2 + \cdots + 1x_{2011})^2 \leq (1^2 + \cdots + 1^2)(x_1^2 + x_2^2 + \cdots + x_{2011}^2) = 2011(x_1^2 + x_2^2 + \cdots + x_{2011}^2)$. This can be seen to be the best possible bound by setting all $x_i = 2011$. (I think that on a problem like this they may want a proof of Cauchy-Schwarz, a good one to use is looking at the discriminant of $(x_1 z + y_1)^2 + \cdots + (x_n z + y_n)^2$. This is a nonnegative polynomial so its discriminant is nonpositive and we get Cauchy-Schwarz from that.) □

Exercise 4. *For each $x > 0$, the integral*

$$I(x) = \int_0^{\infty} \frac{\sin xt}{1+t} dt$$

exists as a conditionally convergent improper Riemann integral. (You may take this for granted.) Prove that $I(x)$ has a limit in $\mathbb{R}$ as $x \to 0^+$.

Proof #1. (A. Newman) We will show that $I(x)$ converges to 0 as $x \to 0^+$. Let (x_n) be a sequence of positive numbers that converge to zero. We will show that $I(x_n)$ converges to zero as well. To do this consider the sequence of functions $f_n(t) = \frac{\sin x_n t}{1+t}$. We will show that this converges uniformly to the zero function. To do this it will suffice to prove that $\sup_{t\in[0,\infty)} |f_n(t)|$ goes to zero as n goes to infinity. Since $f_n(t)$ is always continuous and defined on the closed interval $[0, \infty)$ and does not increase or decrease toward an asymptote as t goes to infinity, the supremum of $|f_n(t)|$ is the maximum or minimum of $f_n(t)$ (since the function is sometimes negative). Therefore the supremum occurs where $f_n'(t)$ is undefined or zero. The derivative is given by

$$f_n'(t) = \frac{x_n(1+t)\cos xt - (\sin xt)}{(1+t)^2}.$$

So the derivative exists for all nonnegative t and it's zero at t such that $x_n(1+t)\cos(x_n t) = \sin x_n t$. At such t the function value may be given by $f_n(t) = x_n \cos(x_n t)$ so $|f_n(t)| = |x_n \cos(x_n t)| \leq x_n$ and so the supremum indeed goes to zero as x_n goes to zero. Thus $f_n(t)$ converges uniformly to zero and so we may conclude that

$$\lim_{x\to 0^+} \int_0^\infty \frac{\sin xt}{1+t} dt = \int_0^\infty \lim_{x\to 0^+} \frac{\sin xt}{1+t} dt = 0$$

[Proof #2] (O. Khalil) (Same idea but different way of proving uniform convergence) Notice that the function $\sin(y)$ lies below its tangent line at 0 ($L(y) = y$), for $y > 0$. So, we get the inequality $|\sin(y)| \leq |y|$ since sin is odd. So, we can bound each $f_n(t)$ as follows:

$$|f_n(t)| = \left|\frac{\sin(x_n t)}{t+1}\right| \leq \frac{|x_n t|}{t+1} < x_n$$

Hence, we get that $|f_n(t)| \leq |x_n|$. Thus, $\sup_{t\in[0,\infty]} |f_n(t)| \to 0$ as $n \to \infty$ as desired. □

Exercise 5. *Let (a_k) be a sequence of non-negative real numbers. Suppose that for each sequence (x_k) of non-negative real numbers with $\lim_{k\to\infty} x_k = 0$, the series $\sum_{k=1}^\infty a_k x_k$ converges. Prove that the series $\sum_{k=1}^\infty a_k$ converges.*

Proof. (K. Nowland) Suppose toward a contradiction that $\sum a_k$ diverges. Note that $\sum_{k=1}^\infty a_k$ divergent implies $\sum_{k=N}^\infty a_k$ diverges for all natural numbers N. In particular, there exists $N_1 \in \mathbb{N}$ such that $\sum_{k=1}^{N_1} a_k \geq 1$. Further, there exists N_2 such that $\sum_{k=N_1+1}^{N_2} a_k \geq 2$. We continue this to find a strictly increasing sequence $1 = N_0 < N_1 < N_2 < \cdots$ such that $\sum_{k=N_{n-1}+1}^{N_n} a_k \geq n$ for all $n \in \mathbb{N}$. Define (x_k) by $x_{N_{n-1}+1} = x_{N_{n-1}+2} = \cdots = x_{N_n} = \frac{1}{n}$.

Now we show that $x_k \to 0$. Let $\varepsilon > 0$. There exists $K \in \mathbb{N}$ such that $0 < \frac{1}{K} < \varepsilon$. Then for all $k > N_K$, we have $0 < x_k < \frac{1}{K} < \varepsilon$. Therefore $\lim_{k\to\infty} x_k = 0$.

By hypothesis, $\sum_{k=1}^{\infty} a_k x_k < \infty$. But also,

$$\begin{aligned}\sum_{k=1}^{\infty} a_k x_k &= \sum_{k=1}^{N_1} a_k x_k + \sum_{k=N_1+1}^{N_2} a_k x_k + \sum_{k=N_2+1}^{N_3} a_k x_k + \cdots \\ &= \sum_{k=1}^{N_1} a_k + \frac{1}{2} \sum_{k=N_1+1}^{N_2} a_k + \frac{1}{3} \sum_{k=N_2+1}^{N_3} a_k + \cdots \\ &\geq 1 + 1 + 1 + \cdots .\end{aligned}$$

Since the series where each term is 1 diverges, it follows that $\sum a_k x_k$ diverges. The contradiction implies the claim that $\sum a_k$ converges. $\square$

Exercise 6. *For $n = 1, 2, 3, \ldots$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \dfrac{2n^2 x}{e^{n^2 x}}$. Does the sequence (f_n) converge uniformly on $[0,1]$? Justify your answer.*

Proof. (K. Nowland) For any fixed x in $[0,1]$,

$$\frac{2n^2 x}{e^{n^2 x}} \to 0$$

as $n \to \infty$. Thus if the sequence converges uniformly, it converges to uniformly to zero. Note that

$$f_n(1/n^2) = \frac{2}{e}.$$

It follows that

$$\sup_{x \in [0,1]} |f_n(x)| \geq \frac{2}{e},$$

for all n such that the sequence cannot converge uniformly. $\square$

2010 - Autumn

Exercise 1. *Let* $S_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. *Find* $\lim_{n\to\infty} \frac{S_n}{n^{3/2}}$. *Justify your answer.*

Proof. (K. Nowland) We rewrite $S_n/n^{3/2}$ as

$$\frac{S_n}{n^{3/2}} = \sum_{k=1}^{n} \frac{\sqrt{k}}{n^{3/2}} = \frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}.$$

This is a Riemann sum for $\int_0^1 \sqrt{x}dx$. Since $\sqrt{x}$ is continuous on $[0,1]$, the limit as n tends to infinity exists and is the value of the integral. Therefore

$$\lim_{n\to\infty}\frac{S_n}{n^{3/2}} = \int_0^1 \sqrt{x}dx = \frac{2}{3}.$$

□

Exercise 2. *Let* (a_n) *be a sequence of real numbers such that the series* $\sum_{n=1}^{\infty} a_n$ *converges. Prove that*

$$\frac{1}{n}\sum_{k=1}^{n} ka_k \to 0. \tag{13}$$

as $n \to \infty$. *(Hint: For any sequence* (b_n) *of real numbers, if* $b_n \to b \in \mathbb{R}$ *as* $n \to \infty$, *then* $\frac{1}{n}\sum_{k=1}^{n} b_k \to b$ *as* $n \to \infty$. *You may use this fact without proof.)*

Proof. (K. Nowland) Since $\sum a_k$ converges, the Cauchy convergence criterion implies that there exists $N_1 \in \mathbb{N}$ such that $n \geq m \geq N_1$ implies

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon. \tag{14}$$

Also because $\sum a_k < \infty$, $a_k \to 0$ as $k \to \infty$. Thus by the hint, $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$ as $n \to \infty$. This implies

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=m}^{n} a_k = 0$$

for all $m \in \mathbb{N}$ since the sum of the $m-1$ first terms is fixed and finite. Taking $m = 1, 2, \ldots, N_1$. Since N_1 is fixed (depending on ε), this is only finitely many terms. We can therefore find $N_2 \in \mathbb{N}$ with $N_2 > N_1$ such that for all $n \geq N_2 > N_1$,

$$\left|\frac{1}{n}\sum_{k=j}^{n} a_k\right| < \frac{\varepsilon}{N_1} \tag{15}$$

for all $j = 1, 2, \ldots, N_1$. Now consider $\frac{1}{n}\sum_{k=1}^{n} ka_k$ for $n \geq N_2$. We rewrite the sum as

$$\frac{1}{n}\sum_{k=1}^{n} ka_k = \frac{1}{n}\sum_{k=1}^{n} a_k + \frac{1}{n}\sum_{k=2}^{n} a_k + \cdots + \frac{1}{n}\sum_{k=N_1}^{n} a_k + \frac{1}{n}\sum_{k=N_1+1}^{n} a_k + \cdots + \frac{1}{n}\sum_{k=n}^{n} a_k.$$

Applying the absolute value and then the triangle inequality, we use (15) on the first N_1 terms and (14) on the remaining $n - N_1$ terms to obtain

$$\left| \frac{1}{n} \sum_{k=1}^{n} k a_k \right| \le \frac{N_1}{N_1}\varepsilon + \frac{n - N_1}{n}\varepsilon \le 2e.$$

Since ε was arbitrary, this completes the proof. $\square$

Exercise 3. *Let $f : [0,1] \to \mathbb{R}$ be continuous. Suppose f is twice-differentiable on the open interval $(0,1)$ and M is a real constant such that for each $x \in (0,1)$, we have $|f''(x)| \le M$. Let $a \in (0,1)$. Prove that*

$$|f'(a)| \le |f(1) - f(0)| + \frac{M}{2}. \tag{16}$$

Proof. (H. Lyu) By Taylor's theorem, we can expand $f(x)$ at $x = a$. In particular, we have

$$\begin{aligned} f(0) &= f(a) - af'(a) + a^2 \frac{f''(\xi)}{2} \\ f(1) &= f(a) + (1-a)f'(a) + (1-a)^2 \frac{f''(\zeta)}{2} \end{aligned}$$

for some $0 < \xi < a$ and $a < \zeta < 1$. Subtracting and rearranging, we get

$$f'(a) = f(0) - f(1) + a^2 \frac{f''(\xi)}{2} - (1-a)^2 \frac{f''(\zeta)}{2}.$$

Note that the quadratic function $g(x) = 2x^2 - 2x + 1$ has axis at $x = 1/2$ and has absolute maximum on $[0,1]$ at $x = 0$ and 1, which is 1. Since $|f''(x)| < M$ for all $x \in (0,1)$, by triangle inequality we obtain

$$|f'(a)| \le |f(0) - f(1)| + \left| a^2 + (1-a)^2 \right| \frac{M}{2} \le |f(0) - f(1)| + \frac{M}{2}$$

as desired. $\square$

Exercise 4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous, define $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(y) = \int_0^1 f(x,y)dx,$$

and suppose $\partial f/\partial y$ is continuous on $[0,1] \times \mathbb{R}$. Prove that g is differentiable on $\mathbb{R}$ and that

$$g'(y) = \int_0^1 \frac{\partial f}{\partial y}(x,y)dx \tag{17}$$

for all $y \in \mathbb{R}$.

Proof. (H. Lyu) Fix $y \in \mathbb{R}$. Let $h \in \mathbb{R}$. Then by the mean value theorem, we have

$$\frac{g(y+h)-g(y)}{h} = \int_0^1 \frac{f(x,y+h)-f(x,y)}{h}\,dx = \int_0^1 \frac{\partial f}{\partial y}(x, y^*(h))\,dx$$

for some $y^*(h) \in (y-|h|, y+|h|)$. Now for each $h \in \mathbb{R}$, define a function $\phi_h : [0,1] \to \mathbb{R}$ by $\phi_h(x) = \frac{\partial f}{\partial y}(x, y^*(h))$. As h varies, (ϕ_h) defines a family of functions. Notice that $y^*(h) \to y$ as $h \to 0$, so by the continuity of $\frac{\partial f}{\partial y}$, we have

$$\phi_h \longrightarrow \frac{\partial f}{\partial y}(-,y) \quad \text{pointwise as } h \to 0. \tag{1}$$

In fact, this convergence is uniform on $[0,1]$. To see this, restrict the continuous map $\frac{\partial f}{\partial y}$ on the compact domain $[0,1]\times[y-1,y+1]$ to get the uniform continuity there. Now fix $\epsilon > 0$. Then by the uniform continuity there exists $\delta > 0$ such that whenever the two points $(x_1,y_1),(x_2,y_2) \in [0,1]\times[y-1,y+1]$ are within δ in the usual Euclidean distance, then we have

$$\left|\frac{\partial f}{\partial y}(x_1,y_1) - \frac{\partial f}{\partial y}(x_2,y_2)\right| < \epsilon.$$

Hence for any $|h| < \min(\delta, 1/2)$, since $(x,y),(x,y^*(h)) \in [0,1]\times[y-1,y+1]$ and $|(x,y) - (x,y^*(h))| < |h| < \delta$ for all $x \in [0,1]$, we have

$$\left|\phi_h(x) - \frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{\partial f}{\partial y}(x,y^*(h)) - \frac{\partial f}{\partial y}(x,y)\right| < \epsilon.$$

Thus the convergence in (1) is uniform on $[0,1]$. This allows us to switch the limit and integral as follows :

$$g'(y) = \lim_{h\to 0}\int_0^1 \phi_h(x)\,dx = \int_0^1 \lim_{h\to 0}\phi_h(x)\,dx = \int_0^1 \frac{\partial f}{\partial y}(x,y)\,dx.$$

Since $y \in \mathbb{R}$ was arbitrary, this shows the assertion. $\square$

Exercise 5. *Let X be a subgroup of $(\mathbb{R},+)$. (This means that $0 \in X \subseteq \mathbb{R}$ and for all $x,y \in X$, we have $x+y \in X$ and $-x \in X$.) Prove that either X is dense in $\mathbb{R}$ or there exists $c \in \mathbb{R}$ such that $X = c\mathbb{Z}$ where $c\mathbb{Z} = \{ck : k \in \mathbb{Z}\}$.*

Proof. (A. Newman) We have two cases to consider. Either for every $\epsilon > 0$, there is a point of X in $(0,\epsilon)$ or there is ϵ_0 so that for $X \cap (0,\epsilon_0) = \varnothing$. In the first case we will show that X is dense. So let $(a, a+\delta)$ be an open set of $\mathbb{R}$. We can find an x_δ so that $0 < x_\delta < \frac{\delta}{2}$. Therefore there is $n \in \mathbb{Z}$ so that $nx_\delta \in (a, a+\delta)$. So X is dense in $\mathbb{R}$.

Now suppose that there is ϵ_0 so that $(0,\epsilon_0)$ contains no point of X. Let c be the infimum of elements in X that are greater than ϵ_0. We claim that $c \in X$. Let $(x_n)_{n\geq 0}$ be a sequence in X with each $x_n \geq c$ that converges to c. Such a sequence exists since c is an infimum of the elements of X that are at least c. Toward a contradiction suppose $(x_n)_{n\geq 0}$ is never eventually constant at c. Then there is x_1 so that $|x_1 - c| < \frac{\epsilon_0}{4}$, and there is $x_2 < x_1$ so that

$|x_2 - c| < \frac{\epsilon_0}{4}$. Now $0 < x_1 - x_2 = |x_1 - c + c - x_2| \le |x_1 - c| + |x_2 - c| < \frac{\epsilon_0}{4} < \epsilon_0$. But $x_1 - x_2$ is in X, and this contradicts the choice of ϵ_0. So $c \in X$ since (x_n) is eventually the constant sequence c. Thus $c\mathbb{Z}$ is in X. Lastly if there is a point of X between nc and $(n+1)c$ for some n, then there are points $x < y \in X$ so that $x - y < c$, but this contradicts the choice of c since $x - y \in X$ and $x - y \neq 0$ so $x - y \ge \epsilon_0$. And c is the greatest lower bound of a such a set. So $X = c\mathbb{Z}$ □

Exercise 6. *Let $f : [0,1] \to \mathbb{R}$ be Riemann-integrable. For each real number $p > 1$, let*

$$A_p = \int_0^1 px^{p-1} f(x)dx.$$

Let $A = \limsup_{p\to\infty} A_p$ and let $B = \limsup_{x\to 1^-} f(x)$. Prove that $A \le B$. (In case you would like a reminder, here is one way to define the limits superior that appear in this problem: $\limsup_{p\to\infty} A_p = \inf_{q\in(1,\infty)} \sup_{p\in(q,\infty)} A_p$ and $\limsup_{x\to 1^-} f(x) = \inf_{u\in(0,1)} \sup_{x\in(u,1)} f(x)$.)

Proof. (K. Nowland) Note that for any $p > 1$ and any $u \in (0,1)$,

$$\begin{aligned} A_p &= \int_0^u px^{p-1} f(x)dx + \int_u^1 px^{p-1} f(x)dx \\ &\le \int_0^u px^{p-1} f(x)dx + \sup_{x\in(u,1)} f(x). \end{aligned}$$

Now if we take the limit supremum of A_p, we see that

$$\limsup_{p\to\infty} A_p \le \limsup_{p\to\infty} \int_0^u px^{p-1} f(x)dx + \sup_{x\in(u,1)} f(x).$$

If a limit exists, then the limit supremum is equal to it. Since f is Riemann integrable on a bounded interval, it follows that f must be bounded by some number M. We see that

$$|\int_0^u px^{p-1} f(x)dx| \le M \int_0^u px^{p-1} dx = Mu^p.$$

Since $0 < u < 1$, as $p \to \infty$, this tends to zero. Therefore

$$\limsup_{p\to\infty} A_p \le \sup_{x\in(u,1)} f(x).$$

Taking the infimum over all u in $(0,1)$, gives the desired result. □

2010 - Spring

Exercise 1. *Prove that if $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers, with $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$, then*

$$\lim_{n\to\infty} \frac{a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0}{n+1} = AB. \tag{18}$$

Proof. (K. Nowland) We begin by writing

$$\begin{aligned}
\left| \frac{a_0 b_n + \cdots + a_n b_0}{n+1} - AB \right| &= \left| \frac{a_0 b_n + \cdots + a_n b_0 - (n+1)AB}{n+1} \right| \\
&= \left| \frac{(a_0 b_n - AB) + \cdots + (a_n b_0 - AB)}{n+1} \right| \\
&\le \frac{1}{n+1} \sum_{k=0}^{n} |a_k b_{n-k} - AB|,
\end{aligned}$$

where the last line follows from the triangle inequality. Let $\varepsilon > 0$. By the convergence of $\{a_k\}$, there exists $N_1 \in \mathbb{N}$ such that $k \ge N_1$ implies $|a_k - A| < \varepsilon$. Since $\{b_n\}$ converges to B, there exists $N_2 \in \mathbb{N}$ such that $n - k \ge N_2$ implies $|b_{n-k} - B| < \varepsilon$. We rewrite the summation above as

$$\begin{aligned}
\left| \frac{a_0 b_n + \cdots + a_n b_0}{n+1} - AB \right| \le{}& \frac{1}{n+1} \sum_{k=0}^{N_1} |a_k b_{n-k} - AB| \;+\; \frac{1}{n+1} \sum_{k=N_1+1}^{n-N_2-1} |a_k b_{n-k} - AB| \\
&+\; \frac{1}{n+1} \sum_{k=n-N_2}^{n} |a_k b_{n-k} - AB|.
\end{aligned}$$

Since $\{a_k\}$ converges, it is bounded in absolute value in $\mathbb{R}$ by some $M_a > 0$ (and $|A|$ obeys this bound). Let M_b be a similar bound for the sequence $\{b_k\}$. Then each term in the first sum obeys the bound

$$\begin{aligned}
|a_k b_{n-k} - AB| &= |a_k b_{n-k} - a_k B + a_k B - AB| \\
&\le |a_k||b_{n-k} - B| + |B||a_k - A| \\
&\le M_a \varepsilon + 2 M_b M_a.
\end{aligned}$$

Similarly, each term in the third sum obeys

$$\begin{aligned}
|a_k b_{n-k} - AB| &= |a_k b_{n-k} - A b_{n-k} + A b_{n-k} - AB| \\
&\le |b_{n-k}||a_k - A| + |A||b_{n-k} - B| \\
&\le M_b \varepsilon + 2 M_a M_b.
\end{aligned}$$

For the middle terms we do better, since both $k > N_1$ and $n - k > N_2$, and we have

$$\begin{aligned}
|a_k b_{n-k} - AB| &= |a_k b_{n-k} - A b_{n-k} + A b_{n-k} - AB| \\
&\le |b_{n-k}||a_k - A| + |a_k||b_{n-k} - B| \\
&\le M_b \varepsilon + M_a \varepsilon.
\end{aligned}$$

Plugging in these estimates gives

$$\begin{aligned}
\left|\frac{a_0b_n+\cdots+a_nb_0}{n+1}-AB\right| &\leq \frac{1}{n+1}\sum_{k=0}^{N_1}(M_b\varepsilon+2M_bM_a)+\frac{1}{n+1}\sum_{k=N_1+1}^{n-N_2-1}(M_b+M_a)\varepsilon \\
&\quad + \frac{1}{n+1}\sum_{k=n-N_2}^{n}(M_a\varepsilon+2M_aM_b) \\
&= \frac{N_1+1}{n+1}(M_b\varepsilon+2M_bM_a)+\frac{n-N_2-1-N_1}{n+1}(M_a+M_b)\varepsilon \\
&\quad + \frac{N_2+1}{n+1}(M_a\varepsilon+2M_aM_b).
\end{aligned}$$

Since N_1+1 and N_2+1 are constant, we can pick n large enough that $\frac{N_1+1}{n+1}$ and $\frac{N_2+1}{n+1}$ are both less than ε. With n this large and noting that the fractional coefficient of $(M_a+M_b)\varepsilon$ is always less than 1 gives

$$\left|\frac{a_0b_n+\cdots+a_nb_0}{n+1}-AB\right| < [(M_b+M_a)(1+\varepsilon)+4M_aM_b]\varepsilon.$$

Since M_a, M_b do not depend on ε and ε was arbitrary, the desired convergence holds. $\square$

Proof. (S. Meehan) Write $a_n=\alpha_n+A$, where $\alpha_n\to 0$ as $n\to\infty$, and $b_n=\beta_n+B$, where $\beta_n\to 0$ as $n\to\infty$. Observe that:

$$\begin{aligned}
\frac{a_0b_n+a_1b_{n-1}+\cdots+a_{n-1}b_1+a_nb_0}{n+1} &= \frac{(\alpha_0+A)(\beta_n+B)+\cdots+(\alpha_n+A)(\beta_0+B)}{n+1} \\
&= \frac{(n+1)AB+B\sum_{k=0}^n\alpha_k+A\sum_{k=0}^n\beta_k+\sum_{k=0}^n\alpha_k\beta_{n-k}}{n+1} \\
&= AB+B\frac{\sum_{k=0}^n\alpha_k}{n+1}+A\frac{\sum_{k=0}^n\beta_k}{n+1}+\frac{\sum_{k=0}^n\alpha_k\beta_{n-k}}{n+1}
\end{aligned}$$

Note that both the second and third summands converge to 0 via Cesaro's Theorem (since $\beta\to 0$). It suffices to show that the final summand converges to 0. Since $\alpha\to 0$, $\{\alpha_n\}$ is bounded, so there exists $M>0$ such that $|\alpha_n|<M$ for all n. So we have:

$$\begin{aligned}
\left|\frac{\sum_{k=0}^n\alpha_k\beta_{n-k}}{n+1}\right| &\leq \frac{\sum_{k=0}^n|\alpha_k|\,|\beta_{n-k}|}{n+1} \\
&\leq \frac{M\sum_{k=0}^n|\beta_{n-k}|}{n+1},
\end{aligned}$$

which converges to 0 via Cesaro's Theorem. Hence by squeeze theorem, we see that

$$\frac{\sum_{k=0}^n\alpha_k\beta_{n-k}}{n+1}\to 0$$

as n goes to infinity. The result follows. $\square$

Exercise 2. *For each $n \in \mathbb{N}$ let $f_n(x) = \dfrac{nx}{x^2+n^2}$, $x \in \mathbb{R}$. Check whether the sequence (f_n) converges uniformly on $\mathbb{R}$.*

Proof. (O. Khalil, K. Nowland) We claim the sequence doesn't converge uniformly. First, we compute the pointwise limit. Fix some $x \in \mathbb{R}$. Then, we have that $f_n(x) = \frac{x}{\frac{x^2}{n}+n} \to 0$ as $n \to \infty$. Now, observe that for each n, we have that $f_n(n) = \frac{1}{2}$. Thus $\sup_{x\in\mathbb{R}} |f_n(x)| \geq \frac{1}{2}$, such that $f_n(x)$ does not converge uniformly to zero, i.e., given $\varepsilon < 1/2$ we cannot find an N such that $n \geq N$ implies $\sup_{x\in\mathbb{R}} |f_n(x)| < \varepsilon$. $\square$

Exercise 3. *Prove the following special case of the Riemann-Lebesgue lemma: Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that $\lim_{t\to\infty} \int_0^1 f(x) \sin tx dx = 0$.*

Proof. (O. Khalil) Let $\varepsilon > 0$ be fixed. Since f is continuous on $[0.1]$ which is a closed bounded interval, then f is also uniformly continuous on $[0,1]$. Hence, $\exists \delta > 0$, such that $\forall x, y \in [0,1]$, whenever $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now, let $T_1 \in (0,\infty)$ be so that $\frac{2\pi}{T} < \delta$. Since f is continuous on a closed bounded interval, then by the extreme value theorem, f is bounded. Let $M > 0$ be such that $f(x) < M < \infty$. Let $T_2 \in (0,\infty)$ be such that $\frac{2\pi M}{T_2} < \varepsilon/2$. Let $T = \max\{T_1, T_2\}$. Let $t > T$ be fixed. We wish to show that

$$\left| \int_0^1 f(x) \sin tx dx \right| < \varepsilon$$

Let $N \in \mathbb{N}$ be the largest integer so that $\frac{2\pi N}{t} \leq 1$. Write

$$\left| \int_0^1 f(x) \sin tx dx \right| \leqslant \sum_{k=1}^{N} \left| \int_{\frac{2\pi(k-1)}{t}}^{\frac{2\pi k}{t}} f(x) \sin tx dx \right| + \left| \int_{\frac{2\pi N}{t}}^{1} f(x) \sin tx dx \right| \tag{19}$$

Now, the period of $\sin tx = \frac{2\pi}{t}$. Hence, for each k, $\forall x \in [\frac{2\pi(k-1)}{t}, \frac{\pi(2k-1)}{t}]$, (half subinterval), we have that $\sin tx = -\sin(tx + \frac{\pi}{t})$. Hence, we can rewrite the sum in 19 as follows

$$\sum_{k=1}^{N} \left| \int_{\frac{2\pi(k-1)}{t}}^{\frac{2\pi k}{t}} f(x) \sin tx dx \right| = \sum_{k=1}^{N} \left| \int_{\frac{2\pi(k-1)}{t}}^{\frac{\pi(2k-1)}{t}} (f(x) - f(x+\frac{\pi}{t})) \sin tx dx \right| \tag{20}$$

But, since $|x - x - \frac{\pi}{t}| = \frac{\pi}{t} \leq \frac{\pi}{T} < \delta$, then uniform continuity of f gives

$$|f(x) - f(x+\frac{\pi}{t})| < \varepsilon$$

Hence, the right hand side in 20 can be bounded as follows:

$$\begin{aligned}\sum_{k=1}^{N} \left| \int_{\frac{2\pi(k-1)}{t}}^{\frac{\pi(2k-1)}{t}} (f(x) - f(x+\frac{\pi}{t})) \sin tx dx \right| &\leqslant \sum_{k=1}^{N} \int_{\frac{2\pi(k-1)}{t}}^{\frac{\pi(2k-1)}{t}} \left| (f(x) - f(x+\frac{\pi}{t})) \sin tx \right| dx \\ &\leq \sum_{k=1}^{N} \int_{\frac{2\pi(k-1)}{t}}^{\frac{\pi(2k-1)}{t}} \varepsilon dx = \sum_{k=1}^{N} \frac{\varepsilon\pi}{t} = \frac{N\varepsilon\pi}{t} \\ &< \varepsilon/2\end{aligned}$$

The last inequality follows from the choice of N. Now, the remaining part of 19 can be bounded as follows

$$\begin{aligned}\left|\int_{\frac{2\pi N}{t}}^{1} f(x)\sin tx dx\right| &\leq \int_{\frac{2\pi N}{t}}^{1} |f(x)\sin tx|\, dx \leq M(1-\frac{2\pi N}{t})\\ &\leq M\frac{2\pi}{t} < M\frac{2\pi}{T} < \frac{\varepsilon}{2}\end{aligned}$$

The third inequality follows by the choice of N which makes the length of the remainder interval less than the period of $\sin tx$. The fourth inequality follows by the choice of T. Plugging these bounds in 19, we get

$$\left|\int_{0}^{1} f(x)\sin tx dx\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

as desired.

$\square$

Exercise 4. *Prove this version of Cauchy's mean value theorem: Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{21}$$

Proof. (E. Nash) First, consider the case where $g(a) = g(b)$. Then Rolle's Theorem implies there exists some $c \in (a, b)$ such that $g'(c) = 0$. Thus, $(f(b) - f(a))g'(c) = 0 = (g(b) - g(a))f'(c)$ in this case. We therefore consider when $g(a) \neq g(b)$. Consider the function $h : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(x)$. Note that h is a linear combination of f and g as $\frac{f(a)-f(b)}{g(a)-g(b)}$ is a constant, so h is continuous on $[a, b]$ and differentiable on (a, b) with $h'(x) = f'(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g'(x)$. Further note that $h(a) = h(b)$:

$$\begin{aligned}h(b) - h(a) &= \left(f(b) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(b)\right) - \left(f(a) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(a)\right)\\ &= f(b) - f(a) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(b) - g(a))\\ &= f(b) - f(a) - (f(b) - f(a)) = 0.\end{aligned}$$

Thus, $h(b) - h(a) = 0$, so $h(a) = h(b)$. Then Rolle's theorem implies there exists c such that $h'(c) = 0$. This implies that $f'(c) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g'(c) = 0$, which simplifies to $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$, confirming the claim. $\square$

Exercise 5. *Find the limit*

$$\lim_{z\to 0^+} \frac{1}{\ln z}\int_0^1 \frac{\cos t}{z+t} dt.$$

Proof. (E. Nash) We claim $\lim_{z\to 0^+} \frac{1}{\ln z}\int_0^1 \frac{\cos t}{z+t}\,dt = -1$. Let $0<\varepsilon<1$ be given and choose $\delta\in(0,1)$ so that $\cos t > 1-\varepsilon$ for all $t\in[0,\delta]$. Now consider $\frac{1}{\ln z}\int_0^\delta \frac{\cos t}{z+t}\,dt$. We have the following, assuming $z\in(0,1)$:

$$\begin{array}{rcccl}
\frac{1}{\ln z}\int_0^\delta \frac{1}{z+t}\,dt & \le & \frac{1}{\ln z}\int_0^\delta \frac{\cos t}{z+t}\,dt & \le & \frac{1-\varepsilon}{\ln z}\int_0^\delta \frac{1}{z+t}\,dt \\
\frac{1}{\ln z}\left(\ln(z+\delta)-\ln z\right) & \le & \frac{1}{\ln z}\int_0^\delta \frac{\cos t}{z+t}\,dt & \le & \frac{1-\varepsilon}{\ln z}\left(\ln(z+\delta)-\ln z\right) \\
\left(\frac{\ln(z+\delta)}{\ln z}-1\right) & \le & \frac{1}{\ln z}\int_0^\delta \frac{\cos t}{z+t}\,dt & \le & (1-\varepsilon)\left(\frac{\ln(z+\delta)}{\ln z}-1\right)
\end{array}$$

Taking the limit as $z\to 0^+$, we have that $\ln\delta - 1 \le \lim_{z\to 0^+}\frac{1}{\ln z}\int_0^\delta \frac{\cos t}{z+t}\,dt \le (\varepsilon-1)(\ln\delta - 1)$. Letting $\delta\to 0^+$, we see that to prove the claim it is now sufficient to show that $\lim_{z\to 0^+}\frac{1}{\ln z}\int_\delta^1 \frac{\cos t}{z+t}\,dt = 0$ for all $\delta\in(0,1)$. We first note that $0\le \int_\delta^1 \frac{\cos t}{z+t}\,dt \le \int_\delta^1 \frac{\cos\delta}{t}\,dt = -\cos\delta\cdot\ln\delta$, so the integral has a finite value. But $\lim_{z\to 0^+}\frac{1}{\ln z} = 0$. Thus, $\lim_{z\to 0^+}\frac{1}{\ln z}\int_\delta^1 \frac{\cos t}{z+t}\,dt = \lim_{z\to 0^+}\frac{1}{\ln z}\cdot\lim_{z\to 0^+}\int_\delta^1 \frac{\cos t}{z+t}\,dt = 0$, as claimed. This finally implies that $\lim_{z\to 0^+}\frac{1}{\ln z}\int_0^1 \frac{\cos t}{z+t}\,dt = -1$, as claimed. □

Exercise 6. *Show that for all $x\in\mathbb{R}$ and all $n\in\mathbb{N}$ we have*

$$e^x \ge \sum_{j=0}^{2n+1}\frac{x^j}{j!}. \tag{22}$$

Proof. (K. Nowland) Let

$$f_n(x) := e^x - \sum_{j=0}^{2n+1}\frac{x^j}{j!}.$$

We wish to prove that $f_n(x)\ge 0$ for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$. We induct on n. For $n=0$, we want to show that $e^x-1-x\ge 0$. Note that $f_0'(x)=e^x-1$ satisfies $f_0'(x)<0$ for all $x<0$ and $f_0'(x)>0$ for all $x>0$. Sicne $f_0(x)$ is continuously differentiable for all real numbers, this implies that $f_0(x)$ has a global iminimum at $x=0$. Since $f_0'(0)=0$, this proves the base case.

Now suppose $f_{n-1}(x)\ge 0$ for all $x\in\mathbb{R}$. We want to prove that this implies that $f_n(x)\ge 0$. for all $x\in\mathbb{R}$. Note that

$$f_n'(x) = e^x - \sum_{j=0}^{2n}\frac{x^j}{j!}$$

satisfies $f_n'(0)=0$ such that $x=0$ is a critical point of f_n. Taking another derivative gives

$$f_n''(x) = e^x - \sum_{j=0}^{2n-1}\frac{x^j}{j!} \ge 0$$

where the last inequality follows from the inductive hypothesis. Therefore f_n is concave up at all $x\ge 0$. Therefore f_n has an absolute minimum at$x=0$. Since $f_n(0)=0$, this completes the proof. □

2009 - Autumn

Exercise 1. *For any sequence* (a_n) *of positive numbers, prove that*

$$\limsup_{n\to\infty}(a_n)^{1/n} \le \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}. \tag{23}$$

Proof. (K. Nowland) This is one half of d'Alembert's ratio test. A proof of the other half can be found in 2005 Autum number 6. If $\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\infty$, then there is nothing to prove, so suppose that $L := \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}$ is finite. Since $a_n > 0$ for all n, we have that $L \ge 0$. Let $t > L$. By the definition of the limit supremum, there exists $N \in \mathbb{N}$ such that

$$\sup_{n\ge N}\frac{a_{n+1}}{a_n} < t.$$

It follows that

$$\frac{a_{n+1}}{a_n} < t$$

for all $n \ge N$. Let $n > N$ and write

$$a_n = \frac{a_n}{a_{n-1}}\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_{N+1}}{a_N}a_N.$$

We therefore have

$$a_n < t^{n-N+1}a_N.$$

Raising to the $1/n$ power,

$$(a_n)^{1/n} < t\left(\frac{a_N}{t^{N+1}}\right)^{1/n}.$$

Taking the limit supremum as $n\to\infty$ gives

$$\limsup_{n\to\infty}(a_n)^{1/n} \le t.$$

Since $t > L$ was arbitrary, we conclude that (23) must hold. □

Exercise 2. *Prove or disprove:* $f(x) = x\log x$ *is uniformly continuous on*

(i) the interval $(0,1]$*;*

(ii) the interval $[1,\infty)$*.*

Proof. (A. Newman) For part (i), we will show that $f(x)$ is uniformly continuous by showing it can be extended to a continuous function g on the compact interval $[0,1]$. All that is required here it to show that $\lim_{x\to 0^+} x\log x = 0$. By computing the derivative $f'(x) = 1+\log(x)$ we know that f is decreasing on $(0,\frac{1}{e})$ and also $f(x) < 0$ on $(0,\frac{1}{e})$, thus on $(0,\frac{1}{e})$ we know that $|x\log(x)|$ is increasing. So let $\varepsilon > 0$ be given. Find n so that $\frac{n}{e^n} < \min\{\varepsilon,\frac{1}{e}\}$. Now set $\delta = \frac{1}{e^n}$ so $0 < x < \frac{1}{e^n}$ implies that $|x\log x| < |\frac{1}{e^n}\log(\frac{1}{e^n})| = \frac{n}{e^n} < \varepsilon$. And show we prove that by defining $f(0) = 0$ we can extend f to a continous function on a compact set, so f is uniformly continuous on $(0,1]$.

The function f is not uniformly continuous on $[1,\infty)$. On $[1,\infty)$ the function increasing (we see this by checking the derivative). So by the mean value theorem for any $x < y \in [1,\infty)$, we have $\frac{|x\log x - y\log y|}{|x-y|} \geq \log x + 1$. So given $\varepsilon > 0$ there is no δ so that $|x-y| < \delta$ implies $|x\log x - y\log y| < \varepsilon$ since $|x\log x - y\log y| \geq |x-y|(\log x + 1) \to \infty$ as x goes to infinity, regardless of how small $|x-y|$ is. □

Exercise 3. *Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable and satisfy $f(0) = f(1) = 0$. Show that*

$$\int_0^1 |f(x)|^2 dx \leq 4\int_0^1 x^2|f'(x)|^2 dx. \tag{24}$$

Proof. (O. Khalil) The continuity of f and its derivative on $[0,1]$ implies that the functions in 24 are integrable on $[0,1]$.

Now, using Cauchy-Schwarz inequality for integrals, write

$$\int_0^1 f(x)(xf'(x))dx \leqslant \left(\int_0^1 |f(x)|^2 dx\right)^{1/2} \left(\int_0^1 x^2|f'(x)|^2 dx\right)^{1/2} \tag{25}$$

Integrating the left handside by parts taking

$$\begin{aligned} u &= x & dv &= ff'dx = fdf \\ du &= dx & v &= \frac{f^2}{2} \end{aligned}$$

yields

$$\int_0^1 f(x)(xf'(x))dx = \left.\frac{x(f(x))^2}{2}\right|_0^1 - 1/2\int_0^1 |f(x)|^2 dx = 0 - 1/2\int_0^1 |f(x)|^2 dx \tag{26}$$

Plugging this into the left hand-side of 25 and squaring both sides

$$1/4\left(\int_0^1 |f(x)|^2 dx\right)^2 \leq \left(\int_0^1 |f(x)|^2 dx\right)\left(\int_0^1 x^2|f'(x)|^2 dx\right) \tag{27}$$

Now, if $\displaystyle\int_0^1 |f(x)|^2 dx = 0$, then 24 would follow immediately since $x^2|f'(x)|^2 \geq 0$ on $[0,1]$. Otherwise, dividing both sides of 27 by $\displaystyle\int_0^1 |f(x)|^2 dx$ gives 24. □

Exercise 4. *Suppose we define the sine function by the convergent series*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Prove there exists $a > 0$ such that $\sin(a) = 0$.

Proof. (K. Nowland) First we show that $\sin(x)$ converges on all of $\mathbb{R}$. The radius of convergence of the power series is given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}},$$

where c_n is the nth coefficient of x^n and with the convention that if the limit superior is 0 then the power series converges on all of $\mathbb{R}$ and if the limit superior is ∞ then the power series diverges for all nonzero x. If we relabel the c_n as $b_n = c_{2n+1}$, then since the other terms are zero, we have

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n|}}.$$

Note thate $|b_n/b_{n+1}| = 1/(2n+1)(2n+3)$ tends to zero as $n \to \infty$. This implies that $\lim_{n\to\infty} \sqrt[n]{|b_n|}$ exists and is zero by D'Alembert's theorem. Thus $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0$ as well, such that $\sin(x)$ converges on all of $\mathbb{R}$.

Now calculate

$$\sin(1) = \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{5!} - \frac{1}{7!}\right) + \cdots > 0.$$

But also,

$$\sin(4) = \left(\frac{4}{1!} - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!}\right) + \left(-\frac{4^{11}}{11!} + \frac{4^{13}}{13!}\right) + \cdots < 0.$$

Since sine converges for all $x \in \mathbb{R}$, it converges uniformly on all disks $\{x : |x| < r\}$ for $r < \infty$. Since $\sin(x)$ is the uniform limit of continuous functions – the partial sums are polynomials which are continuous – on any bounded disk, $\sin(x)$ is continuous on all of $\mathbb{R}$. By the intermediate value theorem, $\sin(x)$ has a zero in the interval $(1, 4)$. $\square$

Proof. (H. Lyu) By ratio test, the power series converges on the whole real line. Hence it converges uniformly on any compact set $K \subset \mathbb{R}$. In particular, for each $x_0 \in \mathbb{R}$, the series converges uniformly on a compact neighborhood of x_0, and since the partial sums are continuous at x_0, we see that $\sin x$ is continuous at x_0. Since this holds for every $x_0 \in \mathbb{R}$, we see that $\sin x$ is continuous on $\mathbb{R}$. Now we claim that $\sin a = 0$ for some $a \in (1, 4)$. By the intermediate value theorem, it would suffices to check $\sin 1 > 0$ and $\sin 4 < 0$. Indeed, observe that

$$\sin 1 = \left(1 - \frac{1}{3!}\right) + \left(\frac{1}{5!} - \frac{1}{7!}\right) + \cdots > 0.$$

On the other hand, observe that

$$\begin{aligned}
\sin 4 &= \left(1 - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!}\right) - \left(\frac{4^{11}}{11!} - \frac{4^{13}}{13!}\right) - \left(\frac{4^{15}}{15!} - \frac{4^{17}}{17!}\right) - \cdots \\
&= -\frac{268}{405} - \frac{4^{11}}{11!}\left(1 - \frac{4^2}{13 \cdot 12}\right) - \frac{4^{15}}{15!}\left(1 - \frac{4^2}{15 \cdot 14}\right) - \cdots \\
&< 0.
\end{aligned}$$

Therefore by the intermediate value theorem, there exists $1 < a < 4$ such that $\sin a = 0$. $\square$

Exercise 5. *Prove that*

$$\lim_{n\to\infty} \frac{\frac{2}{3}n^{3/2} - \sum_{k=1}^{n} k^{1/2}}{n^{1/2}}$$

exists and determine its value.

Proof. (K. Nowland) We claim that the limit is $-\frac{1}{2}$. Let $f(x) = \sqrt{x}$. The numerator can then be rewritten as

$$\int_1^n f(x)dx + \frac{2}{3} - \frac{1}{2}f(1) - f(2) - \cdots - f(n-1) - \frac{1}{2}f(n) - \frac{1}{2}f(1) - \frac{1}{2}f(n).$$

If we divide by $n^{1/2} = f(n)$, then as n tends to infinity, the $2/3$ and $-1/2f(1)$ terms will vanish. We may therefore ignore these terms in taking the limit. If we can show that

$$\int_1^n f(x)dx - \frac{1}{2}f(1) - f(2) - \cdots - f(n-1) - \frac{1}{2}f(n)$$

is bounded uniformly in n, then the limit after dividing by $n^{1/2} = f(n)$ will be $-\frac{1}{2}$, as claimed.

Let $F(x) = \int_1^x f(t)dt$. Since $f(t) = \sqrt{x}$ is infinitely continuously differentiable on $(0, \infty)$, then so is $F(x)$ and $F'(x) = f(x)$. Using Taylor's theorem with Lagrange remainder, there exist $a_k \in (k, k + 1/2)$ and $b_k \in (k + 1/2, k + 1)$ such that

$$F(k + 1/2) = F(k + 1) - \frac{1}{2}f(k + 1) + \frac{1}{8}f'(b_k),$$

and

$$F(k + 1/2) = F(k) + \frac{1}{2}f(k) + \frac{1}{8}f'(a_k).$$

Subtracting the top from the bottom,

$$F(k + 1) - F(k) = \frac{1}{2}f(k) + \frac{1}{2}f(k + 1) + \frac{f'(a_k) - f'(b_k)}{8}.$$

By the definition of $F(x)$,

$$\int_k^{k+1} f(x)dx - \frac{1}{2}f(k) - \frac{1}{2}f(k + 1) = \frac{f'(a_k) - f'(b_k)}{8}.$$

Note that $f(x) = \sqrt{x}$ has a positive but decreasing derivative on $[1, \infty)$. Therefore we see that

$$0 \le \int_k^{k+1} f(x)dx - \frac{1}{2}f(k) - \frac{1}{2}f(k + 1) \le \frac{f'(k) - f'(k + 1)}{8}.$$

Summing from $k = 1$ to $n - 1$ gives

$$0 \le \int_1^n f(x)dx - \frac{1}{2}f(1) - f(2) - \cdots - f(n-1) - \frac{1}{2}f(n) \le \frac{f'(1) - f'(n)}{8} \le \frac{1}{8}f'(1) = \frac{1}{16}.$$

This proves the claim that the above difference is bounded uniformly in n. The limit is therefore $-\frac{1}{2}$. □

Exercise 6. *Let $f \in C([-1,1])$ and $\int_{-1}^{1} x^{2n} f(x)dx = 0$ for all integers $n \geq 0$. Prove that f is an odd function.*

Proof. (K. Nowland with thanks to O. Khalil) We can rewrite the condition on f as

$$\int_0^1 x^{2n}(f(x) + f(-x))dx = 0$$

for all nonnegative integers n. Note that $f(x)+f(-x)$ is a continuous functions because both $f(x)$ and $f(-x)$ are continuous functions on $[0,1]$. We want to apply the Stone-Weierstrass theorem for some set $\mathcal{A}$. Let $\mathcal{A}$ be the polynomials of degree $2n$ where $n \geq 0$ is an integer. Note that the zero degree polynomials are the scalars. Then $\mathcal{A}$ is closed under addition (the sum of polynomials is the sum of the individual monomials which are all even and addition of monomials either keeps the degree the same or reduces it to zero), closed under multiplication (the monomials in the product will have degrees that are sums of the even degrees of the monomials in the factors such that the product will only contain monomials of even degree), and contains the constants. Note that $x^2 \neq y^2$ if $x \neq y$ are in $[0,1]$, such that $\mathcal{A}$ separates points. By Stone-Weierstrass, $\mathcal{A}$ is dense in $C[0,1])$ under the uniform topology.

Since $|f(x) + f(-x)|$ is continuous on the compact interval $[0,1]$, it is bounded by some constant M. Let $\varepsilon > 0$. Choose $g(x) in \mathcal{A}$ such that $|f(x) + f(-x) - g(x)| < \varepsilon/M$ for all $x \in [0,1]$. We calculate

$$\begin{aligned} |\int_0^1 (f(x)+f(-x))^2 dx &- \int_0^1 g(x)(f(x) - f(-x))dx| \\ &\leq \int_0^1 |f(x) + f(-x) - g(x)||f(x) - f(-x)|dx \\ &< \int_0^1 \frac{\varepsilon}{M} M dx \ = \ \varepsilon. \end{aligned}$$

Since $g(x)$ is a finite sum of even monomials,

$$\int_0^1 g(x)(f(x) + f(-x))dx = 0$$

by hypothesis. Thus

$$\int_0^1 (f(x) + f(-x))^2 dx < \varepsilon.$$

Since ε was arbitrary and the integrand is nonnegative, it must be that $f(x) + f(-x) = 0$, i.e., $f(-x) = -f(x)$. Therefore f is odd, as claimed. $\square$

2009 - Spring

Exercise 1. *Let a_1 and b_1 be positive real numbers. Define*

$$a_n = \sqrt{a_{n-1}b_{n-1}}, \quad b_n = \frac{1}{2}(a_{n-1} + b_{n-1})$$

for integers $n > 1$. Prove that both the sequences (a_n) and (b_n) converge and have the same limit.

Proof. (K. Nowland) By the arithmetic-geometric mean inequality, $a_n \leq b_n$ for all n. In fact, equality holds if and only if $a_{n-1} = b_{n-1}$ such that the sequences are constant at some point if and only if $a_1 = b_1$ and they are constant at all points. It therefore suffices to consider the case where $a_1 \neq b_1$. Again by the arithmetic-geometric mean inequality, $a_n < b_n$ for all $n > 1$, such that it suffices to consider $a_1 < b_1$, since we will be immediately in this case and the first term of the sequence does not change the limit. Then we have $a_n < b_n$ for all $n \in \mathbb{N}$. It follows that

$$a_{n+1} = \sqrt{a_n b_n} > \sqrt{a_n^2} = a_n.$$

Thefore (a_n) is an increasing sequence. Similarly,

$$b_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(b_n + b_n) = b_n,$$

and b_n is a decreasing sequence. Also,

$$a_1 \leq a_n < b_n \leq b_1,$$

such that (a_n) is an increasing sequence bounded above and (b_n) is a decreasing sequence bounded below. Both sequences therefore are convergent. Let a be the limit of the a_n and b the limit of the b_n.

Since $a_n < b_n$ for all n, it follows that $a \leq b$. Suppose toward a contradiction that $a < b$ is a strict inequality. Let $c = \frac{1}{2}(a + b)$. Since $a < b$, it must be that $c < b$. Let $\varepsilon > 0$ be less than $b - c$. Let N be so large that $b < b_n < b - \varepsilon$ for all $n \geq N$. Since (b_n) decreases monotonically, $b < b_n$ for all n. We calculate Then

$$b < b_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(a + b + \varepsilon) = c + \frac{1}{2}\varepsilon < b.$$

This is a contradiction. The contradiction proves that claim that $a = b$. □

Exercise 2. *Prove or disprove: If f is a continuous function on $[0, \infty)$ such that $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{x} = 1$, then f is uniformly continuous on $[0, \infty)$.*

Proof. (E. Nash) We claim this statement is not true. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined so that $f\left(\displaystyle\sum_{k=1}^{m} \frac{1}{k}\right) = \displaystyle\sum_{k=1}^{m} \frac{1}{k} + (-1)^m$. Then let f vary linearly between the points

$\left(\sum_{k=1}^{m} \frac{1}{k}, f\left(\sum_{k=1}^{m} \frac{1}{k}\right)\right)$ and $\left(\sum_{k=1}^{m+1} \frac{1}{k}, f\left(\sum_{k=1}^{m+1} \frac{1}{k}\right)\right)$ for all $m \in \mathbb{N}$ and let f be uniformly 0 on the interval $[0,1)$. The function f is well-defined on $[0,\infty)$ because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Further, f is continuous because it is piecewise linear and its piecewise intervals all agree on their endpoints. Visually, the graph of f oscillates above and below the line $y = x$ more and more rapidly as $x \to \infty$.

We claim f is a counterexample to the statement. To see this, note that $x-1 \leq f(x) \leq x+1$ for all $x \in [0,\infty)$. Thus, $\lim_{x\to\infty} \frac{x-1}{x} \leq \lim_{x\to\infty} \frac{f(x)}{x} \leq \lim_{x\to\infty} \frac{x+1}{x}$, so $\lim_{x\to\infty} \frac{f(x)}{x} = 1$. But for arbitrary $\delta > 0$, we may choose $n \in \mathbb{N}$ so that $\frac{1}{2n} < \delta$. Then $\left|\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{2n-1} \frac{1}{k}\right| = \frac{1}{2n} < \delta$. But $\left|f\left(\sum_{k=1}^{2n} \frac{1}{k}\right) - f\left(\sum_{k=1}^{2n-1} \frac{1}{k}\right)\right| = \frac{1}{2n} + 2 > 2$. This implies that f is not uniformly continuous, as claimed. $\square$

Exercise 3. *Given that* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$, *evalute* $\displaystyle\int_0^1 \frac{\log x}{1+x} dx$.

Proof. (K. Nowland) Note that for $0 \leq x < 1$, the geometric series $\sum_{n=0}^{\infty} (-x)^n$ converges to $\frac{1}{1+x}$. We can therefore rewrite the integrand as

$$\frac{\log x}{1+x} = \log x - x \log x + x^2 \log x + \cdots + (-x)^n \log x + \cdots .$$

Though we are not free to interchange the integrand with an infinite summation without uniform convergence, we can do so for any finite number of terms. Thus it suffices to (prove the existence of and) evaluate

$$\int_0^1 \log x dx + \int_0^1 \sum_{n=1}^{\infty} (-x)^n \log x dx.$$

Since $\log x$ is infinite at zero, we excise the origin with an ε-ball and take the limit as $\varepsilon \to 0^+$. We see that

$$\int_\varepsilon^0 \log x dx = (x \log x - x|_\varepsilon^1 = -1 - \varepsilon \log \varepsilon - \varepsilon.$$

As ε tends to zero from above, $\varepsilon \log \varepsilon$ tends to zero. (This can be seen by from expanding $\log(1+x)$ about zero but is not proven here). We must still evaluate

$$\int_0^1 \sum_{n=1}^{\infty} (-x)^n \log x dx.$$

Our goal is to prove uniform convergence of the interior such that we may interchange the integral and summation. It is sufficient to prove that the series converges uniformly absolutely. Consider the partial sum of the absolute values

$$S_N(x) = -\sum_{n=1}^{N} x^n \log x.$$

This is well-defined on $[0,1]$ since $x^n \log x \to 0$ as $x \to 0^+$ since $n > 0$. Note that $S_N(0) = S_N(1) = 0$. For $0 < x < 1$, the convergence of the geometric series implies that the S_N

converge pointwise on the compact interval $[0,1]$. Moreover, the convergence is monotonic, i.e., $S_N(x) < S_{N+1}$ for all $N \geq 1$. Dini's theorem implies that the convergence is uniform. With the uniform convergence, it follows that we may exchange the integral and infinite summation:

$$\int_0^1 \sum_{n=1}^{\infty}(-x)^n \log x dx = \sum_{n=1}^{\infty}(-1)^n \int_0^1 x^n \log x dx.$$

Note that x^n and $\log x$ are both differentiable on $(0,1)$. Integrating by parts,

$$\int_0^1 x^n \log x dx = \frac{x^{n+1}}{n+1} \log x \Big|_0^1 - \frac{1}{n+1} \int_0^1 x^n dx = -\frac{1}{(n+1)^2}.$$

The valuation at zero of the boundary terms is again justified by the vanishing of $x \log x$ as $x \to 0^+$. We therefore have

$$\sum_{n=1}^{\infty}(-1)^n \int_0^1 x^n \log x dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2} = 1 - \frac{\pi^2}{12}.$$

Thus

$$\int_0^1 \frac{\log x}{x+1} dx = -\frac{\pi^2}{12}.$$

□

Exercise 4. *Let* $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $n \in \mathbb{N}$. *Prove or disprove: the sequence* (f_n) *is uniformly convergent on* $[0,1]$.

Proof. (O. Khalil, K. Nowland) We claim this statement is false. First, we calculate the pointwise limit. Fix $x \in [0,1]$. Then, we have that $\frac{nx}{1+n^2x^2} = x\frac{1}{\frac{1}{n}+nx^2} \to 0$ as $n \to \infty$. Hence, we wish to show that $\sup_{x\in[0,1]} |f_n(x) - 0| = \sup_{x\in[0,1]} |f_n(x)| \nrightarrow 0$ as $n \to \infty$. Since $f_n(1/n) = 1/2$, it must be that $\sup_{x\in[0,1]} |f_n(x)| \geq 1/2$ for all $n \in \mathbb{N}$. Thus the sequence cannot converge uniformly to zero. □

Exercise 5. *Prove the following integral form of the Mean Value Theorem: If* f *and* g *are continuous on* $[a,b]$ *and* $g(x) \neq 0$ *for all* $x \in (a,b)$, *then there exists* $c \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx. \tag{28}$$

Proof. (O. Khalil) By the continuity of g on $[a,b]$ and the intermediate value theorem along with the fact that $g \neq 0$ on (a,b), we have that g doesn't change sign on $[a,b]$. Hence, we may assume that $g \geq 0$ on $[a,b]$ (the negative case follows similarly). Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. Since f is continuous and $[a,b]$ is compact, then m and M are finite. Moreover, by the extreme value theorem, $\exists c_1, c_2 \in [a,b]$, such that $f(c_1) = m$ and $f(c_2) = M$. Hence, since $g(x)$ is non-negative, we have that $\forall x \in [a,b]$, $mg(x) \leq f(x)g(x) \leq Mg(x)$. And so, we get

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx \tag{29}$$

Let $I = \int_a^b g(x)dx$. If $I = 0$, then the preceding inequality gives that $\int_a^b f(x)g(x)dx = 0$ and 28 follows trivially. Now, assume $I \neq 0$. Then, $I > 0$ by the non-negativity of g. Dividing 29 through by I gives

$$m = f(c_1) \leq \frac{1}{I}\int_a^b f(x)g(x)dx \leq M = f(c_2) \tag{30}$$

Now, applying the intermediate value theorem to f, we get that $\exists c \in [\min\{c_1, c_2\}, \max\{c_1, c_2\}] \subseteq [a, b]$, such that

$$f(c) = \frac{1}{I}\int_a^b f(x)g(x)dx$$

And, so 28 follows. If $g < 0$ on (a, b), then the inequalities in 29 and 30 will be reversed but the rest will be the same. $\square$

Exercise 6. *Let $f : [0, 1] \to \mathbb{R}$ be continuous. Determine $\lim_{n\to\infty} \int_0^1 nx^n f(x)dx$. Prove your answer.*

Proof. (K. Nowland) We claim that the limit is $f(1)$. Note that as n tends to infinity, that $\frac{n}{n+1} \to 1$. Thus it suffices to show that for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N$ implies that

$$|\int_0^1 nx^n f(x)dx - \frac{n}{n+1}f(1)| < \varepsilon.$$

We calculate

$$\int_0^1 nx^n f(x)dx - \frac{n}{n+1}f(1) = \int_0^1 nx^n(f(x) - f(1))dx.$$

Since f is continuous on the compact set $[0, 1]$, there exists $M > 0$ such that $|f(x)| < M$ for all $x \in [0, 1]$. By (left) continuity at 1, there exists δ satisfying $0 < \delta < 1$ such that $|f(x) - f(1)| < \varepsilon/2$ if $1 - \delta < x < 1$. Then we see that

$$\begin{aligned}
|\int_0^1 nx^n(f(x) - f(1))dx| &\leq \int_0^1 nx^n|f(x) - f(1)|dx \\
&= \int_0^{1-\delta} nx^n|f(x) - f(1)|dx + \int_{1-\delta}^1 nx^n|f(x) - f(1)|dx \\
&< 2M\int_0^{1-\delta} nx^n dx + \varepsilon/2\int_{1-\delta}^1 nx^n dx \\
&< 2M\frac{n}{n+1}(1-\delta)^{n+1} + \varepsilon/2\int_0^1 nx^n dx \\
&< 2M(1-\delta)^{n+1} + \frac{n}{n+1}\varepsilon/2 \\
&< 2M(1-\delta)^{n+1} + \varepsilon/2.
\end{aligned}$$

Note that $0 < 1 - \delta < 1$. Let N be so large that $(1 - \delta)^{n+1} < \varepsilon/4M$ for all $n \geq N$. Then we have

$$|\int_0^1 nx^n f(x)dx - \frac{n}{n+1}f(1)| < \varepsilon,$$

as desired. $\square$

2008 - Autumn

Exercise 1. *Find the limit:*

$$\lim_{n\to\infty} n(\sqrt[n]{n}-1).$$

Proof. (O. Khalil) Since the function $x \mapsto \ln x$ is infinitely differentiable on $(0,\infty)$, with 2^{nd} derivative equal to $\frac{-1}{x^2} < 0$, this function is concave and hence is below its tangent line at every point greater than 0. In particular, it is below its tangent at $x = 1$. Since $(\ln x)'|_1 = 1$, and $\ln 1 = 0$, this gives the following equation for the tangent line at $x = 1$, $L(x) = x - 1$. Therefore, we get the following inequality for all $x > 0$:

$$\ln x \leq x - 1$$

Hence, for each $n \in \mathbb{N}$, we have that

$$\ln(\sqrt[n]{n}) \leq \sqrt[n]{n} - 1$$

And, thus, multiplying both sides by n, we get

$$\ln n \leq n(\sqrt[n]{n} - 1)$$

Since the left-hand side goes to ∞ as $n \to \infty$, then so does the right-hand side. $\square$

Exercise 2. *Find the value of the series:*

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}.$$

Proof # 1. (K. Nowland) Set $S_n(x) = \sum_{k=0}^{n} x^k$. Note that

$$S_n(x) = \frac{1-x^{n+1}}{1-x}.$$

If $0 < x < 1$, $S_n(x)$ converges pointwise as $n \to \infty$. Differentiating,

$$S_n'(x) = \sum_{k=1}^{n} kx^{k-1} = \frac{1}{(1-x)^2} - \frac{x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x}.$$

This converges pointwise for $0 < x < 1$ since $(n+1)x^n \to 0$ as $n \to \infty$. Differentiating again,

$$S_n''(x) = \sum_{k=2}^{n} k(k-1)x^{k-2} = \frac{2}{(1-x)^3} - \frac{2x^{n+1}}{(1-x)^3} - \frac{(n+1)x^n}{(1-x)^2} - \frac{n(n+1)x^{n-1}}{1-x}.$$

As above, this converges pointwise for $0 < x < 1$. Calculate

$$\begin{aligned}
\sum_{k=1}^{n} k^2 x^k &= x + \sum_{k=2}^{n} k^2 x^k \\
&= x + x^2 \sum_{k=2}^{n} k^2 x^{k-2} \\
&= x + x^2 \sum_{k=2}^{n} k(k-1)x^{k-2} + x^2 \sum_{k=2}^{n} k x^{k-2} \\
&= x + x^2 S_n''(x) + x \sum_{k=2}^{n} k x^{k-1} \\
&= x^2 S_n''(x) + x S_n'(x).
\end{aligned}$$

Taking $0 < x < 1$ and letting n tend to infinity, we see that

$$\sum_{k=1}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}.$$

Plugging in $x = 1/2$, we see that

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k} = \frac{1/2}{1/8} + \frac{1/2}{1/4} = 6.$$

[Proof # 2] (O. Khalil)

Let $S_n = \sum_1^n \frac{k^2}{2^k}$, for each $n \in \mathbb{N}$. Set $A_n = \sum_1^n \frac{1}{2^k}$ and note that $A_n = 1 - \frac{1}{2^n}$. Now, use summation by parts to write (some simplification steps are left out but the calculation should be clear)

$$\begin{aligned}
S_n &= n^2 A_n + \sum_{k=1}^{n-1} A_k(k^2 - (k+1)^2) \\
&= n^2 - \frac{n^2}{2^n} - \sum_1^{n-1}(2k+1) + \sum_1^{n-1} \frac{2k+1}{2^k} \\
&= \frac{-n^2}{2^n} + 1 + 2\sum_1^{n-1} \frac{k}{2^k} + A_{n-1} \\
&= \frac{-n^2}{2^n} + 1 + 2T_{n-1} + A_{n-1}
\end{aligned}$$

where $T_n = \sum_1^n \frac{k}{2^k}$. Apply sumation by parts again to T_n to get

$$\begin{aligned}
T_n &= nA_n + \sum_1^{n-1} A_k(k - (k+1)) \\
&= nA_n - (n-1) + A_{n-1} = 1 - \frac{n}{2^n} + A_{n-1}
\end{aligned}$$

Hence, since $\frac{n}{2^n} \to 0$ and $A_n \to 1$ as $n \to \infty$, we get that $T_n \to 2$. Thus, since $\frac{n^2}{2^n} \to 0$, we get that $S_n \to 6$ as $n \to \infty$. $\square$

Exercise 3. *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Prove that f is infinitely differentiable on $\mathbb{R}$.

Proof. (R. Garrett) Clearly, f is C^∞ on all of $\mathbb{R} \setminus \{0\}$, as $f^{(n)}(x) = 0$ for $x < 0$ and by induction $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x}}$ for $x > 0$: indeed, $f'(x) = \frac{-1}{x}e^{\frac{-1}{x}}$, and since $(\frac{1}{x})' = \frac{-1}{x^2}$ and P_n being a polynomial implies that P_n' is a polynomial, we have if $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x}}$ then $f^{(n+1)}(x) = P_n'(\frac{1}{x}) \cdot (\frac{-1}{x^2})e^{\frac{-1}{x}} + P_n(\frac{1}{x})(\frac{-1}{x})e^{\frac{-1}{x}} = (\text{polynomial in } \frac{1}{x})e^{\frac{-1}{x}}$, as desired.

Now we show infinite differentiability at 0 by induction.

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1}{x}e^{-1/x} = \lim_{x \to \infty} xe^{-x} = 0$$

since exponential growth dominates polynomial growth and

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0$$

so $f'(0) = 0$. Now, suppose for a natural number n $f^{(n)}(x) = 0$. Then,

$$\lim_{x \to 0^+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0^+} \frac{1}{x}P_n(\frac{1}{x})e^{\frac{-1}{x}} = \lim_{x \to \infty} \hat{P}_n(x)e^{-x} = 0$$

since exponential growth dominates polynomial growth (we're setting $\hat{P}_n(\frac{1}{x}) = \frac{1}{x}P_n(\frac{1}{x})$). So, by induction we have $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, and we have $f \in C^\infty(\mathbb{R})$.

$\square$

Exercise 4. *Let*

$$I = \int_0^\infty \frac{\sqrt{x}\cos x}{x + 100}dx.$$

Is I convergent?

Proof. (A. Newman) This is a direct application of Dirichlet's Test for integrals since $\int_0^N \cos x dx$ is bounded by 1 for all $N \in \mathbb{R}^+$, and $\frac{\sqrt{x}}{x+100}$ decreases to zero as x goes to infinity. So I will give a proof of a version Dirichlet's test for integrals.

Claim: If g, f are continouous real-valued functions on $[0, \infty)$, with f differentiable and decreasing and $f(x) \to 0$ as $x \to \infty$ and $\int_0^N g(x)dx \le M$ for some constant M for all $N \in \mathbb{R}^+$, then $\int_0^\infty f(x)g(x)dx$ converges.

Proof of claim. Using integration by parts we have for a fixed $N \in \mathbb{R}^+$, $\int_0^N f(x)g(x) = f(x)G(x)|_0^N - \int_0^N G(x)f'(x)dx$ where $G(x)$ is an antiderivative of $g(x)$. So

$$\begin{aligned}\left|\int_0^N f(x)g(x)\right| &= \left|f(x)G(x)|_0^N - \int_0^N G(x)f'(x)dx\right| \\ &\le \left|f(x)G(x)|_0^N\right| + \left|\int_0^N G(x)f'(x)dx\right| \\ &\le |f(N)G(N) - f(0)G(0)| + M(f(N) - f(0)) \\ &\to |f(0)G(0)| + M(-f(0)) \text{ as } N \to \infty.\end{aligned}$$

Thus $\left|\int_0^\infty f(x)g(x)dx\right|$ is convergent. □

Exercise 5. *Suppose that $f_n \in C[0,1]$ for every n, $f_n(x) \ge f_{n+1}(x)$ for every n and x, and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for some function $f \in C[0,1]$. Show that f_n converges to f uniformly on $[0,1]$.

Proof. (O. Khalil) This is an instance of Dini's theorem. Let $g_n(x) = f_n(x) - f(x)$ for each $n \in \mathbb{N}$ and each $x \in [0,1]$. So, $g_n \ge 0$ for all x and all n and $g_n(x) \downarrow 0$. Also, g_n is continuous for each n. We need to show that $\sup_{x\in[0,1]} g_n(x) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be fixed. Let $G_n = \{x \in [0,1] : g_n(x) < \varepsilon\}$ for each n. Then, we have that $G_n \subseteq G_{n+1}$ for each n. Moreover, $G_n = g_n^{-1}[(-\infty, \varepsilon)]$ and so G_n is open for each n by the continuity of g_n. Moreover, since $g_n(x) \downarrow 0$ for each $x \in [0,1]$, then there exists some $N \in \mathbb{N}$ such that for all $n > N$, we have that $g_n(x) < \varepsilon$. Thus, $x \in G_n$ for all $n > N$. Therefore, $[0,1] = \bigcup_{n=1}^\infty G_n$. But, since $[0,1]$ is compact, then there exist finitely many natural numbers $n_1, \cdots, n_k$ such that $[0,1] = \bigcup_{j=1}^k G_{n_j} = G_{n_k}$, since the G_n's is an increasing sequence of sets.

Now, let $N = n_k$. Then, for all $x \in [0,1] = G_N$, we have that $g_N(x) < \varepsilon$ by construction. Hence, for all $n > N$, we have that $g_n(x) \le g_N(x) < \varepsilon$. Therefore, for all $n > N$, $\sup_{x\in[0,1]} g_n(x) \le \varepsilon$. Since, ε was arbitrary, then $\sup_{x\in[0,1]} g_n(x) \to 0$ as desired.

□

Exercise 6. *Let $f : [-1,1] \to \mathbb{R}$ be continuous. Prove that*

$$\int_{-1}^1 \frac{uf(x)}{u^2 + x^2} dx \to \pi f(0), \tag{31}$$

as $u \to 0^+$.

Proof. (O. Khalil) Note the following

$$\int_{-1}^1 \frac{uf(0)}{u^2 + x^2} dx = \int_{\frac{-1}{u}}^{\frac{1}{u}} \frac{f(0)}{1 + t^2} dt = f(0)(\arctan(\frac{1}{u}) - \arctan(\frac{-1}{u}))$$

where we used a change of variable $t = ux$. Hence, we get that $\int_{-1}^{1} \frac{uf(0)}{u^2+x^2}dx \to \pi f(0)$ as $u \to 0$. Hence, it suffices to show that

$$\lim_{u\to 0^+} \int_{-1}^{1} \frac{uf(x)}{u^2 + x^2}dx - \int_{-1}^{1} \frac{uf(0)}{u^2 + x^2}dx = 0$$

Let $\varepsilon > 0$ be fixed. Since f is continuous at 0, let $\delta > 0$ be such that if $|x| \leq \delta$, then $|f(x) - f(0)| < \varepsilon/2\pi$. We may assume that $\delta < 1$. Also, by continuity of f on $[-1, 1]$, we have that f is bounded with some constant $M > 0$. Also, we have that $\frac{u}{u^2+\delta^2} \to 0$ as $u \to 0$. So, let u be such that $\frac{u}{u^2+\delta^2} < \frac{\varepsilon}{8M}$.

Let $A = [-\delta, \delta]$ and $B = [-1, 1] \setminus [-\delta, \delta]$. Then, we have the following

$$\begin{aligned}
\left|\int_B \frac{u(f(x) - f(0))}{u^2 + x^2}dx\right| &\leq \int_B \frac{u|f(x) - f(0)|}{u^2 + x^2}dx \\
&\leq \int_B \frac{u(|f(x)| + |f(0)|)}{u^2 + \delta^2}dx \\
&< 2M\frac{\varepsilon}{8M}\int_{-1}^{1} dx = \varepsilon/2
\end{aligned}$$

Also, we have that

$$\begin{aligned}
\left|\int_A \frac{u(f(x) - f(0))}{u^2 + x^2}dx\right| &\leq \int_A \frac{u|f(x) - f(0)|}{u^2 + x^2}dx \\
&< \frac{\varepsilon}{2\pi}\int_A \frac{u}{u^2 + x^2}dx \\
&= \frac{\varepsilon}{2\pi}2|\arctan \delta/u| \\
&\leq \frac{\varepsilon}{2}
\end{aligned}$$

where we used the fact that the function arctan is bounded by $\pi/2$.

Combining the above 2 estimates gives us that

$$\left|\int_{-1}^{1} \frac{uf(x)}{u^2 + x^2}dx - \int_{-1}^{1} \frac{uf(0)}{u^2 + x^2}dx\right| < \varepsilon$$

Since ε was arbitrary, we get the desired result.

$\square$

2008 - Spring

Exercise 1. *For each $n \in \mathbb{N}$, let*

$$S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n}.$$

Prove that $S_n \to 3$ as $n \to \infty$.

Proof # 1. (K. Nowland) We want to show that

$$\sum_{k=1}^{\infty} \frac{2k-1}{2^k} = 3.$$

Let

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

Note that this converges pointwise for $0 < x < 1$ to $1/(1-x)$. Differentiating,

$$S_n'(x) = \sum_{k=1}^{n} kx^{k-1} = \frac{1}{(1-x)^2} - \frac{x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x}.$$

This converges pointwise to $1/(1-x)^2)$ for $0 < x < 1$. We calculate

$$\sum_{k=1}^{n}(2k-1)x^k = 2x\sum_{k=1}^{n} kx^{k-1} - \sum_{k=1}^{n} x^k = 2xS_n'(x) - S_n(x) + 1$$

For $0 < x < 1$, this converges pointwise as $n \to \infty$ to

$$\sum_{k=1}^{\infty}(2k-1)x^k = \frac{2x}{(1-x)^2} - \frac{1}{1-x} + 1.$$

Plugging in $x = 1/2$ gives

$$\sum_{k=1}^{\infty} \frac{2k-1}{2^k} = \frac{2(1/2)}{1/4} - \frac{1}{1/2} + 1 = 3.$$

[Proof # 2](O. Khalil) Check 2^{nd} proof of 08A2. The idea is to use summation by parts with series whose terms are products of 2 sequences, one of which is easily summable. □

Exercise 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and suppose $f''(x) > 0$ for all $x \in \mathbb{R}$. Prove that f is strictly convex on $\mathbb{R}$; in other words, prove that for all $s, t \in (0,1)$ with $s + t = 1$ and for all $u, v \in \mathbb{R}$ with $u \neq v$, we have*

$$f(su + tv) < sf(u) + tf(v).$$

Proof. (K. Nowland) Without loss of generality, suppose $u < v$. We apply the mean value theorem in the interval $[u, su + tv]$ to see that there exists $x \in (u, tu + sv)$ such that

$$f(su + tv) - f(u) = f'(x)[(s-1)u + tv].$$

Similarly, there exists $y \in (su + tv, v)$ such that

$$f(v) - f(su + tv) = f'(y)[(1-t)v - su].$$

Multiplying the first equation by s and adding it to the second equation multiplied by $-t$ gives

$$f(su + tv) - sf(u) - sf(v) = f'(x)[s(s-1)u + tsv] - f'(y)[t(1-t)v - tsu].$$

We have used the fact that $s + t = 1$. To prove convexity, it suffices to show that the right-hand side of the above is strictly negative. We rewrite this as

$$f'(x)[s(s-1)u + tsv] < f'(y)[t(1-t)v - tsu].$$

Since f'' is strictly positive, $f'(x) < f'(y)$, such that it suffices to show

$$0 \leq s(s-1)u + tsv \leq t(1-t)v - tsu.$$

Since $1 - t = s$ and $s - 1 = -t$ and $v > u$, the above is equivalent to

$$0 \leq -tsu + tsv \leq tsv - tsu,$$

which holds trivially. This completes the proof. □

Exercise 3. *Let (z_n) be a sequence of non-zero complex numbers. Suppose that*

$$\limsup_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| < 1.$$

Prove that $\sum_{n=1}^{\infty} z_n$ converges.

Proof. If $\limsup_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| < 1$, then there exists N so that for all $n \geq N$ one has $\left| \frac{z_{n+1}}{z_n} \right| < (1-\varepsilon)$ for some $(1-\varepsilon)$ between $\limsup_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right|$ and 1. Thus for a fixed k one has $|z_{N+k}| \leq (1-\varepsilon)^k |z_N|$. So $\sum_{k=0}^{\infty} |z_{N+k}| \leq |z_N| \sum_{k=0}^{\infty} (1-\varepsilon)^k < \infty$ by the geometric series test, and since $\sum_{n=0}^{N-1} |z_n|$ is fixed and finite we have that $\sum_{n=0}^{\infty} z_n$ is absolutely convergent, so in particular it is convergent. □

Exercise 4. *Prove the following special case of the Riemann-Lebesgue Lemma: Let $f : [0,1] \to \mathbb{R}$ be continuous. Then*

$$\lim_{t\to\infty} \int_0^1 f(x)\cos(tx)dx = 0. \tag{32}$$

Proof. (H. Lyu) To outline the idea, we start from the observation that the integral of $f(x)\cos(tx)$ over one period will be small if f is almost constant on that interval. Since f is continuous on the compact domain $[0,1]$, it is uniformly continuous. So as t goes to large, the interval of one period for $\cos(tx)$ becomes small, so f becomes almost constant there.

Let $\epsilon > 0$. Since f is continuous on compact interval $[0,1]$, it is bounded and uniformly continuous. Let $M > 0$ be a bound of f. So we can choose $\delta > 0$ such that whenever $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon/8$. Next, choose $t > \max(2\pi/\delta, 4M/\epsilon)$. For each $k \in \mathbb{N}$, denote $I_k = [(k-1)\frac{2\pi}{t}, k\frac{2\pi}{t}]$. Let $N(t)$ be the largest possible k such that $I_k \subset [0,1]$. Write the unit interval as the disjoint union of these intervals and the remainder, i.e.,

$$[0,1] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_{N(t)} \sqcup I_R. \tag{33}$$

Clearly every sub-intervals above has length $\leq 2\pi/t < \delta$. Now we estimate the integral. Suppose $f(x_0) = 0$ for some $x_0 \in I_k$. Then by uniform continuity of f, we have $|f| \leq \epsilon/4$ on I_k. Hence

$$\left|\int_{I_k} f(x)\cos(tx)\,dx\right| \leq \int_{I_k} |f(x)\cos(tx)|\,dx \leq \frac{\epsilon}{4}\int_{I_k} |\cos(tx)|\,dx = \frac{\epsilon}{4t}\int_0^1 |\cos(x)|\,dx = \frac{\epsilon}{4t} \tag{34}$$

On the other hand, suppose $f > 0$ on I_k. Then also by uniform continuity, there is $M_k > 0$ such that $M_k \leq f \leq M_k + \epsilon/4$. Note that $\cos(tx)$ is positive on the first and last quarter, and negative on the second and third quarter on each I_k. Denote the jth quarter of I_k as I_k^j, for $j = 1,2,3,4$. For each function $g : [0,1] \to \mathbb{R}$, define $g^+(x) = \max(0, g(x))$ and $g^-(x) = \max(0, -g(x))$. Then

$$\begin{aligned}
\left|\int_{I_k} f(x)\cos(tx)\,dx\right| &= \left|\int_{I_k} (f(x)\cos(tx))^+\,dx - \int_{I_k} (f(x)\cos(tx))^-\,dx\right| && (35)\\
&= \left|\int_{I_k} f(x)(\cos(tx))^+\,dx - \int_{I_k} f(x)(\cos(tx))^-\,dx\right| && (36)\\
&\leq \left|\int_{I_k^1\cup I_k^4} (M_k + \epsilon/4)\cos(tx)\,dx - \int_{I_k^2\cup I_k^3} M_k\cos(tx)\,dx\right| && (37)\\
&= \frac{2(M_k+\epsilon/4)}{t}\int_0^{\pi/2}\cos(x)\,dx - \frac{2M_k}{t}\int_0^{\pi/2}\cos(x)\,dx = \frac{\epsilon}{2t}. && (38)
\end{aligned}$$

The similar estimation holds if $f < 0$ on I_k. Therefore we have, for each $1 \leq k \leq N(t)$, that

$$\left|\int_{I_k} f(x)\cos(tx)\,dx\right| \leq \frac{\epsilon}{2t}. \tag{39}$$

Lastly, the integral over the remainder interval I_R can be estimated trivially :

$$\left|\int_{I_R} f(x)\cos(tx)\,dx\right| \leq \int_{I_k} |f(x)\cos(tx)|\,dx \leq M\int_{I_k} |\cos(tx)|\,dx \leq Ml(I_R) \leq M\frac{2\pi}{t} \leq \frac{\epsilon}{2}. \tag{40}$$

Now since $N(t) \cdot \frac{2\pi}{t} \leq 1$, combining the previous estimations we obtain

$$\left| \int_0^1 f(x)\cos(tx)\,dx \right| \leq \sum_{k=1}^{N(t)} \left| \int_{I_k} f(x)\cos(tx)\,dx \right| + \left| \int_{I_R} f(x)\cos(tx)\,dx \right| \tag{41}$$

$$\leq \frac{N(t)\epsilon}{2t} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon. \tag{42}$$

This shows the assertion.

□

Exercise 5. *Let $p \in (-1, 0]$. Prove that there exists a convergent sequence (α_n) in $\mathbb{R}$ such that for each $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} k^p = \frac{1}{1+p} n^{1+p} + \alpha_n. \tag{43}$$

Proof #1. (K. Nowland) Note that the above holds trivially for $p = 0$ if we take $\alpha_n = 0$ for all $n \in \mathbb{N}$. Suppose $-1 < p < 0$. We write

$$\begin{aligned}
\sum_{k=1}^{n} k^p &= 1 + \sum_{k=2}^{n} \int_{k-1}^{k} x^p dx + \sum_{k=2}^{n} \int_{k-1}^{k} (k^p - x^p) dx \\
&= \int_1^n x^p dx + 1 + \sum_{k=2}^{n} \int_{k-1}^{k} (k^p - x^p) dx \\
&= \frac{1}{p+1} k^{p+1} - \frac{1}{p+1} + 1 + \sum_{k=2}^{n} \int_{k-1}^{k} (k^p - x^k) dx.
\end{aligned}$$

Let α_n be the last three terms in the above. To show α_n converges, it suffices to show that the series

$$\sum_{k=2}^{n} \int_{k-1}^{k} (k^p - x^p) dx$$

converges as $n \to \infty$. Let $\lceil x \rceil$ be the least integer greater than or equal to x. Then

$$\sum_{k=2}^{n} \int_{k-1}^{k} (k^p - x^p) dx = \int_1^n (\lceil x \rceil^p - x^p) dx.$$

Since $-1 < p < 0$, the integrand is nonpositive for $x \geq 0$, such that it suffices to show that the integral is bounded below, as this will show that the decreasing sequence α_n is bounded below. With p negative, $\lceil x \rceil^p \geq (x+1)^p$. Therefore

$$\int_1^n (\lceil x \rceil^p - x^p) dx \geq \int_1^n [(x+1)^p - x^p] dx.$$

But then

$$\begin{aligned}
\int_1^n [(x+1)^p - x^p]dx &= \int_1^n (x+1)^p dx - \int_1^n x^p dx \\
&= \int_2^{n+1} x^p dx - \int_1^n x^p dx \\
&= \int_n^{n+1} x^p dx - \int_1^2 x^p dx \\
&= \int_n^{n+1} x^p dx - \frac{2^{p+1}}{p+1} + \frac{1}{p+1} \\
&\geq -\frac{2^{p+1}}{p+1} + \frac{1}{p+1}.
\end{aligned}$$

Since the integral is bounded below, the claim is proven.

[Proof #2] (O. Khalil) For $p = 0$, the claim holds trivially. So, we may assume $p < 0$. Let $S_n = \sum_1^n k^p$. Let $I_n = \int_1^n x^p dx$. Since $p < 0$, the function $f(x) = x^p$ is decreasing on $(0, \infty)$ and hence the integral test gives that

$$I_{n+1} \leq S_n \leq 1 + I_n$$

Since the function $f(x)$ is non-negative on $[1, \infty]$, we have that $I_n \leq I_{n+1}$. Moreover, we have that $I_n = \frac{n^{p+1}}{p+1} - \frac{1}{p+1}$. Combining these calculations, we get

$$-\frac{1}{p+1} \leq S_n - \frac{n^{p+1}}{p+1} \leq 1 - \frac{1}{p+1}$$

Let $\alpha_n = S_n - \frac{n^{p+1}}{p+1}$. Then, we have that α_n is bounded. Moreover, we have that

$$\begin{aligned}
\alpha_{n+1} - \alpha_n &= (n+1)^p - \frac{(n+1)^{p+1} - n^{p+1}}{p+1} \\
&= (n+1)^p - \int_n^{n+1} x^p dx
\end{aligned}$$

But, since the function x^p is decreasing, we have that $\int_n^{n+1} x^p dx \geq (n+1)^p(n+1-n) = (n+1)^p$. Hence, we get that $\alpha_{n+1} - \alpha_n \leq 0$. So, α_n is monotonically decreasing and bounded and thus is convergent. Note that we have that $S_n = \frac{n^{p+1}}{p+1} + \alpha_n$, which completes the claim. □

Exercise 6. *Let $f : [0, 2] \to \mathbb{R}$ be continuously differentiable. (Use the appropriate one-sided derivatives at 0 and at 2.) For each $n \in \mathbb{N}$, define $g_n : [0, 1] \to \mathbb{R}$ by $g_n(x) = n(f(x + 1/n) - f(x))$. Prove that the sequence (g_n) converges uniformly on $[0, 1]$.*

Proof. (H. Lyu) We show that the sequence (g_n) converges uniformly to f' on $[0, 1]$. First observe that for each $x \in [0, 1]$ and $n \in \mathbb{N}$, there exists $\xi_{x,n} \in (x, x + \frac{1}{n})$ such that

$$g_n(x) = n(f(x + 1/n) - f(x)) = f'(\xi_{x,n})$$

by the mean value theorem. Now fix $\epsilon > 0$. Since f' is continuous on the compact interval $[0,1]$, it is uniformly continuous, so there exists $\delta > 0$ such that $|f'(x) - f'(y)| < \epsilon$ whenever $|x - y| < \delta$. Let $n > 1/\delta$. Then for every $x \in [0,1]$, since $|x - \xi_{x,n}| < 1/n < \delta$, we have

$$|f'(x) - g_n(x)| = |f'(x) - f'(\xi_{x,n})| < \epsilon.$$

This shows $\|f' - g_n\|_u < \epsilon$ whenever $n > 1/\delta$. Thus $g_n \to f'$ uniformly on $[0,1]$. $\square$

2007 - Autumn

Exercise 1. *Let (a_k) be a decreasing sequence of positive real numbers. Suppose that the series $\sum_{m=1}^{\infty} m a_{m^2}$ converges. Prove that the series $\sum_{k=1}^{\infty} a_k$ converges.*

Proof. (K. Nowland) Note that the distance between consecutive squares is $(m+1)^2 - m^2 = 2m+1 \leq 3m$. Since the sequence is decreasing, we have

$$\sum_{k=1}^{n} a_k \leq \sum_{m=1}^{m} (2m+1) a_{m^2} \leq 3 \sum_{m=1}^{\infty} m a_m^2 < \infty.$$

The partial sums are increasing, since each term is positive. The comparison test implies that $\sum a_k$ converges. $\square$

Exercise 2. *Let $f(x) = xe^{-x^2} \int_0^x e^{s^2} ds$ for all $x \in \mathbb{R}$. Is f bounded on $\mathbb{R}$?*

Proof. (R. Garrett) First, we notice the following: for any $x > 0$, $\int_0^x \frac{s}{x} e^{s^2} ds \leq \int_0^x e^{s^2} ds \leq \int_0^x e^{sx} ds$. These inequalities hold since e^t is an increasing function and $s \in [0, x]$, which means $\frac{s}{x} \leq 1$ and $sx \leq x^2$. Evaluating the integrals on the left and right, we obtain $\frac{1}{2x}(e^{x^2} - 1) \leq \int_0^x e^{s^2} ds \leq \frac{1}{x}(e^{x^2} - 1)$ for all $x > 0$. Since $\frac{1}{2x}(e^{x^2} - 1) \leq \int_0^x e^{s^2} ds \leq \frac{1}{x}(e^{x^2} - 1)$ for all $x > 0$, multiplying by xe^{-x^2} we get $0 \leq \frac{1}{2}(1 - e^{-x^2}) \leq xe^{-x^2} \int_0^x e^{s^2} ds \leq (1 - e^{-x^2}) \leq 1$. So, $f(x)$ is bounded for all $x > 0$. The quantity is 0 when $x = 0$, so it remains to consider the case when $x < 0$. When $x < 0$, $f(x) = -(-x)e^{-(-x)^2} \int_0^x e^{s^2} ds = -(-x)e^{-(-x)^2}(-1) \int_0^{-x} e^{s^2} ds = f(-x)$, where we obtained these equalities by the symmetry of e^{-x^2}, and as a consequence we obtain boundedness when $x < 0$. Thus, $f(x)$ is bounded on all of $\mathbb{R}$.

$\square$

Exercise 3. *Let $f \in C([a, b])$, where $a, b \in \mathbb{R}$ with $a < b$. Let*

$$M = \sup \{|f(x)| : x \in [a, b]\}$$

and for each $n \in \mathbb{N}$, let

$$I_n = \left(\int_a^b |f(x)|^n dx \right)^{1/n}.$$

Prove that $\lim_{n \to \infty} I_n = M$.

Proof. (O. Khalil) Since $|f|$ is continuous and $[a, b]$ is compact, then M is finite by the extreme value theorem. Moreover, the EVT gives that M is attained i.e. there exists $y \in [a, b]$ such that $|f(y)| = M$. Now, for each $n \in \mathbb{N}$, we have that

$$I_n \leq \left(\int_a^b M^n dx \right)^{1/n} = M(b-a)^{1/n}$$

Hence, we get that

$$\limsup_{n \to \infty} I_n \leq \lim_{n \to \infty} M(b-a)^{1/n} = M$$

Now, let $\varepsilon > 0$ be arbitrary. By the continuity of $|f|$ at y, there exists $\delta > 0$ ($\delta \neq 0$) such that $\forall x \in (y - \delta, y + d) \cap [a, b]$, we have that $|f(x)| - M| < \varepsilon$. And, hence, we have that for all such x, $|f(x)| > M - \varepsilon$. Let $a_1 = max\{a, y - \delta\}$ and $b_1 = min\{b, y + \delta\}$. Note that $b_1 - a_1 > 0$ and that $b_1 \neq a_1$. Hence, for each $n \in \mathbb{N}$, we get the following

$$I_n \geq \left(\int_{a_1}^{b_1} |f(x)|^n dx \right)^{1/n} \geq \left(\int_{a_1}^{b_1} (M - \varepsilon)^n dx \right)^{1/n} = (M - \varepsilon)(b_1 - a_1)^{1/n}$$

Therefore, we have that

$$\liminf_{n\to\infty} I_n \geq \lim_{n\to\to\infty} (M - \varepsilon)(b_1 - a_1)^{1/n} = M - \varepsilon$$

But, since ε was arbitrary, we get that

$$\liminf_{n\to\infty} I_n \geq \lim_{\varepsilon\to 0^+} M - \varepsilon = M$$

Combining these estimates

$$\liminf_{n\to\infty} I_n \geq M \geq \limsup_{n\to\infty} I_n$$

And, hence, $\lim_{n\to\infty} I_n$ exists and is equal to M as desired. $\square$

Exercise 4. *Let $f : (0, \infty) \to \mathbb{R}$ be C^1. Suppose that f has at least one zero and that $f(x) \to 0$ as $x \to \infty$.*

(a) Prove that f' has at least one zero.

(b) Suppose in addition that f is C^2 and that f'' has only finitely many zeros. Prove that $f'(x) \to 0$ as $x \to \infty$.

Proof. (O. Khalil)

(a) Let $x_o(0, \infty)$ be such that $f(x_o) = 0$. By continuity of f, there exists $\delta_1 > 0$ so that for all $x \in (x_o - \delta_1, x_o + \delta_1)$, we have $|f(x)| < 1$. Let $x_2 \in (x_o, x_o + \delta_1)$. Similarly, there exists $\delta_2 > 0$ such that for all $x \in (x_o - \delta_2, x_o + \delta_2)$, we have that $|f(x)| < |f(x_2)|$. Let $x_1 \in (x_o - \delta_2, x_o$. Hence, $x_1 < x_2$. Now, since $f(x) \to 0$ as $x \to \infty$, let $x_3 > x_2$ be large enough so that $|f(x_3)| < |f(x_2)|$. Now, suppose that $f(x_2) > 0$. Then, we have that $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$. Thus, the mean value theorem gives the following

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c_1) > 0$$
$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(c_2) < 0$$

for some $c_1 \in (x_1, x_2)$ and $c_2 \in (x_2, x_3)$. Hence, by the intermediate value theorem, since f' is continuous, there exists $c \in (c_1, c_2)$ such that $f'(c) = 0$ as desired.

(b) Let $\varepsilon > 0$ be fixed. We wish to show that $|f'(x)| < \varepsilon$ for sufficiently large x. Let $x_o = \max_{x\in(0,\infty)} f''(x) = 0$. x_o exists and is finite since f'' has only finitely many zeros. Hence, by the continuity of f'', it doesn't change sign after x_o and therefore f' is monotone on (x_o, ∞). By the mean value theorem, for each $n \in \mathbb{N}$, there exists $c_n \in (n, n+1)$ such that $|f(n) - f(n+1)| = |f'(c_n)|$. But, since$f(x) \to 0$ as $x \to \infty$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, we have that $|f(n)| < \varepsilon/2$. Hence, we get that for all $n > N$,

$$|f'(c_n)| \leq |f(n)| + |f(n+1)| < \varepsilon$$

Now, for all $x > N+1$, we have that there exists $n > N$ such that $x \in [c_n, c_{n+1}]$. So, by the monotonicity of f' on $[c_n, c_{n+1}]$, we get that either $-\varepsilon < f'(c_n) \leq f'(x) \leq f'(c_{n+1}) < \varepsilon$ or $\varepsilon > f'(c_n) \geq f'(x) \geq f'(c_{n+1}) > -\varepsilon$. Either way, we get that $|f(x)| < \varepsilon$ for all $x > N+1$ as desired.

□

Exercise 5. *Prove or disprove: For each uniformly continuous function $f : [0, \infty) \to \mathbb{R}$, if the improper Riemann integral $\int_0^\infty f(t)dt$ converges, then $\lim_{x\to\infty} f(x) = 0$.*

Proof. (O. Khalil) (See Ex. 6, Spring 06 for a shorter proof).

Let $\varepsilon > 0$ be fixed. We wish to show that $|f(x)| < \varepsilon$. By uniform continuity, $\exists \delta > 0$ such that for all $x, y > 0$, whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$. Also, by the Cauchy convergence criterion, since $\int_0^\infty f(t)dt$ converges, then there exists $x_o > 0$ such that for all $x > y > x_o$, we have that $\left|\int_x^y f(t)dt\right| < \frac{\varepsilon\delta}{2}$.

Now, let $x > x_o + \delta$. Since f is continuous on $[x - \delta/2, x + \delta/2]$, then by the extreme value theorem, there exists $c, b \in [x - \delta/2, x + \delta/2]$ such that

$$\begin{aligned} f(c) &= \inf_{t\in[x-\frac{\delta}{2},x+\frac{\delta}{2}]} f(t) \\ f(b) &= \sup_{t\in[x-\frac{\delta}{2},x+\frac{\delta}{2}]} f(t) \end{aligned}$$

Hence, we get the following

$$\begin{aligned} \frac{\varepsilon\delta}{2} &> \left|\int_{x-\delta/2}^{x+\delta/2} f(t)dt\right| \geq \int_{x-\delta/2}^{x+\delta/2} f(t)dt \geq \int_{x-\delta/2}^{x+\delta/2} f(c)dt = f(c)\delta \\ -\frac{\varepsilon\delta}{2} &< \int_{x-\delta/2}^{x+\delta/2} f(t)dt \leq \int_{x-\delta/2}^{x+\delta/2} f(b)dt = f(b)\delta \end{aligned}$$

But, since $\delta > 0$, then, $f(c) < \varepsilon/2$ and $f(b) > -\varepsilon/2$. Moreover, we have that $|c - x| < \delta$ and $|b - x| < \delta$. Thus, we have that $f(x) - f(c) \leq |f(x) - f(c)| < \varepsilon/2$ and that $f(x) - f(b) > -\varepsilon/2$ Hence, we get that

$$\begin{aligned} f(x) &< f(c) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon \\ f(x) &> f(b) - \varepsilon/2 > -\varepsilon/2 - \varepsilon/2 = -\varepsilon \end{aligned}$$

Therefore, we get that $|f(x)| < \varepsilon$ as desired. □

Exercise 6. *Let $\mathscr{F}$ be the collection of all twice continuously differentiable functions f on $\mathbb{R}$ satisfying $f \geq 0$ on $\mathbb{R}$ and $f''(x) \leq 1$ on $\mathbb{R}$. Find a constant $C \in (0, \infty)$ such that for each $f \in \mathscr{F}$ and for each $x \in \mathbb{R}$, we have*

$$f'(x)^2 \leq Cf(x). \tag{44}$$

Justify the value you find for C.

Proof. (O. Khalil) Let $x \in \mathbb{R}$ and $f \in \mathscr{F}$ be fixed. Let $h \in \mathbb{R}$ be arbitrary. Then, Taylor's expansion with Lagrange remainder gives

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(c)}{2}$$

for some c between x and $x+h$. Since $f \geq 0$ for all x, then we get that

$$0 \leq f(x) + hf'(x) + \frac{h^2 f''(c)}{2} \leq f(x) + hf'(x) + \frac{h^2}{2}$$

But, the above inequality is valid for any h. Hence, as a polynomial in h, it has at most one real solution and so its discriminant is ≤ 0. Thus, we get that

$$f'(x)^2 - 2f(x) \leq 0$$

Rearranging, we get

$$f'(x)^2 \leq 2f(x)$$

Since x and f were arbitrary, and letting $C = 2$ (doesn't depend on a particular choice of f), then 44 is verified. $\square$

2007 - Spring

Exercise 1. *Let $f : [0,\infty) \to [0,\infty)$ be decreasing and suppose that $\int_0^\infty f(x)dx$ converges. Prove that*

$$\lim_{h\to 0^+}\left(h\sum_{n=1}^{\infty} f(nh)\right) = \int_0^\infty f(x)dx. \tag{45}$$

Proof. (O. Khalil) Let $\varepsilon > 0$ be fixed. By the Cauchy criterion, since $\int_0^\infty f(x)dx$ converges, then there exists $N \in \mathbb{R}$ such that for all $x_o > N$

$$\int_{x_o}^\infty f(x)dx < \varepsilon/4$$

Let $\delta > 0$ be such that $2\delta f(0) < \varepsilon/2$. Let $0 < h < \delta$ and let $K \in \mathbb{N}$ be large enough so that $Kh \geq N$. Now, write

$$\begin{aligned}
\left|h\sum_{n=1}^{\infty} f(nh) - \int_0^\infty f(x)dx\right| &= \left|\sum_{n=1}^{\infty}\left(hf(nh) - \int_{(n-1)h}^{nh} f(x)dx\right)\right| \\
&= \left|\sum_{n=1}^{\infty}\int_{(n-1)h}^{nh}\left(f(nh) - f(x)\right)dx\right| \\
&\leq \sum_{n=1}^{\infty}\int_{(n-1)h}^{nh}|f(nh) - f(x)|dx \\
&= \sum_{n=1}^{K}\int_{(n-1)h}^{nh}(f(x) - f(nh))dx + \sum_{n=K+1}^{\infty}\int_{(n-1)h}^{nh}(f(x) - f(nh))dx
\end{aligned}$$

where the absolute values were dropped in the last step since f is decreasing. Moreover, on each interval of the form $[(n-1)h, nh]$, since f is decreasing, we have that $f(x) \geq f(nh)$. Plugging these estimates to bound the above expression

$$\begin{aligned}
\left|h\sum_{n=1}^{\infty} f(nh) - \int_0^\infty f(x)dx\right| &\leq \sum_{n=1}^{K}\int_{(n-1)h}^{nh}(f((n-1)h) - f(nh))dx + \sum_{n=K+1}^{\infty}\int_{(n-1)h}^{nh}2f(x)dx \\
&= \sum_{n=1}^{K} h(f((n-1)h) - f(nh)) + \int_{Kh}^{\infty} 2f(x)dx \\
&= h(f(0) - f(Kh)) + \int_{Kh}^{\infty} 2f(x)dx \\
&\leq 2hf(0) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

where in the last step we used the Cauchy criterion along with the fact that $Kh > N$ and that $f(0) - f(Kh) < 2f(0)$ along with the choice of δ. Hence, $\lim_{h\to 0^+}\left(h\sum_{n=1}^{\infty} f(nh)\right) = \int_0^\infty f(x)dx$ as desired. □

Proof. (S. Meehan; This proof was inspired by laziness, trying to avoid the above solution.) Let $f : [0, \infty) \to [0, \infty)$ be decreasing and suppose that $\int_0^\infty f(x)dx$ converges. Since $f(x)$ is both positive and decreasing, we have via the integral test:

$$hf(h) + \int_{2h}^{\infty} f(x)dx \geq h \sum_{n=1}^{\infty} f(nh) \geq \int_{h}^{\infty} f(x)dx.$$

Clearly as $h \to 0^+$, we have

$$\int_{2h}^{\infty} f(x)dx, \int_{h}^{\infty} f(x)dx \to \int_{0}^{\infty} f(x)dx.$$

Also, note that $hf(h) \to 0$. Hence the result follows (via squeeze theorem). □

Exercise 2. *Let $C[0,1]$ denote the set of continuous real-valued functions on $[0,1]$. Let $p \in (1, \infty)$. For each continuous function $f : [0,1] \to \mathbb{R}$, let*

$$\|f\| = \left(\int_0^1 |f|^p \right)^{1/p}.$$

Prove Minkowski's inequality in this setting, namely

$$\|f + g\| \leq \|f\| + \|g\| \tag{46}$$

for all $f, g \in C[0,1]$.

Proof #1. (H. Lyu) (This elegant proof is due to Prof. N. Falkner) Let $\alpha = \|f\|$ and $\beta = \|g\|$. Notice that $\alpha, \beta < \infty$, since any continuous function on $[0,1]$ is bounded by the extreme value theorem, the integral over the function on the compact interval $[0,1]$ is finite, so its norm is also finite. If $\alpha = 0$, then $\int_0^1 |f(x)|^p \, dx = 0$, and since $x \mapsto |f(x)|$ is continuous and nonnegative on $[0,1]$, this implies $|f(x)| = 0$ for all $x \in [0,1]$, so f is identically zero. Then the assertion is trivial. Similarly, $\beta = 0$ leads to a trivial case. So we may assume $\alpha, \beta > 0$. Further, we may assume $\alpha + \beta = 1$ by scaling; for instance, consider $f/(\alpha + \beta)$ instead of f and $g/(\alpha + \beta)$ instead of g. Now let $\phi = |f|/\alpha$ and $\psi = |g|/\beta$. Then clearly they are continuous, and $\int_0^1 \phi^p = \int_0^1 \psi^p = 1$. Also note that $|f + g| \leq |f| + |g| = \alpha\phi + \beta\psi$. Since the map $x \mapsto x^p$ is increasing and convex on $[0, \infty)$, and since $\alpha + \beta = 1$, we have

$$|f + g|^p \leq (|f| + |g|)^p \leq \alpha\phi^p + \beta\psi^p.$$

Now this gives

$$\|f + g\|^p = \int_0^1 |f + g|^p \leq \alpha \int_0^1 \phi^p + \beta \int_0^1 \psi^p = \alpha + \beta = 1.$$

Therefore

$$\|f + g\| \leq 1 = \alpha + \beta = \|f\| + \|g\|.$$

□

Proof #2. (K. Nowland) This is a more time-consuming way to do the above proof, but is another way to do it. The first thing to do is prove Young's inequality, then prove Hölder's inequality, then prove Minkowski's inequality. Young's inequality says that for $a, b \geq 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where $1/p + 1/q = 1$. If either a or b is zero, this is trivial. If neither are zero, this follows from the concavity of the logarithm.

$$\log(a^p/p + a^q/q) \geq (1/p)\log a^p + (1/q)\log b^q = \log ab.$$

Then, for Hölder, we rescale and f and g (nonzero, because otherwise it is trivial), by $f/\|f\|_p$ and $g/\|g\|_q$ such that we only need to show that

$$\int_0^1 |f(x)g(x)|dx \leq 1.$$

By Young's inequality,

$$\int_0^1 |f(x)g(x)|dx \leq (1/p)\int_0^1 |f(x)|^p dx + (1/q)\int_0^1 |g(x)|dx = \frac{1}{p} + \frac{1}{q} = 1,$$

as desired. Then if $p > 1$, its conjugate is $p/(p-1) > 1$. Again we can scale so that $\|f\| + \|g\| = 1$ by dividing the original f and g by the sum of their norms.

$$\begin{aligned}
\int_0^1 |f(x)+g(x)|^p dx &\leq \int_0^1 |f+g||f+g|^{p-1}dx \\
&\leq \int_0^1 (|f|+|g|)|f+g|^{p-1}dx \\
&\leq (\|f\| + \|g\|)\left(\int_0^1 |f+g|^p dx\right)^{1-1/p} \\
&= \left(\int_0^1 |f+g|^p dx\right)^{1-1/p}.
\end{aligned}$$

Dividing by the term on the right gives the desired result. □

Exercise 3. *Let $f : [0,1] \to \mathbb{R}$ be continuous. Define $\varphi : [0,1] \to \mathbb{R}$ by*

$$\varphi(x) = \int_0^x e^{-xt} f(t)dt.$$

Prove that φ is differentiable and find φ'.

Proof. (H. Lyu) We show that

$$\varphi'(x) = e^{-x^2} f(x) - \int_0^x te^{-xt} f(t)\,dt. \tag{1}$$

Fix $x, h \in \mathbb{R}$. Note that

$$\frac{\varphi(x+h)-\varphi(x)}{h} = \frac{1}{h}\int_x^{x+h} e^{-(x+h)t} f(t)\,dt + \int_0^x \frac{e^{-ht}-1}{h} e^{-xt} f(t)\,dt. \tag{2}$$

Let $F(y) = \int_0^y e^{-(x+h)t} f(t)\,dt$. Then F is differentiable and $F'(y) = e^{-(x+h)y} f(y)$. So by the mean value theorem,

$$\frac{1}{h}\int_x^{x+h} e^{-(x+h)t} f(t)\,dt = \frac{F(x+h)-F(x)}{h} = e^{-(x+h)\xi_{x,h}} f(\xi_{x,h}) \tag{3}$$

for some $\xi_{x,h} \in (x-|h|, x+|h|)$. As $h \to 0$, $\xi_{x,h} \to x$ and since f is continuous, $f(\xi_{x,h}) \to f(x)$. Thus we have

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} e^{-(x+h)t} f(t)\,dt = \lim_{h\to 0} e^{-(x+h)\xi_{x,h}} f(\xi_{x,h}) = e^{-x^2} f(x). \tag{4}$$

It remains to deal with the limit of the second term in (2) as $h \to 0$. It suffices to show that the integrand converges to $t \mapsto -te^{-xt} f(t)$ uniformly on $[0,1]$, as $h \to 0$, since then we can switch the integral and limit to get the desired convergence. To this end, write

$$\int_0^x \frac{e^{-ht}-1}{h} e^{-xt} f(t)\,dt = -\int_0^x \frac{e^{-ht}-1}{-ht} te^{-xt} f(t)\,dt. \tag{5}$$

Denote $g_h(t) = \frac{e^{-ht}-1}{-ht}$. We claim that $g_h \to 1$ uniformly on $[0,1]$ as $h \to 0$. Let $\epsilon > 0$. Note that since $t \mapsto e^t$ is continuous on the compact interval $[-1,1]$, it is uniformly continuous. So there is $\delta > 0$ such that $|1-e^t|$ whenever $|t| < \delta$. For each $h \in \mathbb{R}$ and $t \in [0,1]$, there is $\xi_{h,t} \in (-|ht|, |ht|)$ such that

$$\frac{e^{-ht}-1}{-ht} = e^{\xi_{h,t}} \tag{6}$$

by mean value theorem. Now let $|h| < \delta$. Then since $t \in [0,1]$, $|ht| \le |h| < \delta$, so $|\xi_{h,t}| < \delta$. Thus

$$\left|1 - \frac{e^{-ht}-1}{-ht}\right| = |1-e^{\xi_{h,t}}| < \epsilon. \tag{7}$$

This shows the claim. Note that this implies the integrand $\frac{e^{-ht}-1}{-ht} te^{-xt} f(t)$ converges to $te^{-xt} f(t)$ as $h \to 0$; the function $(x,t) \to te^{-xt} f(t)$ is continuous from the compact domain $[0,1]^2$ to $\mathbb{R}$, so it is bounded by some number, say, $M > 0$; so for a fixed $\epsilon > 0$,

$$\left|\frac{e^{-ht}-1}{-ht} te^{-xt} f(t) - te^{-xt} f(t)\right| \le \left|\frac{e^{-ht}-1}{-ht} - 1\right| \cdot |te^{-xt} f(t)| \le \epsilon M \tag{8}$$

for all $t \in [0,1]$, whenever $|h|$ is small. Therefore we have

$$\lim_{h\to 0}\int_0^x \frac{e^{-ht}-1}{h} e^{-xt} f(t)\,dt = \int_0^x \lim_{h\to 0} \frac{e^{-ht}-1}{h} e^{-xt} f(t)\,dt = -\int_0^x te^{-xt} f(t)\,dt. \tag{9}$$

This shows the assertion.

□

Remark. [Uniform Convergence and Integration] Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for $n \in \mathbb{N}$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and

$$\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

Proof. Let $\epsilon > 0$ be given. Since $f_n \to f$ uniformly on $[a, b]$, there is $N \in \mathbb{N}$ such that $|f_n - f| \le \epsilon$ for all $n \ge N$ on $[a, b]$. Then for all $n \ge N$, we have

$$\left|\overline{\int_a^b} f_n\, d\alpha - \overline{\int_a^b} f\, d\alpha\right| = \left|\overline{\int_a^b} f_n - f\, d\alpha\right| \le \overline{\int_a^b} |f_n - f|\, d\alpha \le \epsilon(\alpha(b) - \alpha(a)). \tag{1}$$

We have the similar result for the lower Riemann integrals. Now since each $f_n \in \mathcal{R}(\alpha)$, we can choose N large enough so that $\left|\underline{\int_a^b} f_n\, d\alpha - \overline{\int_a^b} f_n\, d\alpha\right| < \epsilon(\alpha(b) - \alpha(a))$. Then for all $n \ge N$, we have

$$\begin{aligned}
\left|\underline{\int_a^b} f\, d\alpha - \overline{\int_a^b} f\, d\alpha\right| &\le \left|\underline{\int_a^b} f_n\, d\alpha - \underline{\int_a^b} f\, d\alpha\right| + \left|\underline{\int_a^b} f_n\, d\alpha - \overline{\int_a^b} f_n\, d\alpha\right| + \left|\overline{\int_a^b} f_n\, d\alpha - \overline{\int_a^b} f\, d\alpha\right| \\
&\le 3\epsilon(\alpha(b) - \alpha(a)).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows $f \in \mathcal{R}(\alpha)$. Then (1) shows $\int f_n\, d\alpha \to \int f\, d\alpha$. ☐

Exercise 4. *Prove that the equation*

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$$

has exactly one solution in $\mathbb{R}$ *if* n *is odd and no solutions if* n *is even.*

Proof. (K. Nowland) Let $P_n(x)$ be the left-hand side of the above equation. Suppose n is even. Note that as $x \to \pm\infty$, $P_n(x) \to \infty$. In particular, $P_n(x)$ attains its infimum. If the infimum is positive, then $P_n(x)$ has no roots. The derivative is

$$P_n'(x) = -1 + x - x^2 + \cdots + (-1)^n x^{n-1} = -\frac{1 - x^n}{1 + x}.$$

The second equality holds only if $x \ne -1$. Note that -1 is not a zero of $P_n'(x)$, since then every term of $P_n'(x)$ will be negative. The only real zero of $P_n'(x)$ is $x = 1$, such that $P_n(x)$ must obtain its infimum at $x = 1$. We see that

$$P_n(1) = (1 - 1) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \frac{1}{n} > 0.$$

Since the infimum of $P_n(x)$ is positive, it follows that $P_n(x)$ has no zero if n is even.

Now let n be odd. Note that $\lim_{x\to-\infty} P_n(x) = \infty$ and $\lim_{x\to\infty} P_n(x) = -\infty$. Since $P_n(x)$ is a polynomial it is continuous. The intermediate value theorem implies that $P_n(x)$ has at least one zero. If $P_n(x)$ has another zero, it must have either a local maximum or local minimum by the continuity of the derivative (it is also a polynomial) and Rolle's theorem. The derivative is

$$P_n'(x) = -1 + x - x^2 + \cdots + (-1)^n x^{n-1} = -\frac{1+x^n}{1+x},$$

where the second equality holds only if $x \neq -1$. Note that -1 is not a zero of $P_n'(x)$ since, as above, every term in $P_n'(x)$ will be negative. Since this is the only possible real zero of $1+x^n$, $P_n'(x)$ has no real zeros and thus $P_n(x)$ has no local maxima or minima. This implies $P_n(x)$ has exactly one real zero when n is odd. This proves the claim. □

Exercise 5. *Let (a_n) be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $na_n \to 0$ as $n \to \infty$.*

Proof. (K. Nowland) Because $\sum a_n < \infty$, the Cauchy condensation theorem implies that $\sum 2^k a_{2^k} < \infty$. It follows that $2^k a_{2^k} \to 0$ as $k \to \infty$. Write $m = 2^k + n$ where $0 \leq n < 2^k$. Then

$$ma_m = (2^k + n)a_m < (2^k + 2^k)a_m = 2(2^k a_m) \leq 2(2^k a_{2^k}),$$

where the last inequality relies on the fact that the sequence consists of decreasing positive numbers. Let $\varepsilon > 0$. Since $2^k a_{2^k} \to 0$ as $k \to \infty$, it follows that there exists $K \in \mathbb{N}$ such that $k \geq K$ implies $2^k a_{2^k} < \varepsilon/2$. If $m \geq 2^K$, taking $m = 2^k + n$ as above,

$$0 < ma_m < 2(2^k a_{2^k}) < \varepsilon.$$

Therefore $ma_m \to 0$ as $m \to \infty$. □

Exercise 6. *Let f be a continuous function on $[0,1]$. Find*

$$\lim_{n\to\infty} n \int_0^1 x^{n+2} f(x)dx.$$

Justify your answer.

Proof. (H. Lyu) We show that the limit is $f(1)$. First observe

$$\int_0^1 nx^{n+2}\,dx = \left[\frac{n}{n+3}x^{n+3}\right]_0^1 = \frac{n}{n+3} < 1.$$

So

$$\begin{aligned} \left|n\int_0^1 x^{n+2}f(x)dx - f(1)\right| &\leq \left|n\int_0^1 x^{n+2}f(x)dx - \frac{n}{n+3}f(1)\right| + \left|\frac{n}{n+3} - 1\right| \cdot |f(1)| \\ &= \left|\int_0^1 nx^{n+2}f(x)dx - \int_0^1 nx^{n+2}f(1)\,dx\right| + \left|\frac{n}{n+3} - 1\right| \cdot |f(1)| \\ &= \left|\int_0^1 nx^{n+2}(f(x) - f(1))dx\right| + \left|\frac{n}{n+3} - 1\right| \cdot |f(1)| \end{aligned}$$

Since $\frac{n}{n+3} \to 1$ as $n \to \infty$, it suffices to show

$$\lim_{n\to\infty} \left| \int_0^1 nx^{n+2} g(x)dx \right| = 0. \tag{1}$$

where $g(x) = f(x) - f(1)$. Let $\epsilon > 0$. Since g is continuous at 1 and $g(1) = 0$, there is $0 < \delta < 1$ such that $|g(x)| < \epsilon/2$ whenever $1-\delta < x < 1$. On the other hand, g is continuous on the compact interval $[0, 1-\delta]$ so there is a bound M_δ for $|g|$ on this interval. Noting that nx^{n+1} is increasing, we have

$$\begin{aligned} \left| \int_0^1 nx^{n+2} g(x)dx \right| &= \left| \int_0^{1-\delta} nx^{n+2} g(x)dx \right| + \left| \int_{1-\delta}^1 nx^{n+2} g(x)dx \right| \\ &\leq (1-\delta)n(1-\delta)^{n+2} M_\delta + \frac{\epsilon}{2} \left| \int_{1-\delta}^1 nx^{n+2}\, dx \right| \\ &\leq (1-\delta)n(1-\delta)^{n+2} M_\delta + \frac{\epsilon}{2}. \end{aligned}$$

Since $n(1-\delta)^{n+2} \to 0$ as $n \to \infty$, there is $N \in \mathbb{N}$ such that for each $n > N$, we have $(1-\delta)n(1-\delta)^{n+2} M_\delta < \epsilon/2$. Thus for each $n > N$, we have

$$\left| \int_0^1 nx^{n+2} g(x)dx \right| \leq \epsilon.$$

This shows (1), and hence the assertion. $\square$

2006 - Autumn

Exercise 1. *Prove the following version of l'Hopital's rule: let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with $g'(x)$ never 0, and suppose that as $x \to \infty$, we have $g(x) \to \infty$ and $f'(x)/g'(x) \to \infty$. Then $f(x)/g(x) \to \infty$ also.*

Proof. (K. Nowland) In the sequel, we assume x is large enough that $g(x) > 0$. This is possible since $g(x) \to \infty$ as $x \to \infty$.

Let $M > 0$ be fixed. It suffices to show that for all x sufficiently large $f(x)/g(x) > M$. Since M is arbitrary, it must be that $f(x)/g(x) \to \infty$.

Let $(x_n) \subset \mathbb{R}$ be any strictly increasing sequence tending to infinity. Since f and g are differentiable on all of $\mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$, there exists $c_n \in (x_{n+1}, x_n)$ such that

$$\frac{f(x_{n+1}) - f(x_n)}{g(x_{n+1}) - g(x_n)} = \frac{f'(c_n)}{g'(c_n)}.$$

Since $x_{n+1} > c_n > x_n$ for all n, c_n is a strictly increasing sequence tending to infinity as $n \to \infty$. Since $f'(x)/g'(x)$ tends to positive infinity as x does, there exists $y \in \mathbb{R}$ such that $f'(x)/g'(x) > M$ for all $x \geq y$. Since (c_n) tends to infinity, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\frac{f(x_{n+1}) - f(x_n)}{g(x_{n+1}) - g(x_n)} = \frac{f'(c_n)}{g'(c_n)} > M.$$

Therefore as $n \to \infty$,

$$\frac{f(x_{n+1}) - f(x_n)}{g(x_{n+1}) - g(x_n)} \to \infty$$

as $n \to \infty$. By Cesáro-Stolz, $f(x_n)/g(x_n) \to \infty$ as $n \to \infty$. By the continuity of f/g, this is independent of the choice of sequence such that $f(x)/g(x) \to \infty$.

It would probably be a good idea to prove Cesàro-Stolz in this case. Let (a_n) and (b_n) be sequences such that $0 < b_1 < b_2 < \cdots$ with $b_n \to \infty$ as n tends to infinity. Suppose

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = +\infty.$$

Then $a_n/b_n \to \infty$. Let $A_n = a_{n+1} - a_n$ and $B_n = b_{n+1} - b_n$. Let $M > 0$. Then there exists $N \in \mathbb{N}$ such that for $n \geq N$, $A_n/B_n > M$. Note then that

$$A_n > MB_n,$$

for all $n \geq N$. Thus

$$A_{n+k} + A_{n+k-1} + \cdots + A_n > M(B_{n+k} + B_{n+k-1} + \cdots + B_n),$$

such that

$$\frac{A_{n+k} + A_{n+k-1} + \cdots + A_n}{B_{n+k} + B_{n+k-1} + \cdots + B_n} > M$$

for $n \geq N$ and all $k \in \mathbb{N}$. Write

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{A_n + A_{n-1} + \cdots + A_N + A_{N_1} + \cdots + A_1}{B_n + B_{n-1} + \cdots + B_N + B_{N-1} + \cdots + B_1}, \\ &= \frac{A_1 + \cdots + A_{N-1}}{b_n} + \frac{A_N + \cdots + A_n}{B_N + \cdots + B_n}\left(1 - \frac{B_1 + \cdots + B_{N-1}}{b_n}\right).\end{aligned}$$

Note that the first term tends to zero as $n \to \infty$. Let $\varepsilon > 0$. Then there exists $N' \geq N$ such that the first term is less than ε is absolute value for all $n \geq N'$. Similarly, the second term in the parentheses can be made less than ε in absolute value for $n \geq N''$ for some $N'' \geq N' \geq N$. Thus for $n \geq N''$,

$$\frac{a_n}{b_n} > -\varepsilon + M(1 - \varepsilon).$$

Since ε was arbitrary, this implies that $a_n/b_n > M$ for n sufficiently large. This completes the proof of Cesàro-Stolz. □

Exercise 2. *Define $f : [0,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0,1] \cap \mathbb{Q}, \end{cases}$$

where p and q are coprime integers. Prove that f is Riemann integrable.

Proof. (K. Nowland) By the Lebesgue integrability theorem, Riemann integrability of f on $[0,1]$ is equivalent to f being bounded and being discontinuous on a set of Lebesgue measure zero. Clearly f is bounded by 1. If we can show that f is continuos at $x \in [0,1] \setminus \mathbb{Q}$, then we are done, as the rational points have zero measure. To prove that the rational points have zero measure, let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}$. Let $\varepsilon > 0$. Define the intervals $I_n = (q_n - \varepsilon 2^{-n-1}, q_n + \varepsilon 2^{-n-1})$. Since

$$\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon$$

by the convergence of the geometric series. Since ε was arbitrary, the outer regularity of Lebesgue measure implies that $\mathbb{Q}$ is a set of Lebesgue measure zero.

Now let $x \in [0,1]$ be irrational and let $\varepsilon > 0$. We must find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| = |f(y)| < \varepsilon$. Since $f = 0$ on the irrationals, it suffices to show that $0 < f(y) < \varepsilon$ for rational y sufficiently close to x. Let N be so large that $1/N < \varepsilon$. Then for all $q > N$ we have $0 < f(p/q) < \varepsilon$ where p and q are coprime. Let δ be the distance from x to the nearest rational with denominator less than N. This distance is strictly positive since there are only finitely many such rational numbers in $[0,1]$, and x is irrational. Then $|x - y| < \delta$ implies that if y is rational, then $0 < f(y) < \varepsilon$. Therefore f is continuous at x. Since x was an arbitrary irrational number in $[0,1]$,, we conclude that f is continuous at all $x \in [0,1] \setminus \mathbb{Q}$ which completes the proof that f is Riemann integrable. □

Exercise 3. *Let $a_k \geq 0$ for each $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} \frac{\sqrt{a_k}}{k}$ also converges.*

Proof. (E. Nash) Define a set $S := \{k \in \mathbb{N} : a_k > \frac{1}{k^2}\}$. Then define two new sequences (b_k) and (c_k) so that $b_k := 1_S \cdot a_k$ and $c_k := 1_{\mathbb{N}\setminus S} \cdot a_k$ for all $k \in \mathbb{N}$. Note that $b_k, c_k \geq 0$ and $\frac{\sqrt{a_k}}{k} = \frac{\sqrt{b_k}}{k} + \frac{\sqrt{c_k}}{k}$ for all k. To show that $\sum_{k=1}^{\infty} \frac{\sqrt{a_k}}{k}$ converges, it will therefore be sufficient to show that $\sum_{k=1}^{\infty} \frac{\sqrt{b_k}}{k} + \sum_{k=1}^{\infty} \frac{\sqrt{c_k}}{k}$ converges.

First, consider the series $\sum_{k=1}^{\infty} \frac{\sqrt{b_k}}{k}$. We claim that $a_k \geq \frac{\sqrt{b_k}}{k}$ for all k. If $k \notin S$, then $\frac{\sqrt{b_k}}{k} = 0 \leq a_k$, so suppose $k \in S$. Then because $a_k \geq 0$ and $a_k > \frac{1}{k^2}$, we have $a_k^2 \geq \frac{a_k}{k^2}$. Taking the positive square root of both sides gives $a_k \geq \frac{\sqrt{a_k}}{k}$. As $k \in S$, $a_k = b_k$ so $a_k \geq \frac{\sqrt{b_k}}{k}$, as claimed. Thus, $\sum_{k=1}^{\infty} \frac{\sqrt{b_k}}{k} \leq \sum_{k=1}^{\infty} a_k < \infty$, so the first series converges.

Now consider the series $\sum_{k=1}^{\infty} \frac{\sqrt{c_k}}{k}$. We claim that $\frac{\sqrt{c_k}}{k} \leq \frac{1}{k^2}$ for all $k \in \mathbb{N}$. If $k \in S$, then $c_k = 0$ and the result is immediate, so suppose $k \notin S$. Then $c_k = a_k \leq \frac{1}{k^2}$, so $\frac{\sqrt{c_k}}{k} \leq \frac{\sqrt{1/k^2}}{k} = \frac{1}{k^2}$. Thus, $\sum_{k=1}^{\infty} \frac{\sqrt{c_k}}{k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so this series converges as well. Because $\sum_{k=1}^{\infty} \frac{\sqrt{b_k}}{k}$ and $\sum_{k=1}^{\infty} \frac{\sqrt{c_k}}{k}$ both converge, their sum converges, as desired. $\square$

Proof. (S. Chowdhury) Notice that $(\sqrt{a_k} - \frac{1}{k})^2 \geq 0$. Indeed, we have:

$$a_k + \frac{1}{k^2} \geq 2\frac{\sqrt{a_k}}{k}$$

Taking sums, we find that the terms on the left hand side converge (the first by assumption, and the second by the p-test/integral test/Cauchy condensation test). By comparison, and the fact that all our terms are non-negative, the sum of the right hand side will also converge. $\square$

Exercise 4. *Let (f_n) be a sequence of functions from $[a, b]$ to $\mathbb{R}$, where $a, b \in \mathbb{R}$ with $a < b$. Suppose that for each $c \in [a, b]$, (f_n) is equicontinuous at c and $(f_n(c))$ converges. Prove that (f_n) converges uniformly. (To say that (f_n) is* equicontinuous *at c means that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in [a, b]$, if $|x - c| < \delta$, then for each n, $|f_n(x) - f_n(c)| < \varepsilon$.)*

Proof. (O. Khalil) Let $f(x)$ be the point-wise limit of $f_n(x)$. We begin by showing that f is continuous on $[a, b]$. Let $\varepsilon > 0$ be fixed. Let $c \in [a, b]$. By the equicontinuity of (f_n) at c, $\exists \delta > 0$, such that $\forall x \in [a, b]$ with $|x - c| < \delta$, we have that $\forall n \in \mathbb{N}$, $|f_n(x) - f(c)| < \varepsilon/3$. Now, let $x \in [a, b]$ with $|x - c| < \delta$. Let $N \in \mathbb{N}$ be such that $|f_N(c) - f(c)| < \varepsilon/3$ and $|f_N(x) - f(x)| < \varepsilon/3$. Thus, by the choice of N and equicontinuity, We get

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \varepsilon$$

Hence, f is continuous. Now, we wish to show that for large enough n:

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| \leq \varepsilon$$

Since $[a, b]$ is compact and f is continuous, then f is uniformly continuous and so $\exists\delta_1 > 0$ such that $\forall x, y \in [a, b]$, we have that $|f(x) - f(y)| < \varepsilon/3$ whenever $|x - y| < \delta_1$.

Now, for each $x \in [a, b]$, we have that by the equicontinuity of (f_n) at x, $\exists\delta_x > 0$, such that $\forall n \in \mathbb{N}$, and $\forall y \in [a, b] \cap (x - \delta_x, x + \delta_x)$, we have that $|f_n(x) - f_n(y)| < \varepsilon/3$. We may assume that $\delta_x \leq \delta_1$ for all x. But, then we get that $[a, b] \subseteq \bigcup_{x\in[a,b]}(x - \delta_x, x + \delta_x)$. But, again, the compactness of $[a, b]$ gives us that there exist $x_1, \cdots x_k \in [a, b]$ such that $[a, b] \subseteq \bigcup_1^k (x_i - \delta_{x_i}, x_i + \delta_{x_i})$.

For each i, let $N_i \in \mathbb{N}$ be such that $\forall n > N_i$, we have that $|f_n(x_i) - f(x_i)| < \varepsilon/3$. Let $N = \max\{N_1, \cdots, N_k\}$.

Now, let $x \in [a, b]$ be arbitrary. Then, we have that $x \in (x_i - \delta_{x_i}, x_i + \delta_{x_i})$ for some $1 \leq i \leq k$. Hence, we get that for all $n > N$,

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

where bounding the first term uses the equicontinuity at x_i by the choice of δ_{x_i}, the second term uses pointwise convergence by the choice of N (which is independent of x), and the third term uses uniform continuity of f since $\delta_{x_i} \leq \delta_1$.

Since x was arbitrary, then we get that

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \varepsilon$$

as desired.

$\square$

Exercise 5. *For each $\alpha \in (0, \infty)$, define $f_\alpha : (0, \infty) \to \mathbb{R}$ by $f_\alpha(x) = x^\alpha \log x$. For which values of α is f_α uniformly continuous? Justify your answer.*

Proof. (K. Nowland) We claim that f_α is uniformly continuous for $0 < \alpha < 1$ and not uniformly continuous for $\alpha > 1$. Note that f_α is differentiable on $(0, \infty)$ for all $\alpha > 0$. The derivative is

$$f'_\alpha(x) = x^{\alpha-1}(1 + \alpha \log x).$$

First suppose $\alpha > 1$. Let $\varepsilon > 0$. To disprove uniform continuity, it suffices to show that for any $\delta > 0$, there exist $x, y \in (0, \infty)$ such that $|f(x) - f(y)| > \varepsilon$ with $|x - y| < \delta$. Fix $\delta > 0$. Let x be so large that $y > x$ implies that $|f'_\alpha(y)| > 2\varepsilon/\delta$. This is possible since $f'_\alpha(x) \to \infty$ as $x \to \infty$ for $\alpha > 1$. Let $y = x + \delta/2$. By the mean value theorem, there exists $\delta \in (x, y)$ such that $f_\alpha(y) - f_\alpha(x) = f'_\alpha(c)(y - x) = f'(c)\delta/2$. Since $c > x$,

$$|f_\alpha(y) - f(x)| = |f'_\alpha(c)|\delta/2 > \varepsilon.$$

But $|x - y| = \delta/2 < \delta$. Thus f_α is not uniformly continuous.

Now suppose $0 < \alpha < 1$. To prove uniform continuity, it suffices to show that $|f'_\alpha|$ is bounded. This is because by the mean value theorem,

$$|f(x) - f(y)| < M|x - y|.$$

Given $\varepsilon > 0$, we may choose $\delta < 1/M$, to see that $|f(x)-f(y)| < \varepsilon$ if $|x-y| < 1/M$. To show that f'_α is bounded on $(0, \infty)$, it suffices to show that f'_α is bounded as $x \to 0^+$ and $x \to \infty$. In this case, f'_α will be bounded for $0 < x < y_1$ for some y_1 and also for $x > y_2 > y_1$ for some y_2. By the continuity of f'_α on $[y_1, y_2]$, f'_α is bounded on $[y_1, y_2]$. Taking the maximum of these three bounds will give a bound on f'_α on all of $(0, \infty)$.

To bound $f'_\alpha(x)$ as $x \to \infty$, it suffices to bound $x^{\alpha-1} \log x$ for $x \geq 1$. Note that for any y, $e^y \geq y$. Then for $x > 1$,

$$0 \leq x^{\alpha-1} \log x = \frac{\log x}{e^{(1-\alpha)\log x}} \leq \frac{1}{1-\alpha}.$$

Because $x^{\alpha-1} \to 0$ as $x \to 0^+$, to bound $f'_\alpha(x)$ as x approaches zero from above it suffices to bound $x^{\alpha-1} \log x$ for $0 < x < 1$. Cearly $0 > x^{\alpha-1} \log x$. But as above,

$$0 < -x^{\alpha-1} \log x \leq -\frac{1}{1-\alpha}.$$

Therefore f'_α is also bounded near zero. This completes the proof that f'_α is bounded on all of $(0, \infty)$, which implies that f_α is uniformly continuous in the case $0 < \alpha < 1$. □

Exercise 6. *Consider the integral*

$$I = \int_0^1 \int_0^1 \frac{1}{1-xy} dxdy.$$

Find $a, b \in \mathbb{R}$ such that $I \in [a, b]$ and $b - a < 1/2$. Justify your answer.

Proof. (O. Khalil) Let $\varepsilon \in (0, 1)$. Let $I_\varepsilon = \int_\varepsilon^{1-\varepsilon} \int_0^1 \frac{1}{1-xy} dxdy$. Evaluating the inner integral

$$\int_0^1 \frac{1}{1-xy} dx = \left.\frac{\ln(1-xy)}{-y}\right|_0^1 = \frac{\ln(1-y)}{-y}$$

Now, on $[\varepsilon, 1-\varepsilon]$, we have that $\ln(1-y)$ is defined. Hence, we can use power series expansion around $y = 0$ to write

$$I_\varepsilon = \int_\varepsilon^{1-\varepsilon} \frac{\ln(1-y)}{-y} dy = \int_\varepsilon^{1-\varepsilon} \frac{\sum_{n=1}^\infty \frac{(-1)^{n+1}(-y)^n}{n}}{-y} = \int_\varepsilon^{1-\varepsilon} \sum_{n=1}^\infty \frac{y^{n-1}}{n} \tag{47}$$

Now, using Cauch-Hadamard formula to compute the radius of convergence of the power series in 47, we get

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n+1}}} = 1$$

where we used d'Alembert's theorem to evaluate the limit superior. Hence, on $[\varepsilon, 1-\varepsilon]$, we have that $\sum_{n=1}^{\infty} \frac{y^{n-1}}{n}$ is uniformly convergent. Hence, we can interchange the sum and the integral to get

$$I_\varepsilon = \sum_{n=1}^{\infty} \int_{\varepsilon}^{1-\varepsilon} \frac{y^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{y^n}{n^2}\bigg|_{\varepsilon}^{1-\varepsilon} = \sum_{n=1}^{\infty} \frac{(1-\varepsilon)^n}{n^2} - \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n^2}$$

where both series on the right-hand side converge by the comparison test being bounded above by the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and below by 0.

Moreover, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ has radius of convergence $= 1$ by d'Alembert's theorem and hence converges uniformly on $[0, 1-\varepsilon]$ and so in particular the function $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on this interval. Thus, we get that

$$\lim_{\varepsilon \to 0^+} I_\varepsilon = \lim_{\varepsilon \to 0^+} \left(\sum_{n=1}^{\infty} \frac{(1-\varepsilon)^n}{n^2} - \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n^2} \right) = \sum_{n=1}^{\infty} \lim_{\varepsilon \to 0^+} \left(\frac{(1-\varepsilon)^n}{n^2} - \frac{\varepsilon^n}{n^2} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

where we used the fact that $g(1)$ converges along with Abel's theorem giving that g is left continuous at 1. Hence, we have that

$$I = \lim_{\varepsilon \to 0^+} I_\varepsilon = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

And, so, the claim follows. $\square$

2006 - Spring

Exercise 1. *Determine the radius of convergence and behaviour at the endpoints of the interval of convergence for the series*

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^n.$$

Proof. (K. Nowland) Let a_n be the coefficient of x^n. Then the Cauchy-Hadamard theorem says that the radius of convergence R of $\sum_{n=1}^{\infty} a_n x^n$ is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

If the limit exists, then the limit and limit supremum agree. Note that each of the a_n is strictly positive. Then d'Alembert's ratio test implies that when $\lim_{n\to\infty} a_{n+1}/a_n = l$ exists, then $\lim_{n\to\infty}\sqrt[n]{a_n}$ exists and is also l. Note that

$$\frac{a_{n+1}}{a_n} = 1 = \frac{1}{n(n+1)} \to 1.$$

Therefore the radius of convergence $R = 1$.

The series does not converge at either $x = 1$ nor $x = -1$. For $x = 1$, note that $a_n \geq 1$ such that the sum of n terms is at least n. Since we may take n as large as we like, the series must diverge to infinitey. For $x = -1$, the terms in the series a_n do not converge to zero, which implies that the series must fail the Cauchy convergence criterion for series, as the difference between consecutive partial sums up to n and $n-1$ is a_n. Thus the series converges at neither endpoint. $\square$

Exercise 2. *Let $n \in \mathbb{N}$. Define $P : \mathbb{R} \to \mathbb{R}$ by*

$$P(x) = \frac{d^n}{dx^n}((x^2-1)^n).$$

Clearly P is a polynomial. Prove that the roots of P are all real and lie in the interval $(-1, 1)$.

Proof. (R. Garrett) First, notice that if P is a polynomial and a is a root of P with multiplicity n, then a is a root of P' with multiplicity $n-1$: indeed, if $P(x) = (x-a)^n Q(x)$, then by the product rule $P'(x) = n(x-a)^{n-1}Q(x) + (x-a)^n Q'(x) = (x-a)^{n-1}[nQ(x) + (x-a)Q'(x)]$.

Now, for ease of notation, let $Q(x) = (x^2-1)^n$ and $Q^{(i)}(x)$ denote the ith derivative of $Q(x)$. $Q(x)$ has roots 1 and -1 both of multiplicity n. By Rolle's Theorem, $P'(x_0) = 0$ for some $x_0 \in (-1, 1)$. Moreover, Q' has degree $2n-1$ and, as shown at the start of this proof, 1 and -1 are roots of Q' of multiplicity $n-1$, so x_0 has multiplicity 1 and all roots of Q' are real. Again, by Rolle's Theorem, $Q''(x_1) = 0$ for some $x_1 \in (-1, x_0)$ and $Q''(x_2) = 0$ for some $x_2 \in (x_0, 1)$. By similar reasoning as before, in Q'', 1 and -1 are roots of multiplicity $n-2$ and Q'' has degree $2n-2$, so x_1 and x_2 each have multiplicity 1 and all roots of Q''

are real. We now proceed by induction. Suppose for induction hypothesis that $Q^{(k)}(x)$ has $2n-k$ real roots: 1 and -1 each occurring with multiplicity $n-k$ and remaining roots $-1 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < -1$. Then, by what we first showed, $Q^{(k+1)}(x)$ has degree $2n-k-1$ and $2n-k-1$ real roots: -1 and 1 each occurring with multiplicity $n-k-1$ and (by Rolle's Theorem) additional roots $\beta_0 \in (-1, \alpha_0$, $\beta_i \in (\alpha_i, \alpha_{i+1})$ for $i \in \{1, ..., k-1\}$, and $beta_k \in (\alpha_k, 1)$. Hence, by induction, $P(x)$ has n real roots lying in $(-1, 1)$.

□

Exercise 3. *Let (x_n) be a sequence of strictly positive real numbers. Suppose that $\sum_{n=1}^{\infty} x_n y_n$ converges for each sequence (y_n) of strictly positive numbers such that $y_n \to 0$ as $n \to \infty$. Prove that $\sum_{n=1}^{\infty} x_n < \infty$.*

Proof. (K. Nowland) Suppose toward a contradiction that $\sum x_n$ diverges. Note that it must diverge to $+\infty$ since $x_n > 0$ for all n. We will repeatedly use the fact that if a series diverges, then removing a finite number of terms will not change the fact that the series diverges. Since the series diverges to $+\infty$, there exists N_1 such that $\sum_{n=1}^{N_1-1} x_n > 1$. Similarly, there exists N_2 such that $\sum_{n=N_1}^{N_2-1} x_n > 2$. Continuing in this way, we construct a sequence of strictly increasing natural numbers such that N_k satisfies $\sum_{N_{k-1}}^{N_k-1} x_n > k$. Let (y_n) be sequence such that for $N_{k-1} \leq n \leq N_k - 1$, $y_n = 1/k$. Then $y_n > 0$ for all n and $y_n \to 0$ as $n \to \infty$ since for any $\varepsilon > 0$ there exists k such that $1/k < \varepsilon$ and thus for $n \geq N_{k-1}$ we have $y_n = 1/k < \varepsilon$. Thus $\sum y_n x_n$ converges by assumption. But

$$\begin{aligned} \sum_{n=1}^{\infty} y_n x_n &= \sum_{n=1}^{N_1-1} x_n + \frac{1}{2}\sum_{n=N_1}^{N_2-1} x_n + \frac{1}{3}\sum_{N_2}^{N_3-1} x_n + \cdots \\ &> 1 + 1 + 1 + \cdots . \end{aligned}$$

The series $\sum x_n y_n$ therefore diverges. The contradiction implies that $\sum x_n$ must converge. □

Exercise 4. *Find the limit of $m \sum_{n=m}^{\infty} \frac{1}{n^2}$ as $m \to \infty$. Justify your answer.*

Proof. (K. Nowland) Note that each term in the sequence is defined since the series $\sum n^{-2}$ converges. Since $1/x^2$ is decreasing as $x \to \infty$,

$$\int_m^{k+1} \frac{dx}{x^2} \leq \sum_{n=m}^{k} \frac{1}{n^2} \leq \int_{m-1}^{k} \frac{dx}{x^2}.$$

Since $\int_1^{\infty} \frac{dx}{x^2}$ converges, we may take the limit as $k \to \infty$, which gives

$$m \int_m^{\infty} \frac{dx}{x^2} \leq m \sum_{n=m}^{\infty} \frac{1}{n^2} \leq m \int_{m-1}^{\infty} \frac{dx}{x^2}.$$

Integrating,

$$1 \leq m \sum_{n=m}^{\infty} \frac{1}{m^2} \leq \frac{m}{m-1}.$$

The squeeze theorem implies $\lim_{m\to\infty} m \sum_{n=m}^{\infty} n^{-2} = 1$. □

Exercise 5. *Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n \in (0, \infty)$. Let*

$$G = (a_1 a_2 \cdots a_n)^{1/n} \qquad \text{and} \qquad A = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove that $G \leq A$.

Proof. (K. Nowland) We first prove that claim for $n = 2^k$. We induct on k. For $k = 1$, the claim is that

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2}$$

for $a_1, a_2 > 0$. Multiplying by 2 and squaring both sides, this is the claim that

$$4a_1 a_2 \leq (a_1 + a_2)^2 = a_1^2 + 2a_1 a_2 + a_2^2,$$

or that

$$0 \leq a_1^2 - 2a_1 a_2 + a_2^2 = (a_1 - a_2)^2.$$

Since the square of real numbers is postiive, this proves the base case. Now suppose the statement is true for 2^{k-1} numbers, we want to prove it for 2^k numbers. Grouping the numbers into pairs, the inductive assumption and base case imply

$$\begin{aligned}(a_1 a_2 \cdots a_{2^k-1} a_{2^k})^{1/2^k} &= (a_1 a_2)^{1/2} \cdots (a_{2^k-1} a_{2^k})^{1/2}]^{1/2^{k-1}} \\ &\leq \frac{\sqrt{a_1 a_2} + \cdots + \sqrt{a_{2^k-1} a_{2^k}}}{2^{k-1}} \\ &\leq \frac{a_1 + \cdots + a_{2^k}}{2^k},\end{aligned}$$

as desired. This proves the case for $n = 2^k$ for all $k \in \mathbb{N}$.

Now suppose n is not a power of two. Let k be such that $2^{k-1} < n < 2^k$. Let $A = (a_1 + \cdots + a_n)/n$ and $G = \sqrt[n]{a_1 \cdots a_n}$. Clearly $A > 0$. We pad out the $a_1, \ldots, a_n$ with $2^k - n$ copies of A. Since we have the statement for 2^k positive terms, we have

$$\sqrt[2^k]{a_1 \cdots a_n A^{2^k - n}} \leq \frac{a_1 + \cdots + a_n + (2^k - n)A}{2^k}.$$

This can be rewritten as

$$\sqrt[2^k]{G^n A^{2^k - n}} = G^{\frac{n}{2^k}} A^{1 - \frac{n}{2^k}} \leq \frac{nA + (2^k - n)A}{2^k} = A.$$

But then this is $G^{n/2^k} \leq A^{n/2^k}$ such that $G \leq A$ holds, as desired. $\square$

Exercise 6. *Let $f : [0, \infty) \to \mathbb{R}$. Suppose that f is uniformly continuous and that*

$$\int_0^\infty f(x)dx$$

converges. Prove that

$$\lim_{x \to \infty} f(x) = 0. \tag{48}$$

Proof. (K. Nowland) Suppose toward a contradiction that $f(x) \not\to 0$ as $x \to \infty$. Then there exists $\varepsilon > 0$ such for a sequence $(t_n) \subset [0, \infty)$ tending to infinity $|f(t)| > \varepsilon$. Without loss of generality we may assume that $t_{n+1} - t_n \geq 1$ for all $n \in \mathbb{N}$. By uniform continuity, there exists $\delta > 0$ such that $|t - t_n| < \delta$ implies $|f(t)| \geq \varepsilon/2$. We may suppose that $\delta < 1$. This implies that

$$|\int_{t_n-\delta}^{t_n+\delta} f(t)dt| \geq \varepsilon\delta.$$

Since $t_n \to \infty$ as $n \to \infty$, this implies that the integral fails the Cauchy convergence criterion and does not converge. The contradiction proves the claim.

□

2005 - Autumn

Exercise 1. *Determine whether* $\sum_{n=1}^{\infty} a_n$ *converges, where*

$$a_n = \begin{cases} n^{-1} & \text{if } n \text{ is a square,} \\ n^{-2} & \text{otherwise.} \end{cases}$$

Justify your answer.

Proof. (K. Nowland) The series converges. The series can be rewritten as

$$\sum_{n=1}^{\infty} a_n = \sum_{m \neq n^2} \frac{1}{m^2} + \sum_{m=1}^{\infty} \frac{1}{m^2} \leq 2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{n} k^{-2}$ converges, the comparison test implies $\sum a_n$ converges. To apply the comparison test we used the fact that each term in the original series is positive. $\square$

Exercise 2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable. Suppose that* $|f''(x)| \leq 1$ *for all* x *and that* $f(-1) = f'(-1) = f(1) = f'(1) = 0$. *What is the maximum possibly value for* $f(0)$*? Justify your asnwer.*

Proof. (K. Nowland via stackexchange) By continuity, f restricted to $[-1, 1]$ realizes its maximum. Without loss of generality, we may suppose its maximum is assumed at $a \geq 0$. By Taylor's theorem with second order Lagrange remainder,

$$f(1-h) = f(1) - f'(1)h + \frac{f''(\zeta)}{2}h^2 \leq \frac{1}{2}h^2$$

where for some $\zeta \in (1-h, 1)$. Similarly,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(\xi)}{2}h^2 \geq f(a) - \frac{1}{2}h^2$$

for some $\xi \in (a, a+h)$ and we have used the fact that $f'(a) = 0$ since there is a maximum there. This is not *a priori* valid, but is if we can suppose that $a \neq 1$. This is easy to see, because if the maximum is there, then the maximum is zero. Then $f \leq 0$ on all of $[-1, 1]$. If f is not identically zero, we can replace f with $-f$ to say that the maximum is in fact not at 1 and thus satisfies $f'(a) = 0$ as a local maximum of a differentiable function. Setting $h = (1-a)/2$, we calculate

$$f(a) - \frac{(1-a)^2}{8} \leq f\left(a + \frac{1-a}{2}\right) = f\left(1 - \frac{1-a}{2}\right) \leq \frac{(1-a)^2}{8}.$$

Therefore

$$f(0) \leq f(a) \leq \frac{(1-a)^2}{4} \leq \frac{1}{4}.$$

We can realize this bound with the function

$$f(x) = \begin{cases} \frac{1}{2}(1+x)^2 & x \leq \frac{-1}{2}, \\ \frac{1}{4} - \frac{1}{2}x^2 & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \frac{1}{2}(1-x)^2 & x \geq \frac{1}{2}. \end{cases}$$

This function is not twice differentiable at $\pm\frac{1}{2}$ and thus we must rely on an approximation by smooth functions. The smooth function we use is defined as follows: Let $\varepsilon > 0$ be arbitrary but less than 1/2. Define $g(x)$ to be

$$g(x) = \begin{cases} -1 & x < -\frac{1}{2} - \varepsilon \text{ or } x > \frac{1}{2} + \varepsilon, \\ -\frac{1}{\varepsilon}\left(\frac{1}{2} + x\right) & -\frac{1}{2} - \varepsilon \leq x \leq -\frac{1}{2} + \varepsilon, \\ 1 & -\frac{1}{2} + \varepsilon < x < \frac{1}{2} - \varepsilon, \\ -\frac{1}{\varepsilon}\left(\frac{1}{2} - x\right) & \frac{1}{2} - \varepsilon \leq x \leq \frac{1}{2} + \varepsilon. \end{cases}$$

Let $G(x) = \int_{-1}^{x} g(t)dt$ and $F(x) = \int_{-1}^{x} G(t)dt$. Then F satisfies the required conditions. Using the fact that $f'(-1) = G(-1) = 0$, we see that the difference between $f'(x)$ and $G(x)$ is at most 2ε. Then we see that $|F(0) - f(0)| \leq 2\varepsilon$, using the fact that $F(-1) = f(-1) = 0$. This completes the proof that $\frac{1}{4}$ is sharp, since ε was arbitrary. □

Exercise 3. *Prove or disprove: The series* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin x}{1+n^2x^2}$ *converges uniformly on* $[-\pi, \pi]$.

Proof. (O. Khalil) Let $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin x}{1+n^2x^2}$ and Let $S_n(x)$ denote the partial sums of the series for a given x. Now, we have that

$$\| S_n(x) - S(x) \|_\infty = \sup_{x \in (0,\pi]} \sum_{k=n+1}^{\infty} \frac{\sin x}{1+k^2x^2} \geqslant \sup_{x \in (0,\pi]} \int_{n+1}^{\infty} \frac{\sin x}{1+t^2x^2}\, dt$$

This is because we have that $S(0) = 0$ and the function is odd so it suffices to consider the interval $(0, \pi]$. The absolute value is dropped because the summand is non-negative on $(0, \pi]$. The inequality follows from the integral test since the function $f(t) = \frac{\sin x}{1+t^2x^2}$ is decreasing.

Now, making the substitution $u = tx$, the integral yields $\frac{\sin x}{x}\left(\frac{\pi}{2} - \arctan((n+1)x)\right)$. So, we get that

$$\begin{aligned} \| S_n(x) - S(x) \|_\infty &\geqslant \sup_{x \in (0,\pi]} \frac{\sin x}{x}\left(\frac{\pi}{2} - \arctan((n+1)x)\right) \\ &\geq \lim_{x \to 0^+} \frac{\sin x}{x}\left(\frac{\pi}{2} - \arctan((n+1)x)\right) = \frac{\pi}{2} \end{aligned}$$

So, we have that $\forall n \geq 1, \|S_n(x) - S(x)\|_\infty \geq \frac{\pi}{2} \nrightarrow 0$ as $n \to \infty$. So, the series doesn't converge uniformly on $[-\pi, \pi]$. □

Proof. (H. Lyu) This series does converge pointwise by summation by parts. To show that the convergence is not uniform, let us estimate the Cauchy segment and try to find a lower bound. Let $f(x) = \sin x$. Then $f''(x) = -\sin x < 0$ on $[0, \pi]$. so f is concave down on $[0, \pi]$. Hence the graph of f is above any secant line. In particular, $\sin(x) \geq \frac{2}{\pi}x$ on $[0, \pi/2]$. Let $n, m \in \mathbb{N}$ with $n < m$. Then

$$\frac{\sin x}{1+n^2x^2} + \cdots + \frac{\sin x}{1+m^2x^2} \geq \frac{2}{\pi}\left(\frac{x}{1+n^2x^2} + \cdots \frac{x}{1+m^2x^2}\right)$$

provided $x \in [0, \pi/2]$. Put $m = 2n$ and $x = 1/n$. We may assume n is large enough so that $1/n < \pi/2$. Then we have

$$\begin{aligned}\frac{\sin x}{1+n^2x^2} + \cdots + \frac{\sin x}{1+(2n)^2x^2} &\geq \frac{2}{\pi n}\left(\frac{1}{1+n^2/n^2} + \cdots \frac{1}{1+(2n)^2/n^2}\right) \\ &\geq \frac{2}{\pi n}\left(\frac{1}{5} + \cdots + \frac{1}{5}\right) \\ &\geq \frac{2}{\pi n}\frac{n+1}{5} \geq \frac{2}{5\pi}.\end{aligned}$$

This holds all $n > 2/\pi$. Thus the sequence of the partial sums is not uniformly Cauchy. Therefore the series does not converge uniformly. (”Stan estimates”) □

Exercise 4. *Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and has a local maximum at each point in $[0,1]$. Prove that f is constant.*

Proof. (O. Khalil) $[0,1]$ is compact and f is continuous, so by the extreme value theorem, f attains its infimum at some point $x_o \in [0,1]$. Fix some $a \in [0,1]$ such that there exists some $b \in [0,1]$, $x_o \in (a,b)$ and $f(x_o) \geq f(x)$ for all $x \in (a,b)$. The existence of such a, b is guaranteed by assumption that f has local maximum at every point. Define the following set

$$B_a = \{b : b \in [0,1], b > a, x_o \in (a,b) \text{ and } \forall x \in (a,b), f(x_o) \geq f(x)\}$$

Let $b \in B_a$. Observe that for each $x \in (a,b)$, we have $f(x_o) \geq f(x)$ by construction and that $f(x_o) \leq f(x)$ by the fact that $f(x_o)$ is the infimum of f on $[0,1]$. Hence, f is constant on (a,b). Moreover, since this gives us that the left-handside limit of f at b is $f(x_o)$, then by continuity of f, we have that $f(b) = f(x_o)$. This holds for each $b \in B_a$.

Let $\beta = \sup B_a$. Since both B_a is bounded, then β is finite and $\beta \leq 1$. We claim that β belongs to B_a. Suppose not. Then, there exists $x \in (a,\beta)$ such that $f(x) > f(x_o)$. But, then for all $b \in (x,\beta)$, we have that $b \notin B_a$. Hence, for all $b \in (x,\beta)$, we have that $b > \sup B_a = \beta$, a contradiction.

Now, suppose that $\beta < 1$. Then, by the observation that $f(b) = f(x_o)$ for each $b \in B_a$, we have that $f(\beta) = f(x_o)$. But, f has a local maximum at β by assumption, so we can find some $b \leq 1$ such that $\beta \in (a,b)$ with $f(x_o) = f(\beta) \geq x$ for all $x \in (a,b)$. But, then $b \in B_a$ and $b \gneq \beta$, which is a contradiction. Therefore, $b = 1$.

Now, if we fix $b = 1$ and consider the set

$$A_1 = \{a : a \in [0,1], a < 1, x_o \in (a,1] \text{ and } \forall x \in (a,1], f(x_o) \geq f(x)\}$$

Following the same argument as above, we find that $\inf A_1 = 0$ and that $0 \in A_1$. Thus, $f(x_o)$ is both the supremum and infimum on $[0,1]$ and therefore f is constant. $\square$

Proof. (H. Lyu) Since f is continuous on the compact set $[0,1]$, by extreme value theorem it attains absolute minimum, say, m, at some $c \in [0,1]$. Let $K = f^{-1}[\{m\}]$. Since $c \in K$, K is nonempty. It suffices to show that $K = [0,1]$. Since f is continuous and K is the pull-back of a singleton, which is closed in $\mathbb{R}$, K is closed in $[0,1]$. Now since $[0,1]$ is connected, any nonempty subset of $[0,1]$ which is both open and closed must be the whole space $[0,1]$. As we know K is closed in $[0,1]$, it suffices to show that K is also open in $[0,1]$. To this end, let $y \in K$. Since f has a local minimum at y, there is $\epsilon > 0$ such that f has absolute maximum at y on $U := (y-\epsilon, y+\epsilon) \cap [0,1]$. But since $y \in K$, f has absolute minimum on $[0,1]$ at y. Thus f must be constant on the relative ϵ-ball U of y. But since $f(y) = m$, f must be identically m on U. Hence $U \subset K$. So y is an interior point of K. Since $y \in K$ was arbitrary, K is open in $[0,1]$. Thus $K = [0,1]$. This shows the assertion. $\square$

Exercise 5. *Prove or disprove: For each unbounded open set $U \subseteq (0,\infty)$, the function f defined by $f(x) = x^2$ is not uniformly continuous on U.*

Proof. (K. Nowland) The claim is false. We provide an unbounded open set U such that f is uniformly continuous on U. The key to the construction is to provide an increasing sequence of open intervals which decrease in size rapidly enough that x^2 does not have room to change much on each interval.

Let $U_n = (n - 1/n^2, n + 1/n^2)$ for $n \geq 2$. We claim that $f(x) = x^2$ is uniformly continuous on $U = \cup_{n=2}^{\infty} U_n$. Clearly this set is unbounded. It is open as it is the union of open sets.

To prove uniform continuity, let $\varepsilon > 0$ be fixed. We must find $\delta > 0$ such that $|x-y| < \delta$ implies that $|f(x) - f(y)| \leq \varepsilon$. Note that on any U_n, the distance between $f(x)$ and $f(y)$ is bounded by $(n+1/n^2)^2 - (n-1/n^2)^2$. We calculate

$$(n+1/n^2)^2 - (n-1/n^2)^2 = \frac{(n^3+1)^2 - (n^3-1)^2}{n^4} = \frac{n^4 + 2n^3 + 1 - n^4 + 2n^2 - 1}{n^4} = \frac{4}{n}.$$

Let N be so large that $4/n < \varepsilon$ for all $n \geq N$. It follows that for any choice of $\delta < 1/2$ that $|f(x) - f(y)| < \varepsilon$ for all $x, y \geq N - 1/N^2$ and $|x-y| < \delta$. This is because $|x-y| < \delta$ with $x, y \in U$ implies that x and y are in the same U_n for some n such that $|f(x) - f(y)| < 4/n < \varepsilon$. Thus we can choose any $\delta < 1/2$ that works for the compact interval $[1, N+1/N^2]$. Since any continuous function is uniformly continuous on a compact set, there exists such a δ.w $\square$

Exercise 6. *Let (a_n) be a sequence of strictly positive real numbers. Prove that*

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{1/n}. \tag{49}$$

Proof. (O. Khalil) This is part of D'Alembert's theorem. Let $l = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \in [0,\infty)$, since (a_n) are strictly positive. If $l = 0$, then there is nothing to prove, so suppose $l > 0$ and let $t \in (0,l)$. Now, we have that

$$t < \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \sup_{n>0}(\inf_{m\geq n} \frac{a_{m+1}}{a_m})$$

So, $\exists N > 0$, such that $t < \inf\limits_{m \geq N} \dfrac{a_{m+1}}{a_m}$. Hence, $\forall n > N$, we have that $a_n = \dfrac{a_n}{a_{n-1}} \dots \dfrac{a_{N+1}}{a_N} a_N \geq a_N t^{n-N}$. Taking n^{th} root, we get that $a_n^{1/n} = \left(\dfrac{a_N}{t^N}\right)^{1/n} t$. But, since $\lim\limits_{n \to \infty} \left(\dfrac{a_N}{t^N}\right)^{1/n} = 1$, we find that $\liminf\limits_{n \to \infty} a_n^{1/n} > t$. Since $t \in (0, l)$ was arbitrary, 49 follows.

$\square$

2005 - Spring

Exercise 1. *Let*

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

for each $n \in \mathbb{N}$. Prove that x_n converges as $n \to \infty$.

Proof. (K. Nowland) Our strategy is to use partial summation. Let $A_k = k$ for $k \geq 0$. Then

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &= \sum_{k=1}^{n} \frac{A_k - A_{k-1}}{k} \\
&= 1 + \sum_{k=1}^{n-1} \frac{A_k}{k} - \sum_{k=2}^{n} \frac{A_{k-1}}{k} \\
&= 1 + \sum_{k=1}^{n-1} \frac{A_k}{k} - \sum_{k=1}^{n-1} \frac{A_k}{k+1} \\
&= 1 + \sum_{k=1}^{n-1} A_k \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= 1 + \sum_{k=1}^{n-1} A_k \int_k^{k+1} \frac{1}{x^2} dx \\
&= 1 + \int_1^n \frac{x - \{x\}}{x^2} dx,
\end{aligned}$$

where $\{x\} = x - [x]$ and $[x]$ is the greatest integer less than or equal to x. Breaking up the integral and evaluating, we see that

$$x_n = 1 - \int_1^n \frac{\{x\}}{x^2} dx.$$

Note that $0 \leq \{x\} \leq 1$, such that

$$\int_1^n \frac{\{x\}}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx = 1.$$

Since the integral will be strictly increasing as $n \to \infty$ but is bounded above, we conclude that the integral converges. Thus x_n converges (to the Euler-Mascheroni constant). $\square$

Proof. (S. Chowdhury, credit to Hanbaek Lyu)

Notice that we have

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x} dx$$

So in particular,

$$x_n = \sum_{k=1}^{n} \frac{1}{k} - \log n \geq \log(n+1) - \log n = \log(\frac{n+1}{n})$$

Thus x_n is bounded below. We just need to show that it is decreasing.

$$x_n - x_{n-1} = \frac{1}{n} - \log n + \log(n-1) = \frac{1}{n} + \log(1 - \frac{1}{n})$$

Consider the graph of $\log(1-x)$; it is always below the graph of $y = -x$. Replacing $x = \frac{1}{n}$, we get:

$$\log(1 - \frac{1}{n}) \leq -\frac{1}{n} \Rightarrow \log(1 - \frac{1}{n}) + \frac{1}{n} \leq 0$$

Thus x_n is indeed decreasing and bounded below. So it converges.

$\square$

Exercise 2. *Prove or disprove: For each continuous function $f : [0, \infty) \to \mathbb{R}$, if $\lim_{t\to\infty} f(t) = \infty$, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt = \infty. \tag{50}$$

Proof. (O. Khalil) Let T_n be an arbitrary strictly increasing sequence such that $T_n \to \infty$ as $n \to \infty$. Let $a_n = \int_{T_0}^{T_n} f(x)dx$. Now, let $M > 0$ be fixed. Since $f(x) \to \infty$ as $x \to \infty$, then $\exists x_o > 0$ such that for all $x > x_o$, $f(x) > M$. Also, since $T_n \to \infty$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $T_n > M$. Now, for all $n > N$, we have the following

$$\frac{a_{n+1} - a_n}{T_{n+1} - T_n} = \frac{\int_{T_{n-1}}^{T_n} f(x)dx}{T_{n+1} - T_n} \geq \frac{M(T_{n+1} - T_n)}{T_{n+1} - T_n} = M$$

Since M was arbitrary, then

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{T_{n+1} - T_n} = \infty$$

Hence, by Cesaro-Stolz theorem, we have that

$$\lim_{n\to\infty} \frac{a_n}{T_n} = \infty$$

But, since T_n was arbitrary, then 50 is verified.

To prove this instance of Cesaro-Stolz theorem, let a_n and b_n be 2 sequences such that b_n is increasing and unbounded and such that

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \infty$$

Let $M > 0$ be fixed. Then, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} > M$$

Multiplying through by $b_{n+1} - b_n > 0$,

$$a_{n+1} - a_n > M(b_{n+1} - b_n)$$

Summing from N to $n+1$

$$\sum_{N}^{n+1}(a_{n+1} - a_n) = a_{n+1} - a_N > \sum_{N}^{n+1} M(b_{n+1} - b_n) = M(b_{n+1} - b_N)$$

Dividing by b_{n+1}, we get

$$\frac{a_{n+1} - a_N}{b_{n+1}} > M\left(1 - \frac{b_N}{b_{n+1}}\right) \tag{51}$$

Now, as $n \to \infty$, $\frac{a_N}{b_{n+1}} \to 0$ and $\left(1 - \frac{b_N}{b_{n+1}}\right) \to 1$. Hence, there exists $L \in \mathbb{N}$ such that for all $k > L$ we have that

$$\frac{-1}{2} < \frac{a_N}{b_{k+1}} < \frac{1}{2}$$
$$\frac{1}{2} < \left(1 - \frac{b_N}{b_{k+1}}\right) < 1 + \frac{1}{2}$$

Plugging these estimates in 51, we get that for all $n > K$,

$$\frac{a_{n+1}}{b_{n+1}} > \frac{M-1}{2}$$

But, M was arbitrary and so $\frac{a_n}{b_n} \to \infty$ as desired. □

Proof #2. (K. Nowland) This is more direct than Osama's proof, but is essentially the same. Let $M > 0$ be fixed. Since M is arbitrary, it suffices to prove that for all T large enough, $\frac{1}{T}\int_0^T f(t)dt \geq M$. Let $\varepsilon > 0$ be fixed. Since $f(t)$ tends to infinity as t tends to zero, there exists $T_1 > 0$ such that $t \geq T_1$ implies $f(t) \geq M$. Since f is continuous on $[0, \infty)$, the integral $\int_0^T f(t)dt$ exists for all $T \geq 0$, and in particular exists and is finite for $T = T_1$. Let $T_2 \geq T_1 \geq 0$ be such that

$$\frac{1}{T_2}\int_0^{T_1} f(t)dt \geq -\varepsilon.$$

Now let $T \geq T_2 \geq T_1$. We calculate

$$\begin{aligned}\frac{1}{T}\int_0^T f(t)dt &= \frac{1}{T}\int_0^{T_0} f(t)dt + \int_{T_1}^T f(t)dt \\ &\geq -\varepsilon + \frac{T - T_1}{T}M.\end{aligned}$$

Since $(T - T_1)/T$ tends to 1 as T tends to infinity, let $T_3 \geq T_2 \geq T_1 \geq 0$ be such that $T \geq T_3$ implies $(T - T_1)/T \geq (1 - \varepsilon)$. Then for all such T we have

$$\frac{1}{T}\int_0^T f(t)dt \geq (1 - \varepsilon)M - \varepsilon.$$

Since ε was arbitrary, we conclude that for T sufficiently large,

$$\frac{1}{T}\int_0^T f(t)dt \geq M,$$

as desired. $\square$

Exercise 3. *Prove of disprove: For each function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0, and for each strictly decreasing sequence (a_n) in $(0, \infty)$ such that $\lim_{n\to\infty} a_n = 0$, we have*

$$\lim_{n\to\infty} \frac{f(a_n) - f(a_{n+1})}{a_n - a_{n+1}} = f'(0). \tag{52}$$

Proof. (E. Nash) We claim this statement is not true. To see this, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then we claim f is differentiable at 0. Let $\varepsilon > 0$ be given and suppose $|x| < \varepsilon$ and $x \neq 0$. If $x \in \mathbb{Q}$, then $\left|\frac{f(x)-f(0)}{x-0}\right| = |x| < \varepsilon$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\left|\frac{f(x)-f(0)}{x-0}\right| = 0 < \varepsilon$. Thus, f is differentiable at 0 with $f'(0) = 0$.

Now construct a sequence (a_n) as follows. Let $a_1 = 1$ and let a_2 be some irrational number in the interval $\left(\frac{1}{2}, 1\right)$. Let $a_3 = \frac{1}{2}$ and let a_4 be some irrational number in the interval $\left(\frac{1}{3}, \frac{1}{2}\right)$. Continue in this way so that for each $k \in \mathbb{N}$, $a_{2k-1} = \frac{1}{k}$ and $a_{2k} \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ is irrational. Note that (a_n) is a strictly decreasing sequence of positive numbers with limit 0. Now observe the following for some $k \in \mathbb{N}$:

$$\frac{f(a_{2k-1}) - f(a_{2k})}{a_{2k-1} - a_{2k}} = \frac{\frac{1}{k^2} - 0}{\frac{1}{k} - a_{2k}} > \frac{\frac{1}{k^2}}{\frac{1}{k} - \frac{1}{k+1}} = \frac{k+1}{k} > 1$$

From this, it follows that we cannot have $\lim_{n\to\infty} \frac{f(a_n)-f(a_{n+1})}{a_n-a_{n+1}} = 0 = f'(0)$ as the sequence $\left(\frac{f(a_n)-f(a_{n+1})}{a_n-a_{n+1}}\right)$ contains a subsequence that is always greater than 1. $\square$

Exercise 4. *Let f be an n times continuously differentiable real-valued function on $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Suppose that the nth derivative of f satisfies $f^{(n)}(x) > 0$ for each $x \in [a, b]$. Prove that f has at most n zeros in $[a, b]$.*

Proof. (K. Nowland) The above follows from the general fact that if $g : [a, b] \to \mathbb{R}$ is differentiable and has at least k (distinct) zeros, then g' must have at least $k - 1$ zeros. This is guaranteed by Rolle's theorem, which says that if $a \leq x < y \leq b$ are such that $g(a) = g(b)$, then there exists $c \in (a, b)$ such that $g'(c) = 0$. The fact that $x < c < y$ guarantees that distinct consecutive pairs of zeroes will lead to distinct zeros of the derivative. If we iterate the above procedure, we see that if f has at least k zeros in $[a, b]$ and $k \geq n$, then $f^{(n)}$ has at least $k - n$ zeros. Since $f^{(n)}$ has no zeros on $[a, b]$, it must be that f has at most n zeros on $[a, b]$. $\square$

Exercise 5. *Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that*

$$\int_0^1 f(x)g'(x)dx = 0$$

for each continuously differentiable function $g : [0,1] \to \mathbb{R}$ satisfying $g(0) = 0 = g(1)$. Prove that f must be a constant function.

Proof. (O. Khalil) Let $F(x) = \int_0^x f(t)dt$. Then, since $f(x)$ is continuous, the fundamental theorem of calculus implies that $F(x)$ is also continuous with $F'(x) = f(x)$. Hence, F is continuously differentiable. Let $c = F(1)$. Let $g(x) = F(x) - cx$. Thus, $g(x)$ is continuously differentiable with $g(1) = 0 = g(0)$. Therefore, by assumption, we have that

$$\int_0^1 f(x)g'(x)dx = \int_0^1 f(x)(f(x) - c)dx = \int_0^1 f(x)^2 dx - c\int_0^1 f(x)dx = 0$$

Thus, we find that

$$\int_0^1 f(x)^2 dx = c\int_0^1 f(x)dx = \left(\int_0^1 f(x)dx\right)^2 \tag{53}$$

But, the continuous Cauchy-Schwarz's inequality gives

$$\left(\int_0^1 f(x)h(x)dx\right)^2 \leq \int_0^1 f(x)^2 dx \int_0^1 h(x)^2 dx$$

for any continuous function $h(x) : [0,1] \to \mathbb{R}$ and equality holds if and only if $f(x) = \lambda h(x)$ for all $x \in [0,1]$ and some constant $\lambda \in \mathbb{R}$. But, then, letting $h(x) = 1$, we have that $\int_0^1 h(x)^2 dx = 1$. Thus, given 53, we have that $\exists \lambda \in \mathbb{R}$ such that $f(x) = \lambda$ for all $x \in [0,1]$ as desired.

The continuous Cauchy-Schwarz inequality can be proven as follows: for any $\lambda \in \mathbb{R}$, we have that $(\lambda f(x) - h(x))^2 \geq 0$ for all x. Hence, we have that

$$\int_0^1 (\lambda f(x) - h(x))^2 \geq 0$$

Expanding the above expression, we get

$$\lambda^2 \int_0^1 f(x)^2 dx - 2\lambda \int_0^1 f(x)h(x)dx + \int_0^1 h(x)^2 dx \geq 0$$

The above expression can be considered as a polynomial in λ. Hence, a polynomial is ≥ 0 if and only if it has at most one real solution if and only if its discriminant is ≤ 0. Computing the discriminant gives the C-S inequality. Moreover, if equality occurs, then this implies that $\exists \lambda \in \mathbb{R}$ such that the polynomial vanishes. But, then, we get that

$$\int_0^1 (\lambda f(x) - h(x))^2 = 0$$

And, thus, the integrand has to vanish identically yielding that $f(x) = \lambda h(x)$ for all $x \in [0,1]$ □

Exercise 6. *Prove that*

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \log\left(1 + \frac{x}{n}\right)$$

is defined and differentiable on the open interval $-1 < x < \infty$.

Proof. (O. Khalil) First, observe that the sequence $\log\left(1+\frac{x}{n}\right)$ goes monotonically to 0 as $n \to \infty$ for all $x \in (-1,\infty)$. Hence, the alternating series test gives that $\sum_{n=1}^{\infty}(-1)^{n+1}\log\left(1+\frac{x}{n}\right)$ converges for all $x \in (-1,\infty)$ and so f is defined.

Now, let $x \in (-1,\infty)$ be fixed. Let $[a,b] \subset (-1,\infty)$ be such that $x \in [a,b]$. To show that f is differentiable at x, we begin by showing that the following series converges uniformly on $[a,b]$:

$$g(y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+y}$$

But, for every $y \in [a,b]$, we have the following (note that $a \gneq -1$)

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+y}\right| &= \left|\sum_{n \text{ odd}}^{\infty} \frac{1}{(n+y)(n+y+1)}\right| \\
&\le \sum_{n \text{ odd}}^{\infty} \left|\frac{1}{(n+y)(n+y+1)}\right| \\
&\le \sum_{n \text{ odd}}^{\infty} \left|\frac{1}{(n+a)(n+a+1)}\right| \\
&\le \sum_{n=1}^{\infty} \frac{1}{(n+a)(n+a+1)}
\end{aligned}$$

But, $\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+a+1)}$ is a convergent p-series, with $p = 2$. And, since $a > -1$, then all the terms of the series are positive. Hence, by the Weirstrass M-test, we have that the series in question converges uniformly on $[a,b]$ as desired. Moreover, since the function $(-1)^{n+1}\frac{1}{n+x}$ is continuous for each n, then the uniform limit $g(y)$ is continuous.

To conclude, we wish to show that the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. Using the fact that g converges uniformly on a neighborhood of x, we get the following

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\log\left(1+\frac{x+h}{n}\right) - \log\left(1+\frac{x}{n}\right)}{h} \\
&= \sum_{n=1}^{\infty} \lim_{h\to 0} (-1)^{n+1} \frac{\log\left(1+\frac{x+h}{n}\right) - \log\left(1+\frac{x}{n}\right)}{h} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+x} = g(x)
\end{aligned}$$

x was arbitrary, so f is differentiable $(-1,\infty)$ as desired. $\square$

2004 - Autumn

Exercise 1. *Suppose that (a_n) is a decreasing sequence of positive real numbers such that $\sum a_n$ diverges. Prove that* $\lim_{n\to\infty} \frac{a_1 + a_3 + \cdots + a_{2n+1}}{a_2 + a_4 + \cdots + a_{2n}} = 1.$

Proof. (K. Nowland) First we observe that since the sequence is positive and decreasing, we have the inequalities

$$\begin{aligned} a_1 + a_3 + \cdots + a_{2k-1} &\geq a_2 + a_4 + \cdots + a_{2k}, \\ a_2 + a_3 + \cdots + a_{2k} &\geq a_3 + a_5 + \cdots + a_{2k+1}. \end{aligned}$$

Since adding in a_1 would not affect the convergence of the odd terms, it follows that the sum of the even terms converges if and only if the sum of the odd terms converges. The convergence of either would therefore imply the convergence of $\sum a_n$ for all n. Since this is not the case, it follows that both the sum of the even terms and the sum of the odd terms must be divergent.

Using the fact that the sequence is decreasing,

$$\frac{a_1 + a_3 + \cdots + a_{2n+1}}{a_2 + a_4 + \cdots + a_{2n}} = \frac{a_1}{a_2 + \cdots + a_{2n}} + \frac{a_3 + \cdots + a_{2n+1}}{a_2 + \cdots + a_{2n}} \leq \frac{a_1}{a_2 + \cdots + a_{2n}} + 1.$$

Note that this goes to 1 as $n \to \infty$, by the fact that the sum of even terms diverges to positive infinity. But also,

$$\frac{a_1 + \cdots + a_{2n+1}}{a_2 + \cdots + a_{2n}} \geq \frac{a_1 + \cdots + a_{2n+1}}{a_1 + \cdots + a_{2n-1}} = \frac{a_{2n+1}}{a_3 + \cdots + a_{2n+1}} + 1.$$

This also tends to 1 as $n \to \infty$, since the series of odd terms diverges to positive infinity and a_{2n+1} is bounded above by a_1 (and below by 0). The squeese theorem implies the desired result. $\square$

Exercise 2. *Suppose that f is a C^1 function on $\mathbb{R}$ which has the properties that $\lim_{x\to+\infty} f(x) = A$ and $\lim_{x\to+\infty} f'(x) = B$ for some real numbers A and B. Show that $B = 0$.*

Proof. (O. Khalil) Let $\varepsilon > 0$ be fixed. Since $f(x) \to A$ as $x \to \infty$, there exists $M_1 > 0$ such that for all $x > M_1$, we have that $|f(x) - A| < \varepsilon/4$. Thus, for all $x, y > M_1$, we have that

$$|f(x) - f(y)| \leq |f(x) - A| + |A - f(y)| < \varepsilon/2$$

Similarly, since $f'(x) \to B$, there exists $M_2 > 0$ such that for all $x, y > M_2$, we have that $|f'(x) - f'(y)| < \varepsilon/2$. Let $M = \max\{M_1, M_2\}$. Let $x > M$ be arbitrary. We wish to show that $|f'(x)| < \varepsilon$. By the mean value theorem, we have that

$$\frac{f(x+1) - f(x)}{x + 1 - x} = f'(\theta)$$

for some $\theta \in (x, x+1)$. Moreover, we have that since $x, x+1 > M$, then $|f(x+1) - f(x)| = |f'(\theta)| < \varepsilon/2$. Also, since $x, \theta > M$, then we have that $|f'(x) - f'(\theta)| < \varepsilon/2$. But, then we get that

$$|f'(x)| < |f'(\theta)| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

as desired. $\square$

Exercise 3. *Let $\varphi \in C_0^\infty(-1,1)$ (recall that this means that φ is infinitely differentiable and φ is identically 0 in some neighborhood of -1 and 1). Show that for any natural number N, there exists a constant $C = C_N$ such that*

$$\left| \int_{-1}^{1} e^{i\lambda x} \varphi(x) dx \right| \leq C\lambda^{-N} \tag{54}$$

for all $\lambda > 0$.

Proof. (O. Khalil) First, for ease of notation, we may extend φ so that it is defined on the closed interval $[-1,1]$ by letting $\varphi(1) = 0 = \varphi(-1)$. Since $\varphi = 0$ in some neighborhood of 1 and -1, then the one-sided limits exist at these points and are equal to 0. Hence, φ remains continuous after such extension. Moreover, being constant in a neighborhood of 1 and -1 implies that φ' is identically 0 on some neighborhood of 1 and -1 and that $\lim_{x\to 1^-} \varphi'(x) = \lim_{x \to -1^+} \varphi'(x) = 0$. Hence, we may extend φ' in a similar fashion. By induction, for each $n \in \mathbb{N}$, we may extend $\varphi^{(n)}$ to be defined and continuous on $[-1,1]$ with $\varphi^{(n)}(1) = 0 = \varphi^{(n)}(-1)$.

Now, let $\lambda > 0$ be arbitrary and fixed and let $f(x) = e^{i\lambda x}$. For each $n \in \mathbb{N} \cup \{0\}$, integration by parts gives

$$\begin{aligned}
\left(\frac{-1}{i\lambda}\right)^n \int_{-1}^{1} f(x)\varphi^{(n)}(x)dx &= \left(\frac{-1}{i\lambda}\right)^n \left(\frac{1}{i\lambda} f(x)\varphi^{(n)}(x) \Big|_{-1}^{1} - \frac{1}{i\lambda}\int_{-1}^{1} f(x)\varphi^{(n+1)}(x)dx \right) \\
&= \left(\frac{-1}{i\lambda}\right)^{n+1} \int_{-1}^{1} f(x)\varphi^{(n+1)}(x)dx
\end{aligned}$$

Now, fix a natural number N. Induction on the above expression gives that

$$\int_{-1}^{1} f(x)\varphi(x)dx = \left(\frac{-1}{i\lambda}\right)^N \int_{-1}^{1} f(x)\varphi^{(N)}(x)dx$$

Let $C_N = \int_{-1}^{1} \left|\varphi^{(N)}(x)\right| dx$. C_N is finite since $\varphi^{(N)}(x)$ is bounded being continuous on a closed bounded interval. Hence, we get the following

$$\begin{aligned}
\left| \int_{-1}^{1} e^{i\lambda x}\varphi(x)dx \right| &= \left| \left(\frac{-1}{i\lambda}\right)^N \int_{-1}^{1} e^{i\lambda x}\varphi^{(N)}(x)dx \right| \\
&\leq \lambda^{-N} \int_{-1}^{1} \left| e^{i\lambda x}\varphi^{(N)}(x)\right| dx \\
&= \lambda^{-N} \int_{-1}^{1} \left|\varphi^{(N)}(x)\right| dx = C_N \lambda^{-N}
\end{aligned}$$

Since C_N depends only on N and λ was arbitrary, then 54 is verified. $\square$

Exercise 4. *Let f be a differentiable real valued function on $[1,\infty)$ and suppose that $f'(x)/x$ is bounded. Prove that the funtion $f(x)/x$ is uniformly cntinuous on $[1,\infty)$.*

Proof. See Spring 2012, Exercise 3 (with the difference that the function here is defined on $[1,\infty)$ instead of $(1,\infty)$ which makes the problem easier). $\square$

Exercise 5. *Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{x+n^{7/5}}$ for $x \geq 0$.*

(a) Find $\lim_{x\to+\infty} f(x)$.

(b) Find $\lim_{x\to+\infty} \frac{\log f(x)}{\log x}$.

Proof. (K. Nowland)

(a) We claim that $\lim_{x\to+\infty} f(x) = 0$. Note that by comparison with the p-series $\sum n^{-7/5}$, the given series converges for all $x \geq 0$. Fix $x_0 \geq 0$. Let $\varepsilon > 0$. By convergence, there exists N such that

$$\left|\sum_{n=N+1}^{\infty} \frac{1}{x_0+n^{7/5}}\right| < \varepsilon/2.$$

Note that each term decreases as x gets larger, such that this estimate is valid for all $x \geq x_0$. Also, since each term tends to zero as x tends to infinity, there exists $x_1 \geq x_0$ such that the first N terms of the seris (a finite number) satisfies

$$\left|\sum_{n=1}^{N} \frac{1}{x+n^{7/5}}\right| < \varepsilon/2$$

for all $x \geq x_1$. Thus we see that for $x \geq x_1$,

$$\left|\sum_{n=1}^{\infty} \frac{1}{x+n^{7/5}}\right| \leq \left|\sum_{n=1}^{N} \frac{1}{x+n^{7/5}}\right| + \left|\sum_{n=N+1}^{\infty} \frac{1}{x+n^{7/5}}\right| < \varepsilon.$$

Thus $\lim_{x\to+\infty} f(x) = 0$, as claimed.

(b) The limit is $-12/7$. By the integral test,

$$\int_1^{\infty} \frac{dt}{x+t^{7/5}} \leq f(x) \leq \int_0^{\infty} \frac{dt}{x+t^{7/5}}.$$

Examining the upper bound, we calculate

$$f(x) \leq \frac{1}{x}\int_0^{\infty} \frac{dt}{1+(t/x^{5/7})^{7/5}} = \frac{1}{x^{12/7}}\int_0^{\infty} \frac{du}{1+u} = \frac{C}{x^{12/7}},$$

where C is the value of the convergent integral. Similarly, we can bound the bottom to see that

$$\frac{C}{x^{12/7}} - \int_0^{1} \frac{dt}{x+t^{7/5}} \leq f(x) \leq \frac{C}{x^{12/7}}.$$

Since the integral term vanishes as x goes to infinity, we conclude that $\lim_{x\to\infty} \frac{\log f(x)}{\log x} = -\frac{12}{7}$.

□

Exercise 6. *If f is a differentiable strictly increasing function $[0,1]$, can the set $\{x : f'(x) = 0\}$ be uncountable? (You have to justify your answer, of course.)*

Proof. (H. Lyu) This solution is due to Donald, who was the Analysis Qual prep TA.

The answer is yes, i.e., there is a differentiable, strictly increasing function f on $[0,1]$ whose critical points form an uncountable set. The useful way of constructing a differentiable function is to use the fundamental theorem of calculus :

$$f(x) = \int_0^x g(t)\,dt \tag{1}$$

where g is a Riemann integrable function on $[0,1]$. Since f must be differentiable and strictly increasing, it is required that g is nonnegative, continuous, and the set $\{g = 0\}$ uncountable, but contains no open interval. The only sparse uncountable subset of $[0,1]$ we should know of for the Qual is the Cantor set C. So we want $g = 0$ on C, and $g > 0$ on C^c. The point is to make such function g continuous. Here, we use the similar construction when Weierstrass constructed his nowhere differentiable continuous function(see p.154, PMA, Rudin), using the fact that the uniform limit of a sequence of continuous functions is continuous. Recall the contraction of Cantor set: Let $C_0 = [0,1]$, and recursively define C_{n+1} to be the set obtained from C_{n+1}, which is a disjoint union of closed intervals, by taking out the open middle third form each of those components of C_n. Then by definition $C = \bigcap C_n$. C consists of the numbers in $[0,1]$ whose trinary expansion contains no digit of 1. So C is uncountable, and contains no open interval. Now define

$$g_n(x) = \mathrm{dist}(x, C_n^c) = \inf\{|x-y| : y \in C_n^c\}.$$

So each g_n is continuous on $[0,1]$, and $0 \le g_n \le 3^{-n}$. Define

$$g(x) = \sum_{n=0}^{\infty} g_n(x).$$

This series converges uniformly on $[0,1]$ by Weierstrass test, since $|g_n| \le 3^{-n}$ and $\sum 3^{-n} < \infty$. Thus g is continuous on $[0,1]$. For this function g, the function f defined by (1) is a desired one.

□

2004 - Spring

Exercise 1. *Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Show that*

$$\lim_{u \to 0^+} \int_{-1}^{1} \frac{uf(x)}{u^2 + x^2} dx = \pi f(0). \tag{55}$$

Proof. See Autumn 2008, exercise 6. $\square$

Exercise 2. *Let (a_n) be a sequence of real numbers. Suppose that the series $\sum a_n y_n$ converges for every sequence (y_n) with $\lim y_n = 0$. Prove that $\sum a_n$ converges absolutely.*

Proof. (K. Nowland) Suppose toward a contradiction that $\sum |a_n|$ diverges. Then there exists $N_1 \in \mathbb{N}$ such that $\sum_{n=1}^{N_1} |a_n| \geq 1$. Similarly, $\sum_{m=N_1+1}^{\infty} |a_m|$ must diverge, such that there exists $N_2 > N_1$ such that $\sum_{m=N_1+1}^{N_2} |a_n| \geq 2$. Continuing in this way, define a sequence b_n by

$$b_{N_{k-1}+1} = b_{N_{K-1}+1} = \cdots = b_{N_k} = \frac{1}{k}.$$

Let $y_n = \text{sign}(a_n) b_n$. By construction,

$$\sum_{n=1}^{\infty} a_n y_n = \sum_{n=1}^{\infty} |a_n| b_n = \sum_{n=1}^{N_1} |a_n| + \frac{1}{2} \sum_{N_1+1}^{N_2} |a_n| + \frac{1}{3} \sum_{N_2+1}^{N_3} |a_n| + \cdots \geq 1 + 1 + 1 + \cdots .$$

It follows that $\sum a_n y_n$ does not converge. But $y_n \to 0$ as $n \to \infty$, such that this contradicts our hypothesis. The contradiction proves the claim. $\square$

Exercise 3. *Define $h(x) = \sqrt{x^2 + 1}$ for $x \in \mathbb{R}$. Is h uniformly continuous on $\mathbb{R}$? Why?*

Proof. (S.Chowdhury)

Claim: f is uniformly continuous. Note that we can write

$$|f(x) - f(y)| = |\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| = |\frac{x^2 + 1 - (y^2 + 1)}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}|$$

This becomes

$$|\frac{x^2 - y^2}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}| = |\frac{(x + y)(x - y)}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}}|$$

We can control $|x - y|$ via δ, so we need to control the other part.

$$|x| < \sqrt{x^2 + 1}, \text{ and } |y| < \sqrt{y^2 + 1} \Rightarrow |x + y| \leq |x| + |y| < \sqrt{x^2 + 1} + \sqrt{y^2 + 1}$$

So we can write:

$$\frac{|x + y|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}} < 1$$

(Note that the bottom term is positive.)

Finally, if we fix $\varepsilon > 0$ and set $\delta = \varepsilon$, we have

$$|x - y| < \delta = \varepsilon$$
$$\Rightarrow |f(x) - f(y)| = \left|\frac{(x+y)(x-y)}{\sqrt{x^2+1} + \sqrt{y^2+1}}\right| < \varepsilon$$

$\square$

Proof. (H. Lyu) Recall that a function is uniformly continuous if it is Lipschitz continuous. Suppose f is Lipschitz continuous on $\mathbb{R}$, i.e., there exists a constant $K \geq 0$ such that for all $x > y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq K|x - y|.$$

Now given $\epsilon > 0$, one has $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$ when $0 < \delta < \epsilon/K$. Thus f is uniformly continuous.

If a function g is differentiable, a useful way to show g is Lipschitz continuous is to show that g' is bounded. Indeed, if there is a constant $M > 0$ such that $|g'| < M$, then by the mean value theorem we have

$$|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$$

for each $x > y \in \mathbb{R}$, where $\xi \in (x, y)$.

Now according to the previous notes, it suffices to show that h' is bounded. Indeed, for all $x \in \mathbb{R}$,

$$|h'(x)| = \left|\frac{x}{\sqrt{x^2+1}}\right| < 1.$$

Thus h is uniformly continuous on $\mathbb{R}$

$\square$

Exercise 4. *Let U be an open set in $\mathbb{R}$. Show that U may be written as a countable (or finite) disjoint union of open intervals.*

Proof. (S.Chowdhury) *StackExchange has several variants on this proof; I'm quoting one of them here.*

Take an open set O in $\mathbb{R}$. Let $\mathcal{I}$ denote the family of all open intervals contained in O. Define an equivalence relation $\sim$ on $I, J \in \mathcal{I}$ by setting $I \sim J$ if there exists a sequence $I = I_0, I_1, \cdots, I_n = J$ such that $I_k \cap I_{k+1}$ is nonempty.

Let $[I]$ denote the equivalence class of I. Then $[I]$ is an open interval, and $\displaystyle\bigcup_{I \in \mathcal{I}} [I]$ is a decomposition of O into pairwise disjoint open intervals.

To show countability, take a rational number q_I in each $[I]$. Let $\mathcal{K}$ denote the set of equivalence classes described above. Then we have a map $f : \mathcal{K} \to \mathbb{Q}$ given by $f([I]) = q_I$. Because the classes are disjoint, the map is injective, and thus we have a *countable* collection of open intervals. $\square$

Exercise 5. *If a function $g : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and g' is one-to-one, prove that g' is monotone.*

Proof. (S.Chowdhury)

This problem requires Darboux's theorem about the intermediate value property of derivatives. Recall that on an interval $[a, b]$, if we have $g'(a) < \lambda < g'(b)$, then there exists $c \in (a, b)$ such that $g'(c) = \lambda$. Here's the proof:

Define $h(x) = g(x) - \lambda x$. Then $h'(x) = g'(x) - \lambda$, and so $h'(b) > 0$ (increasing), $h'(a) < 0$ (decreasing). Thus there exist d, e in (a, b) such that $h(e) < h(b)$ and $h(d) < h(a)$. In particular, h attains its minimum value (extreme value theorem), and it does so at some point $c \in (a, b)$. At this point, we have $h'(c) = 0 = g'(c) - \lambda$.

Back to the proof. For contradiction, assume g' is 1-1 but not monotonic. Then there exist $a < b < c \in \mathbb{R}$ such that (WLOG) $g'(a) < g'(b) > g'(c)$, and either $g'(a) < g'(c)$ or $g'(a) > g'(c)$. Suppose (WLOG) $g'(a) < g'(c)$. Then we have $g'(a) < g'(c) < g'(b)$, and by Darboux's theorem, there exists a point $d \in (a, b)$ such that $g'(d) = g'(c)$. But this is a contradiction because g' is 1-1.

$\square$

Exercise 6. *Let the function φ be continuous on $[0, 1]$ with*

$$\int_0^1 \varphi(x)dx = 0 \qquad \text{and} \qquad \int_0^1 x\varphi(x)dx = 1.$$

Prove that $|\varphi(x)| \geq 4$ for some $x \in [0, 1]$.

Proof. (H. Lyu) Suppose for contrary that $|\varphi| < 4$. Define

$$f(x) = \int_0^x \varphi(t)\, dt.$$

Then f is differentiable on $(0, 1)$ with $f' = \varphi$, $f(0) = 0$ and $f(1) = \int_0^1 \varphi(x)\, dx = 0$. The second condition implies $|\int_0^1 f(x)\, dx| = 1$, since by integrating by parts

$$1 = \int_0^1 t\varphi(t)\, dt = [tf(x)]_0^1 - \int_0^1 f(x)\, dx = -\int_0^1 f(x)\, dx.$$

Now the idea is the following. Since $|f'| = |\varphi| < 4$, and $f(0) = f(1) = 0$, the graph of $y = f(x)$ must be enclosed by the four lines of "maximal" slope, namely, $y = \pm 4x$ and $y = \pm 8 \mp 4x$. But then the net area under $y = f(x)$ must be strictly less than 1, contrary to the fact that $|\int_0^1 f(x)\, dx| = 1$. To be more precise, we claim that $|f(x)| < |4x|$ on $[0, 1/2]$

and $|f(x)| < |8-4x|$ on $[1/2, 1]$. First suppose for contrary that $|f(y)| \geq |4y|$ for some $y \in [0, 1/2]$. Then by the mean value theorem, there exists $\xi_y \in (0, 1/2)$ such that

$$|\varphi(\xi_y)| = \left|\frac{f(y) - f(0)}{y - 0}\right| = \left|\frac{f(y)}{y}\right| \geq \left|\frac{4y}{y}\right| \geq 4,$$

contrary to the assumption that $|\varphi| < 4$. The similar argument works for the second part of our claim. Recall that if ϕ is a continuous non-negative function on $[0, 1]$ and $\int_0^1 \phi = 0$, then $\phi = 0$. This implies that if g, h are continuous functions on $[0, 1]$ such that $g(a) < h(a)$ for some $a \in [0, 1]$, then $\int_0^1 g < \int_0^1 h$. In particular, we have

$$\int_0^{1/2} |f(x)|\,dx < \int_0^{1/2} 4x\,dx, \qquad \int_{1/2}^1 |f(x)|\,dx < \int_0^{1/2} 8 - 4x\,dx.$$

Therefore we obtain

$$1 = \left|\int_0^1 f(x)\,dx\right| \leq \int_0^1 |f(x)|\,dx < \int_0^{1/2} 4x\,dx + \int_{1/2}^1 8 - 4x\,dx = 1/2 + 1/2 = 1,$$

which is a contradiction. This shows that $|\varphi(x)| \geq 4$ for some $x \in [0, 1]$. $\square$

2003 - Autumn

Exercise 1. *Prove or disprove: $g(x) = \sin(e^x)$ is uniformly continuous on $\mathbb{R}$.*

Proof. (S. Chowdhury) Claim: g is not uniformly continuous. To see this, consider the sequences given by

$$x_n = \ln(2n\pi + \frac{\pi}{2}),\ y_n = \ln(2n\pi + \frac{3\pi}{2})$$

Observe that we have

$$y_n - x_n = \ln \frac{2n\pi + \frac{3\pi}{2}}{2n\pi + \frac{\pi}{2}} = \ln(1 + \frac{\pi}{2n\pi + \frac{\pi}{2}})$$

So $y_n - x_n \to 0$.

On the other hand, $|f(y_n) - f(x_n)| = |\sin(2n\pi + \frac{3\pi}{2}) - \sin(2n\pi + \frac{\pi}{2})| = 2$. Set $\varepsilon = 2$; this contradicts uniform continuity. □

Exercise 2. *Suppose (f_n) is a sequence of functions on $[0, 1]$ which converges pointwise to a continuous function f and suppose that for each n, the function f_n is increasing on $[0, 1]$. Does it follow that $f_n \to f$ uniformly? Justify your answer.*

Proof. (S. Chowdhury) Claim: $f_n \to f$ uniformly.

First note that f is continuous on a compact domain, hence it is uniformly continuous. Fix ε; this gives us a δ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Now we partition $[0, 1]$ into intervals of length $< \frac{\delta}{2}$, with each endpoint a rational number. Thus we have a partition $P = \{0 = q_0, q_1, \cdots, q_k = 1\}$. Now consider

$$|f_n(x) - f(x)| = |f_n(x) - f_n(q_i) + f_n(q_i) - f(x)| \leq |f_n(x) - f_n(q_i)| + |f_n(q_i) - f(x)|$$

where we have taken q_i to be the smallest rational in P that is greater than x.

Controlling $|f_n(x) - f_n(q_i)|$: We have $q_{i-1} < x \leq q$ and we also know that f_n is increasing. So we can write

$$\begin{aligned} |f_n(x) - f_n(q_i)| < |f_n(q_{i-1}) - f_n(q_i)| &= |f_n(q_{i-1}) - f(q_{i-1}) + f(q_{i-1}) - f(q_i) + f(q_i) - f_n(q_i)| \\ &\leq |f_n(q_{i-1}) - f(q_{i-1})| + |f(q_{i-1}) - f(q_i)| + |f(q_i) - f_n(q_i)| \end{aligned}$$

The first and third terms here can be controlled by pointwise convergence at the rationals; we just need to choose a large n that works for all $q_1, \cdots, q_k$. The second term can be controlled by the uniform continuity of f.

Controlling $|f_n(q_i) - f(x)|$: Write this as

$$\begin{aligned} |f_n(q_i) - f(x)| &= |f_n(q_i) - f(q_i) + f(q_i) - f(x)| \\ &\leq |f_n(q_i) - f(q_i)| + |f(q_i) - f(x)| \end{aligned}$$

We have already seen that the first term can be controlled by pointwise convergence (for all q_i), and the second term can be controlled by uniform continuity.

We can control $|f_n(x) - f(x)|$ for all $x \in [0,1]$, and so we have uniform convergence. $\square$

Exercise 3. *Let $a_n \downarrow 0$ with $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $na_n \to 0$.*

Proof. (S. Chowdhury) Apparently this problem is a classic, and there are at least two ways to solve it. One uses even and odd subsequences (and is a little ad-hoc, I think), and the other uses the Cauchy condensation test. We will use the condensation test to show that $\sum na_n$ converges.

$\sum a_n$ converges (bounded series of nonnegative terms), so by the CCT, we know $\sum 2^n a_{2^n}$ converges and $2^n a_{2^n} \to 0$. Consider the values of k for which $2^n < k < 2^{n+1}$. Because the terms are decreasing, we also have $a_{2^{n+1}} \leq a_k \leq a_{2^n}$. So we can write:

$$\begin{aligned} 2^n a_{2^{n+1}} < ka_{2^{n+1}} \leq ka_k \leq ka_{2^n} < 2^{n+1} a_{2^n} \\ \Rightarrow \frac{1}{2}(2^{n+1} a_{2^{n+1}}) \leq ka_k \leq 2 \cdot (2^n a_{2^n}) \end{aligned}$$

The terms in parentheses go to zero, so by the squeeze theorem, the term in the middle also goes to zero. Thus we conclude that na_n goes to zero. $\square$

Exercise 4. *Construct (or prove the existence of) a continuous function f on $[0, \infty)$ such that the improper integrals $\int_0^\infty f(x)dx$ and $\int_0^\infty xf(x)dx$ are both well-defined and equal to zero but $\int_0^\infty |f(x)|dx = \infty$.*

Proof. (K. Nowland) $f(x) = \sin x/(x \log x)$ is such a function (after removing the singulairty near the origin in a continuous way). Since $\sin x/(x \log x)$ and $\sin x/\log x$ satisfy the condition that $\int_0^N \sin x dt$ is bounded for all N and $1/x \log x$ and $1/\log x$ go to zero as as x goes to infinity, the integrals we wish to converge will certainly converge.

On the other hand, $|\sin x/x \log x|$ may not have a convergent integral. If we break up the integral into intervals of length π, we see that the integral of $|\sin x|$ over any period will be the same. But then the integral of $1/x \log x$ diverges, as the antiderivative is $\log \log x$. $\square$

Exercise 5. *Suppose f' exists and is decreasing on $[0, \infty)$ and $f(0) = 0$. Prove that $\frac{f(x)}{x}$ is decreasing in $(0, \infty)$.*

Proof. (S. Chowdhury) Set $g(x) = \frac{f(x)}{x}$. Then $g'(x) = \frac{xf'(x) - f(x)}{x^2}$. The denominator is positive; we want to show that the numerator is negative.

Apply the MVT on $(0, x)$; we get

$$\begin{aligned} f(x) - f(0) &= xf'(c) \\ \Rightarrow f'(c) &= \frac{f(x)}{x} \\ \text{But } f' \text{ is decreasing, so } f'(x) < f'(c) &= \frac{f(x)}{x} \\ \Rightarrow xf'(x) - f(x) &< 0 \\ \Rightarrow g'(x) &< 0 \\ \Rightarrow \frac{f(x)}{x} &\text{ is decreasing.} \end{aligned}$$

$\square$

Exercise 6. *Let $f \in C^2[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Let $m = (a + b)/2$, the midpoint of the interval $[a, b]$. Prove that there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)dx = (b - a)f(m) + \frac{1}{24}f''(c)(b - a)^3. \tag{56}$$

Proof. (O. Khalil) Define the function $F(x) = \int_a^x f(t)dt$. Since f is twice continuously differentiable on $[a, b]$, then by the Fundamental Theorem of Calculus, F is 3-times continuously differentiable on (a, b).

Using Taylor's expansion with Lagrange remainder, write

$$F(b) = F(m) + (b - m)f(m) + \frac{(b - m)^2}{2}f'(m) + \frac{(b - m)^3}{6}f''(c_1) \tag{57}$$

$$F(a) = F(m) + (a - m)f(m) + \frac{(a - m)^2}{2}f'(m) + \frac{(a - m)^3}{6}f''(c_2) \tag{58}$$

for some $c_1 \in (m, b)$ and $c_2 \in (a, m)$. Now, observe that $b - m = \frac{b-a}{2}$ and $a - m = \frac{a-b}{2}$. And, so, we have that $(b - m)^2 = (a - m)^2$.

Substituting these calculations and substracting 58 from 57, we get

$$\int_a^b f(x)dx = f(m)(b - a) + \frac{(b - a)^3}{24}\left(\frac{f''(c_1) + f''(c_2)}{2}\right) \tag{59}$$

But, since f'' is continuous on $[a, b]$ and since $\frac{f''(c_1)+f''(c_2)}{2}$ lies between $f''(c_1)$ and $f''(c_2)$, then, by the intermediate value theorem, we have that there exists $c \in (c_1, c_2) \subset (a, b)$ such that $f''(c) = \frac{f''(c_1)+f''(c_2)}{2}$. Substituting in 59, we get 56 as desired. $\square$

2003 - Spring

Exercise 1. *Find a triangle ABC of maximum area if $A = (-1, 1)$, $B = (2, 4)$, and*

$$C \in \left\{(x, y) : y = x^2, -2 \leq x \leq 2\right\}.$$

(E. Nash) This problem has two valid proofs. We present them both.

Proof #1. Draw a sketch and guess. You will be correct. □

Proof #2. Note that this problem is equivalent to considering $A = (-1, -1)$, $B = (2, 2)$, and

$$C \in \{(x, y) : y = x^2 - 2, -2 \leq x \leq 2\}.$$

We have just shifted every point down by two. Now if $C = (x, x^2 - 2)$ for some $x \in [-2, 2]$, the area of the triangle ABC is one half the length of the segment AB times the length of the perpendicular line from $(x, x^2 - 2)$ to the subspace spanned by $(1, 1)$. Thus, maximizing the area of the triangle is equivalent to maximizing the length of this perpendicular. We must first calculate the projection of the vector $(x, x^2 - 2)$ onto the subspace spanned by $(1, 1)$. The unit vector in the direction of $(1, 1)$ is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, so the projection is

$$\left[(x, x^2 - 2) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right] * \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\frac{x^2}{2} + \frac{x}{2} - 1, \frac{x^2}{2} + \frac{x}{2} - 1\right).$$

Subtracting this projection vector from the vector $(x, x^2 - 2)$ gives the perpendicular vector we are seeking:

$$(x, x^2 - 2) - \left(\frac{x^2}{2} + \frac{x}{2} - 1, \frac{x^2}{2} + \frac{x}{2} - 1\right) = \left(-\frac{x^2}{2} + \frac{x}{2} + 1, \frac{x^2}{2} - \frac{x}{2} - 1\right).$$

Set $p_1(x) = -\frac{x^2}{2} + \frac{x}{2} + 1 = -\frac{1}{2}(x-2)(x+1)$, $p_2(x) = \frac{x^2}{2} - \frac{x}{2} - 1 = \frac{1}{2}(x-2)(x+1)$, and $p(x) = p_1(x)^2 + p_2(x)^2$. Then the length of the perpendicular vector is $\sqrt{p(x)}$. To maximize this function on the interval $[-2, 2]$, we take the derivative and set it equal to 0: $\frac{p'(x)}{2\sqrt{p(x)}} = 0$. Note first that $p(x) = 0$ only when both $p_1(x) = 0$ and $p_2(x) = 0$, so the derivative of $\sqrt{p(x)}$ is undefined at $x = -1$ and $x = 2$ and we must consider these as critical points when calculating the extrema. Now we find when $p'(x) = 0$, which will coincide with the zeroes of $\frac{p'(x)}{2\sqrt{p(x)}}$:

$$p'(x) = 2p_1(x)\left(\frac{1}{2} - x\right) + 2p_2(x)\left(x - \frac{1}{2}\right) = 2(x-2)(x+1)\left(\frac{1}{2} - x\right)$$

Thus, we have critical points at $x = -1, \frac{1}{2}, 2$, and the endpoint -2. Clearly $\sqrt{p(-1)} = \sqrt{p(2)} = 0$, so we must consider $\frac{1}{2}$ and the endpoint $x = -2$. We have the following:

$$\sqrt{p\left(\frac{1}{2}\right)} = \sqrt{\left(-\frac{1}{8} + \frac{1}{4} + 1\right)^2 + \left(\frac{1}{8} - \frac{1}{4} - 1\right)^2} = \sqrt{\left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2}$$

$$\sqrt{p(-2)} = \sqrt{(-2-1+1)^2 + (2+1-1)^2} = \sqrt{2^2+2^2}.$$

Thus, $\sqrt{p(-2)} \geq \sqrt{p\left(\frac{1}{2}\right)}$ and so $\sqrt{p(x)}$ attains a maximum at $x = -2$ on the interval $[-2, 2]$. Recasting this in the context of the original problem, the triangle of maximum area therefore has vertices $A = (-1, 1)$, $B = (2, 4)$, and $C = (-2, 4)$. $\square$

Exercise 2. *Let f_n be differentiable on $[0, 1]$ and suppose:*

(a) For each $n \in \mathbb{N}$ and each $x \in [0, 1]$, $|f_n'(x)| \leq 1$;

(b) For each $q \in \mathbb{Q} \cap [0, 1]$, the sequence of numbers $(f_n(q))$ converges.

Prove that the sequence of functions (f_n) converges uniformly on $(0, 1)$.

Proof. (S. Chowdhury) Use the Cauchy criterion for uniform convergence; we want to get

$$|f_n(x) - f_m(x)| < \varepsilon$$

for any $x \in [0, 1]$, given n, m large enough. Write this as:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f_n(q) + f_m(q) - f_m(x) + f_n(q) - f_m(q)| && (60) \\ &\leq |f_n(x) - f_n(q) + f_m(q) - f_m(x)| + |f_n(q) - f_m(q)| && (61) \end{aligned}$$

Note that we can control the second term in (61) by the assumption that $\{f_n\}$ converges on the rationals. So we need to try and control the first term.

Try MVT on the function $f_n - f_m$; for any $q, x \in [0, 1]$, we have

$$\begin{aligned} |f_n(x) - f_m(x) - f_n(q) + f_m(q)| &= |f_n'(c) - f_m'(c)||x - q| \\ &\leq 2 \cdot \frac{\varepsilon}{4} \end{aligned}$$

The "2" follows from the assumption that f_n' is bounded, and the $\frac{\varepsilon}{4}$ appears because we can choose q as close to x as we want.

Now we can control (61) and make it smaller than ε; this concludes the proof. $\square$

Exercise 3. *Let f be twice-differentiable on $\mathbb{R}$ and suppose there are constants $A, C \in [0, \infty)$ such that for each $x \in \mathbb{R}$, $|f(x)| \leq A$ and $|f''(x)| \leq C$. Prove that there is a constant $B \in [0, \infty)$ such that for each $x \in \mathbb{R}$, $|f'(x)| \leq B$.*

Proof. (S. Chowdhury) Use Taylor expansion about α.

$$f(\beta) - P(\beta) = f''(\gamma)\frac{(\beta - \alpha)^2}{2}$$

For our case, we set $\alpha = x$ and $\beta = x + 2h$.

$$\begin{aligned}
f(x+2h) - f(x) - f'(x)(x+2h-x) =& f''(\gamma) \cdot 2h^2 \\
\Rightarrow f'(x) =& \frac{f(x+2h) - f(x) - f''(\gamma)2h^2}{2h} \\
\Rightarrow \left|f'(x)\right| =& \frac{1}{2h}\left|f(x+2h) - f(x) - 2h^2 f''(\gamma)\right| \\
\leq& \frac{1}{2h}\left|f(x+2h) - f(x)\right| + h\left|f''(\gamma)\right| \\
<& \frac{1}{2h}2A + hC \text{ (the bounds come from assumption)} \\
=& \frac{A}{h} + hC
\end{aligned}$$

This gives us a bound and concludes the proof. ◻

Exercise 4. *Let f be a continuous function on $[0,1]$. Determine*

$$\lim_{n\to\infty} n \int_0^1 e^{n(x-1)} f(x) dx.$$

Proof. (K. Nowland) This is a typical d-function approximation. Note that $ne^{n(x-1)}$ tends to zero pointwise on $[0,1)$ but tends to infinity at 1, such that we would guess that the limit is $f(1)$. Note that

$$\frac{f(1)}{1-e^{-n}} \to f(1)$$

as $n \to \infty$, such that it suffices to show that the limit of the above is the same as this limit, i.e., for n large enough, the difference between the two can be made arbitrarily small. (To be precise, this would require a triangle inequality.) Let $\varepsilon > 0$ be fixed. Since f is continuous, there exists $\delta > 0$ such that $1 - \delta < x \leq 1$ implies $|f(x) - f(1)| < \varepsilon/2$. Since f is continuous on the compact interval $[0,1]$, it is bounded in absolute value by some constant $M > 0$. We calculate

$$\begin{aligned}
\left|\int_0^1 ne^{n(x-1)} f(x)dx - \frac{f(1)}{1-e^{-n}}\right| &= \left|\int_0^1 ne^{n(x-1)}(f(x) - f(1))dx\right| \\
&\leq \int_0^1 ne^{n(x-1)}|f(x) - f(1)|dx \\
&= \int_0^{1-\delta} ne^{n(x-1)}|f(x) - f(1)|dx + \varepsilon/2 \int_{1-\delta}^1 ne^{n(x-1)}dx \\
&\leq 2M \int_0^{1-\delta} ne^{n(x-1)}dx + \varepsilon/2 \int_0^1 ne^{n(x-1)}dx \\
&\leq 2M(e^{-n\delta} - e^{-n}) + \varepsilon/2(1 - e^{-n}) \\
&\leq 2Me^{-n\delta} + \varepsilon/2.
\end{aligned}$$

Since $e^{-n\delta} \to 0$ as $n \to \infty$, let n be so large that $e^{-n\delta} < M\varepsilon/4$, such that we have

$$\left| \int_0^1 n e^{n(x-1)} f(x) dx - \frac{f(1)}{1 - e^{-n}} \right| < \varepsilon,$$

as desired. □

Exercise 5. *Either prove the following statement, or disprove it by giving a counterexample: For each nonnegative continuous function f on $[0, \infty)$, if the improper Riemann integral $\int_0^\infty f dx$ converges, then $\int_0^\infty f^3 dx$ converges.*

Proof. (K. Nowland) The statement is false. We build a continuous function $f(x)$ as follows. Let $f(x) = n$ for $x \in [n - .5n^{-3}, n + .5n^{-3}]$ for all $n \geq 2$. At the endpoints of each such interval, liniearly connect $f(x)$ to zero at a distance of $.5n^{-3}$ form the end of the interval. Let f be zero elsewhere. Then the integral of f over the interval $[n - n^{-3}, n + n^{-3}]$ is bounded by $2/n^2$, since f is bounded by n on these intervals but the length of the interval is only $2/n^3$. Since the series $\sum_{n=1}^\infty n^{-2}$ converges, this proves that $\int_0^\infty f dx$ converges.

On the other hand, $f^3(x) = n^3$ on $[n - .5n^{-3}, n + .5n^{-3}]$, such that the integral just over this interval is 1. Since $\int_0^\infty f^3 dx$ is greater than the sum of the integrals over the inifnitely many such intervals, $\int_0^\infty f^3 dx$ must diverge. □

Exercise 6. *Let K be a compact subset of $\mathbb{R}^n$ and let f be a map from K to K. Consider the graph of f:*

$$G_f = \{(x, f(x)) : x \in K\}.$$

Prove that if G_f is a closed subset of $K \times K$, then f is continuous.

Proof. (S. Chowdhury) First, we recall a general result: Suppose we have a vector-valued function

$$g(x) = \big(f_1(x), f_2(x), \cdots f_n(x)\big)$$

Then g is continuous iff each f_i is continuous. Proof: use the inequality

$$\big|f_i(x) - f_i(y)\big| \leq \big|g(x) - g(y)\big| = \Big[\sum_{i=1}^n \big|f_i(x) - f_i(y)\big|^2\Big]^{\frac{1}{2}}$$

If each f_i is continuous, then we can control the $|f_i(x) - f_i(y)|$ terms and hence control $|g(x) - g(y)|$. On the other hand, if g is continuous, then we can control each $|f_i(x) - f_i(y)|$ by the inequality given above.

Now we discuss the main proof. Define

$$\begin{aligned} g : K &\to G_f \\ x &\mapsto (x, f(x)) \end{aligned}$$

This map is both onto and 1-1, and it makes sense to define g^{-1} by

$$g^{-1}((x, f(x))) = x$$

Note that g^{-1} is continuous. Given $\varepsilon > 0$, set $\delta = \varepsilon$. Then

$$\begin{aligned} \left|(x, f(x)) - (y, f(y))\right| &< \delta \\ \Rightarrow \left[(x-y)^2 + (f(x) - f(y))^2\right]^{\frac{1}{2}} &< \delta \\ \Rightarrow x - y < \delta = \varepsilon \end{aligned}$$

Next, we claim that g is continuous. We will use the result that a function is continuous if the preimages of closed sets are closed. Observe that K compact implies $K \times K$ is compact (general result: a product space is compact iff each component space is compact - does this need to be proved in the qual?), and if G_f is closed, then G_f is compact (closed subsets of compact sets are compact). Take a closed set V in G_f. V is compact, so $g^{-1}(V)$ is compact (continuous image of a compact set) and hence closed (Heine-Borel). Thus g is continuous, and by the first result about vector-valued functions, we conclude that f is continuous. $\square$

2002 - Autumn

Exercise 1. *Determine whether the sequence of functions*

$$F_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}, \quad n = 1, 2, 3, \ldots,$$

converges uniformly on the whole real line.

Proof. (K. Nowland) The sums do not converge uniformly on all of $\mathbb{R}$. Note that

$$|F_{n+1}(x) - F_n(x)| = \frac{|x|^{2n+1}}{(2n+1)!}.$$

With n fixed, this tends to infinity as $x \to \pm\infty$. Therefore

$$\sup_{x \in \mathbb{R}} |F_{n+1}(x) - F_n(x)| = \infty$$

for any $n \in \mathbb{N}$. Therefore the sequence is not uniformly Cauchy on $\mathbb{R}$ whence it is not uniformly convergent on $\mathbb{R}$. $\square$

Exercise 2. *Prove that the sequence*

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, \quad n = 1, 2, 3, \ldots,$$

converges as $n \to \infty$. (Here $\ln n$ means the natural logarithm of n.)

Proof. See 05S1. $\square$

Exercise 3. *Let K be a compact subset of $\mathbb{R}^2$ and let $f : K \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous.*

Proof. (O. Khalil) Suppose by way of contradiction that f is not uniformly continuous on K. Then, there exists $\varepsilon > 0$ and points $x_n, y_n \in K$ such that $|x_n - y_n| < 1/n$ for each $n \in \mathbb{N}$, but $|f(x_n) - f(y_n)| \geq \varepsilon$. But, since $\mathbb{R}^2$ is Euclidean and K is compact, then K is closed and bounded (by the Heine-Borel theorem). Hence, in particular, the sequence (x_n) is bounded. But, then, by the Bolzano-Weirstrass theorem,(x_n) has a convergent subsequence (x_{n_j}) with limit x. Similarly, (y_{n_j}) has a convergent subsequence $(y_{n_{j_k}})$ with limit y. Since K is closed, then $x, y \in K$. Also, by continuity of f, we have that $f(x_{n_{j_k}}) \to f(x)$ and $f(y_{n_{j_k}}) \to f(y)$. Moreover, since $|f(x_n) - f(y_n)| > \varepsilon$ for all n, then $|f(x) - f(y)| \geq \varepsilon > 0$ and so $f(x) \neq f(y)$. But, since $|x_n - y_n| < 1/n$ for all n, then we have that

$$y_{n_{j_k}} - \frac{1}{n_{j_k}} < x_{n_{j_k}} < \frac{1}{n_{j_k}} + y_{n_{j_k}}$$

for all k. Thus, taking the limit as $k \to \infty$, we get that $x = y$. Hence, we have that $f(x) = f(y)$, a contradiction. Therefore, f is uniformly continuous as desired. $\square$

Exercise 4. *Let $f : (a, b) \to \mathbb{R}$ be a convex function. (To say that f is* convex *means that for all $x_0, x_1 \in (a, b)$ and all $t \in [0, 1]$, we have $f((1-t)x_0 + tx_1) \le (1-t)f(x_0) + tf(x_1)$.) Prove that the right hand derivative of f exists and is finite at every point of (a, b). (Of course the same is true for the left hand derivative, although you are not asked to prove this.)*

Proof. (K. Nowland) The key to this proof is drawing pictures to understand the situation. I have not drawn them as that is difficult in LaTeX. Let $x \in (a, b)$. To show that the right hand derivative exists, we must show that the limit

$$\lim_{y \to x^+} \frac{f(x) - f(y)}{x - y}$$

exists. To show this we show the following: That $[f(x) - f(y)]/[x - y]$ is decreasing in y and bounded below.

First we show that the sequence is bounded below. Let $a < w < x < y$. Then we claim that

$$\frac{f(x) - f(w)}{x - w} \le \frac{f(y) - f(x)}{y - x} = \frac{f(x) - f(y)}{x - y}.$$

Since $w < x < y$, it follows that there exists $\lambda \in (0, 1)$ such that $x = \lambda w + (1 - \lambda)y$. It is an easy calculation to see that $\lambda = (y - x)/(y - w)$. By convexity, we have

$$f(x) \le \frac{y - x}{y - w} f(w) + \frac{x - w}{y - w} f(y).$$

Using the fact that $1 = \lambda + (1 - \lambda)$, we can rewrite this as

$$\frac{y - x}{y - w} f(x) + \frac{x - w}{y - w} f(x) \le \frac{y - x}{y - w} f(w) + \frac{x - w}{y - w} f(y).$$

Rearranging,

$$\frac{y - x}{y - w}(f(x) - f(w)) \le \frac{x - w}{y - w}(f(y) - f(x)).$$

clearing the denominator,

$$(y - x)(f(x) - f(w)) \le (x - w)(f(y) - f(x)).$$

Dividing by $y - x$ and $x - w$ gives the desired bound from below.

Now we show that the sequence is decreasing. Let z be such that $x < y < z$. We want to show that

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(x) - f(z)}{x - z}$$

As above, there exists $\lambda \in (0, 1)$ such that $y = \lambda x + (1 - \lambda)z$. A quick calculation shows that $\lambda = (z - y)/(z - x)$. By convexity,

$$f(y) \le \frac{z - y}{z - x} f(x) + \frac{y - x}{z - x} f(z).$$

As, above we rewrite the left hand side and obtain

$$\frac{z-y}{z-x}f(y)+\frac{y-x}{z-x}f(y)\leq\frac{z-y}{z-x}f(x)+\frac{y-x}{z-x}f(z).$$

Rearranging and clearing the denominator gives

$$(z-y)(f(y)-f(x))\leq(y-x)(f(z)-f(y)).$$

Dividing through,

$$\frac{f(y)-f(x)}{y-x}\leq\frac{f(z)-f(y)}{z-y}.$$

After multiplying top and bottom of each side by -1, we have the desired inequality. Thus we have shown that the difference quotient is decreasing in y and bounded below, such tha the limit exists. This limit is the right hand derivative, $\square$

Exercise 5. *Let $f:[0,\infty)\to\mathbb{R}$ be uniformly continuous and suppose that the improper Riemann integral $\int_0^\infty f(x)dx$ converges. Prove that $f(x)\to 0$ as $x\to\infty$.*

Proof. (See 07A5 and 06S6 for 2 different solutions) $\square$

Exercise 6. *Let $f:\mathbb{R}\to\mathbb{R}$ be increasing. Prove that f has at most a countable number of discontinuities.*

Proof. (O. Khalil) Let $a\in\mathbb{R}$ be a point of discontinuity for f. We first show that the one-sided limits exist at a. Indeed, let (x_n) be any increasing sequence such that $x_n\to a$, then since f is increasing, we get that the sequence $(f(x_n))$ is increasing and bounded above by $f(a)$. Thus, it converges. Therefore, $f(a^-)=\lim_{x\to a^-}f(x)$ exists in $\mathbb{R}$ and $f(a^-)\leq f(a)$. Similarly, $f(a^+)=\lim_{x\to a^+}f(x)$ exists and $f(a^+)\geq f(a)$. Since f is discontinuous at a, then $f(a^-)\neq f(a^+)$. In particular, $f(a^-)<f(a^+)$. Let $r_a\in\mathbb{Q}\cap(f(a^-),f(a^+))$ be a rational number.

Let b be any other point of discontinuity for f and suppose that $a<b$. Let $c\in(a,b)$. Then, we have that $f(a^-)<f(c)<f(b^+)$. Hence, we get that $(f(a^-),f(a^+))\cap(f(b^-),f(b^+))=\varnothing$. Hence, we have that $r_a\neq r_b$.

Now, let $D=\{x: f \text{ is discontinuous at } x\}$ and define a map $\varphi: D\longrightarrow\mathbb{Q}$ by $\varphi(x)=r_x$. By the above argument, we have that φ is one-to-one and thus the cardinality of D is at most that of $\mathbb{Q}$ which is countable. $\square$

2002 - Spring

Exercise 1. *Prove this form of Dini's theorem: Let* (f_n) *be a sequence of continuous real-valued functions on the closed bounded interval* $[a, b]$*. Suppose that for each* $t \in [a, b]$*, we have* $f_n(t) \geq f_{n+1}(t)$ *for all* n *and* $\lim_{n\to\infty} f_n(t) = 0$*. Prove that* (f_n) *converges uniformly to 0 on* $[a, b]$*.*

Proof. See 08A5. $\square$

Exercise 2. *Let* f *be a differentiable function from* $\mathbb{R}$ *to* $\mathbb{R}$*. Suppose that for each* $x \in \mathbb{R}$*, we have*

$$0 \leq f(x) \leq \frac{1}{1+x^2}.$$

Show that there exists $c \in \mathbb{R}$ *such that*

$$f'(c) = \frac{-2c}{(1+c^2)^2}. \tag{62}$$

Proof. (K. Nowland) Let $g(x) = (1+x^2)^{-1} - f(x)$. This is a nonnegative, real-valued function which is differentiable on all of $\mathbb{R}$. We wish to find c such that $g'(c) = 0$. If $g \equiv 0$, then the statement is obvious, as any c will work. In the sequel we assume this is not the case. Now suppose there exists $x_1 \neq x_2$ such that $g(x_1) = g(x_2)$. Without loss of generality, suppose $x_1 < x_2$. By the mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$g'(c)(x_2 - x_1) = g(x_2) - g(x_1) = 0.$$

Since $x_1 \neq x_2$, we may divide by $x_2 - x_1$ to see that $g'(c) = 0$, as desired.

Note that $0 \leq g(x) \leq (1+x^2)^{-1}$. Since $(1+x^2)^{-1} \to 0$ as $x \to \pm\infty$, $g(x) \to 0$ as $x \to \pm\infty$. Thus g obtains an absolute maximum M for some $y \in \mathbb{R}$. Since we are assuming $g \not\equiv 0$, M is strictly greater than zero. Since $0 < M/2 < M$, and $g(x) \to 0$ as $x \to -\infty$, the intermediate value theorem implies that there exists $x_1 < y$ such that $g(x_1) = M/2$. Similarly, there exists $x_2 > y$ such that $g(x_2) = M/2$. This completes the proof. $\square$

Exercise 3. *Define a function* f *on the interval* $(0, 1)$ *by*

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}.$$

Prove that f *is differentable on* $(0, 1)$*.*

Proof. (O. Khalil) We begin by showing the series $g(x) = \sum_1^\infty \frac{\cos(kx)}{k}$ converges uniformly on any closed subinterval $[a, b]$ contained in $(0, 1)$. Fix $[a, b] \subset (0, 1)$. Let $n \in \mathbb{N}$. Let $g_n(x)$ denote the n^{th} partial sum of $g(x)$ for some $x \in [a, b]$. Let $A_n = \sum_1^n \cos(kx)$ and let $b_n = 1/n$. Summation by parts gives

$$g_n(x) = b_n A_n - \sum_1^{n-1} A_k(b_k - b_{k+1})$$

Using the trig identity $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$, we get the following

$$2\sin(\frac{x}{2})A_n(x) = \sum_1^n 2\sin(x/2)\cos(kx) = \sum_1^n (\sin(x(2k+1)/2) - \sin((x(2k-1)/2))$$

The above sum telescopes giving $2\sin(\frac{x}{2})A_n(x) = \sin(x(2n+1)/2) - \sin(x/2)$. So, we get that

$$|A_n(x)| = \left|\frac{\sin(x(2n+1)/2) - \sin(x/2)}{2\sin(x/2)}\right| \leq \left|\frac{\sin(x(2n+1)/2) - \sin(x/2)}{2\sin(a/2)}\right| \leq \frac{1}{\sin(a/2)}$$

where we used the fact that $\sin(x)$ is increasing on $(0,1)$ and that a is strictly greater than 0. So, the partial sums $A_n(x)$ are uniformly bounded on $[a,b]$. Hence, by Abel-Dirichlet's test for uniform convergence, since $1/k$ decreases uniformly to 0, then g converges uniformly on $[a,b]$. Moreover, $f(x)$ converges for every $x \in (0,1)$ since $|f(x)| \leq \sum_1^n \frac{1}{k^2}$ for every x and using the Weirstrass M-test.

To show that f is differentiable at x, let $[a,b] \subseteq (0,1)$ be a subinterval containing x. Hence, we have the following:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\sum_1^\infty \frac{\sin(k(x+h) - \sin(kx)}{hk^2} \\
&= \sum_1^\infty \lim_{h\to 0}\frac{\sin(k(x+h) - \sin(kx)}{hk^2} \\
&= \sum_1^\infty \frac{\cos(kx)}{k} \\
&= g(x)
\end{aligned}$$

where on the second line, we interchanged the limit and the summation because the series converges uniformly. This shows that $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists for each x and so f is differentiable as desired. □

Exercise 4. *Let f be a continuously differentiable function from the interval $[0,1]$ to $\mathbb{R}$. (Use one-sided derivatives at the endpoints of the interval.) Suppose that $f(1/2) = 0$. Show that*

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f'(x)|^2 dx. \tag{63}$$

Proof. (O. Khalil) Using the Fundamental theorem of Calculus, we can write

$$f(x) = \int_{\frac{1}{2}}^x f'(t)dt$$

Hence, using Cauchy-Shwarz inequality for integrals (see proof in solution of Ex 5, Spring 05), we get that

$$f(x)^2 = \left(\int_{\frac{1}{2}}^x f'(t)dt\right)^2 \leq \int_{\frac{1}{2}}^x f'(t)^2 dt \int_{\frac{1}{2}}^x 1^2 dt = \left(x - \frac{1}{2}\right)\int_{\frac{1}{2}}^x f'(t)^2 dt$$

Let $G(x) = \int_0^x f'(t)^2 dt$. Since $f'(t)^2 \geq 0$ for all t, then $G(x)$ is increasing on $[0,1]$ with supremum $= G(1)$. Also, we have that $\left|x - \frac{1}{2}\right| \leq \frac{1}{2}$ for all $x \in [0,1]$. Hence, we have that

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &\leq \int_0^1 \left| \left(x - \frac{1}{2}\right)(G(x) - G(1/2))\right| dx \\ &\leq \frac{1}{2} 2G(1) \int_0^1 dx = G(1)\end{aligned}$$

as desired. ☐

Exercise 5. *Let f be a continuous function from the interval $[0,1]$ to $\mathbb{R}$. Compute*

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) dx.$$

Justify your answer.

Proof. See 09S6. ☐

Exercise 6. *Let (x_n) be a sequence of real numbers. For each n, let*

$$A_n = \frac{x_1 + \cdots + x_n}{n}.$$

Suppose $x_n \to \infty$ as $n \to \infty$. Show that $A_n \to \infty$ as $n \to \infty$.

Proof. (O. Khalil) Set $C_n = x_1 + \cdots + x_n$ and $b_n = n$. Now, observe that $x_n = \frac{C_n - C_{n-1}}{b_n - b_{n-1}}$. The conclusion follows by Cesaro-Stolz theorem (see proof of this instance of the theorem in solution of Ex 2, Spring 05). ☐

2001 - Autumn

Exercise 1. *Let R be a non-constant rational function $R(x) = \dfrac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0}$, $a_m, b_n \neq 0$. Prove that there exists $x_0 \in \mathbb{R}$ such that R is strictly monotone on (x_0, ∞).*

Proof. (K. Nowland) Rewrite $R(x)$ as $R(x) = f(x)/g(x)$ where f and g are polynomials. The deritivative, defined everwhere $g(x) \neq 0$, is

$$R'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

is also a rational functional. As with $R(x)$, it is defined and continuous anywhere $g(x) \neq 0$. Since $g(x)$ is a polynomial with finite degree, it has only finitely many zeros. If $x_1 < x_1 < \cdots < x_k$ are the $k \leq n$ distinct real zeros of $g(x)$, then first we make $x_0 > 0x_k$. For all $x > x_k$, $R(x)$ is differentiable and continuous and $R'(x)$ is continuous. If $R(x)$ is not monotone on some interval $[a, b]$ with $x_k < a < b$, then it has a maximum or minimum in this interval by continuity. By differentiability, $R'(x) = 0$ at this point. Since $R'(x) = 0$ implies its numerator is zero, the prove the claim it suffices to note that the numerator is a polynomial of finite degree, and therefore only has a finite number of zeros. If x_0 is strictly greater than any of these zeros, then (x_0, ∞) will be a region on which R is monotone. $\square$

Exercise 2. *Find*

(i) $\lim_{n\to\infty} \sin(\pi\sqrt{n^2+1})$;

(ii) $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^2+n})$.

Proof.

(i) (N. DeBoer)

$$\begin{aligned}\sqrt{n^2+1} &= \sqrt{n^2+1} - n + n \\ &= \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} + n \\ &= \frac{1}{\sqrt{n^2+1}+n} + n\end{aligned}$$

Therefore

$$\sin(\pi\sqrt{n^2+1}) = \sin(\frac{\pi}{\sqrt{n^2+1}+n} + n\pi)$$

. By the summation formula for sine the above expression equals

$$\sin(\frac{\pi}{\sqrt{n^2+1}+n})\cos(n\pi) + \cos(\frac{\pi}{\sqrt{n^2+1}+n})\sin(n\pi)$$

Which in turn equals

$$(-1)^n \sin(\frac{\pi}{\sqrt{n^2+1}+n})$$

Since

$$\lim_{n\to\infty} \frac{\pi}{\sqrt{n^2+1}+n} = 0$$

By the continuity of sine

$$\lim_{n\to\infty} \sin(\frac{\pi}{\sqrt{n^2+1}+n}) = 0$$

$$\text{So} \quad \lim_{n\to\infty} (-1)^n \sin(\frac{\pi}{\sqrt{n^2+1}+n}) = 0$$

(ii) (O. Khalil) We wish to show that this sequence converges to 1. Define $f(x) : [1, \infty) \to \mathbb{R}$ by $f(x) = \sin^2(\pi\sqrt{x})$. Then, f is differentiable on $(1, \infty)$, with derivative

$$f'(x) = 2\pi \sin(\pi\sqrt{x}) \cos(\pi\sqrt{x}) \frac{1}{2\sqrt{x}}$$

Hence, we get that

$$|f'(x)| \le 2\pi \left| \frac{1}{2\sqrt{x}} \right| \le \pi$$

In particular, $f(x)$ has a bounded derivative on $(1, \infty)$. Hence, it is uniformly continuous by the mean value theorem.

Now, let $\varepsilon > 0$ be fixed. Let $\delta > 0$ be such that whenever $x, y \in (0, \infty)$ have that $|x - y| < \delta$, we get that $|f(x) - f(y)| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$. Observe that $\sqrt{n^2+n} = \sqrt{(n+1/2)^2 - 1/4}$. Let $n > 8N$. Let $m = n + 1/2$. Note that the function $x \mapsto \sqrt{x}$ is differentiable on the interval $(m^2 - 1/4, m^2)$. So, by the mean value theorem, we have that

$$|\sqrt{m^2 - 1/4} - \sqrt{m^2}| = |\frac{1}{2\theta_m}\frac{1}{4}| \le \frac{1}{8\sqrt{m^2 - 1/4}} \le \frac{1}{8\sqrt{m^2-1}} \le \frac{1}{8m-8} \le \frac{1}{N} < \delta$$

Therefore, we get that $|f(m^2 - 1/4) - f(m^2)| < \varepsilon$. Now, observe the following

$$f(m^2) = \sin^2(\pi(n+1/2)) = (\sin(n\pi)\cos(\pi/2) + \cos(n\pi)\sin(\pi/2))^2 = \cos^2(n\pi)$$

But, we have that $\cos^2(n\pi) = 1$ for all $n \in \mathbb{N}$. Let $a_n = \sin^2(\pi\sqrt{n^2+n})$ and let $m = n + 1/2$. For $n > 8N$, we get the following

$$|a_n - 1| = |f(m^2 - 1/4) - f(m^2)| < \varepsilon$$

ε was arbitrary so, a_n converges to 1 as desired.

□

Exercise 3. *Let f be a continuous function on $[0,1]$. Prove that*

$$\exp\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \exp(f(x))dx. \tag{64}$$

Proof. (K. Nowland) Partition $[0,1]$ into n equally spaced intervals and let $\xi_i \in [\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, \ldots, n$ be any sample points. Sice $\frac{d^2}{dx^2}\exp(x) = \exp(x) > 0$ for all x, $\exp(x)$ is convex. By the discrete Jensen's inequality,

$$\exp\left(\frac{1}{n}\sum_{i=1}^n f(\xi_i)\right) \leq \frac{1}{n}\sum_{n=1}^n \exp(f(\xi_i)).$$

Note that this inequality is uniform in n and the ξ_i. It therefore suffices to show that the left hand side approximates the left hand side of (64) and similarly for the right hand side. Let $\varepsilon > 0$. Since exp and f are continuous, $\exp \circ f$ is continuous, such that for all n large enough and all samples ξ_i as above,

$$\left|\frac{1}{n}\sum_{n=1}^\infty \exp(f(\xi_i)) - \int_0^1 \exp(f(x))dx\right| < \varepsilon.$$

Also, the continuity of f implies that for n large enough and all samples ξ_i,

$$\left|\exp\left(\frac{1}{n}\sum_{i=1}^n f(\xi_i)\right) - \exp\left(\int_0^1 f(x)dx\right)\right| < \varepsilon.$$

This completes the proof.

The proof of the discrete Jensen's inequality can be done by induction on the number of terms in the sum. For two terms, Jensen's inequality holds by the definition of convexity. For the general case, we have $p_1x_1 + \cdots + p_nx_n$ where $\sum p_i = 1$, $p_i > 0$. Write $p_1x_1 + p_2x_2 = rx$ where $r = p_1 + p_2$ and use the inductive assumption. ☐

Exercise 4. *(i) Prove that for every $n = 1, 2, \ldots$ the series $\displaystyle\sum_{k=n+1}^\infty \left(\frac{1}{\sqrt{k-n}} - \frac{1}{\sqrt{k}}\right)$ converges.*

(ii) For each $n \in \mathbb{N}$, let S_n be the sum of the above series. Evaluate $\displaystyle\lim_{n\to\infty} \frac{S_n}{\sqrt{n}}$.

Proof. (K. Nowland)

(i) The first n terms in the series are

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{2n}}.$$

Note that the terms $1/\sqrt{1}$, $1/\sqrt{2}$, ..., $1/\sqrt{n}$ will never be cancelled by subsequent terms in the series. In other words, the sum from $n+1$ to $n+k$ where $k \geq n$ is given by

$$\left(\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}}\right) - \left(\frac{1}{\sqrt{k+1}} + \cdots + \frac{1}{\sqrt{k+n}}\right).$$

Since n is fixed, this is a finite sum of terms which each go to zero such that the series converges to

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

(ii) Since $1/\sqrt{k}$ is decreasing in k, the integral test implies that

$$\int_1^{n+1} \frac{dx}{\sqrt{x}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 1 + \int_1^{n} \frac{dx}{\sqrt{x}}.$$

Integrating,

$$2\sqrt{n+1} - 2 \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

If we divide by $\sqrt{n}$, we see that

$$\frac{2\sqrt{n+1}-2}{\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} \leq 2 - \frac{1}{\sqrt{n}}.$$

Taking the limit as $n \to \infty$ gives $S_n/\sqrt{n} \to 2$.

□

Exercise 5. *Prove the theorem on term-by-term differentiation of power series: if $r > 0$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $-r < x < r$, then $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges for $-r < x < r$, and $f' = g$.*

Proof. (H. Lyu) Let r_f, r_g be the radii of convergence of f and g, respectively. Then by the Cauchy-Hadamard formula, we have

$$r_g = \frac{1}{\limsup_{n\to\infty} (n a_n)^{1/n}} = \frac{1}{\lim_{n\to\infty} n^{1/n} \limsup_{n\to\infty} a_n^{1/n}} = \frac{1}{\limsup_{n\to\infty} a_n^{1/n}} = r_f$$

But since $r_f < r$ by hypothesis, we also have $r_g < r$. It remains to show that $f'(x) = g(x)$ for $|x| < r$.

Now let $f_n(x) = \sum_{k=0}^{n} a_k x^k$. For each n, f_n is differentiable with derivative $f_n' = \sum_{k=1}^{n} k a_k x^{k-1}$. Note that f_n' is the n-th partial sum for g. Recall that the partial sums of a power series converges uniformly on any compact set contained in the domain of convergence. In particular, $f_n \to f$ and $f_n' \to g$ uniformly on any compact set $K \subset (-r, r)$. Since each f_n' is continuous, and since K can be a compact neighborhood of any point in $(-r, r)$, this implies g is continuous on $(-r, r)$. So g is integrable on any closed interval contained in $(-r, r)$. Observe that for any $x \in (-r, r)$, we have

$$\int_0^x g(t)\,dt = \int_0^x \lim_{n\to\infty} f_n'(t)\,dt = \lim_{n\to\infty} \int_0^x f_n'(t)\,dt = \lim_{n\to\infty} (f_n(x) - f_n(0)) = f(x) - f(0),$$

where by the uniform convergence $f_n' \to g$ we can switch the limit and integral, and the last two equalities follow from the fundamental theorem of calculus and the pointwise convergence $f_n(x) \to f(x)$. Then the above calculation yields

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \int_x^{x+h} g(t)\, dt.$$

Hence it suffices to show that the right hand side converges to $g(x)$ as $h \to 0$. Let $G(x) = \int_0^x g(t)\, dt$. Then G is differentiable with derivative g, so by the mean value theorem, for each small enough $h \in \mathbb{R}$, $G(x+h) - G(x) = hg(\xi_h)$ for some ξ_h between x and $x+h$. Hence

$$\frac{1}{h} \int_x^{x+h} g(t)\, dt = \frac{G(x+h) - G(x)}{h} = g(\xi_h).$$

Now as $h \to 0$, $\xi_h \to x$, and by the continuity of g, we get $g(\xi_h) \to g(x)$. Hence $f'(x) = g(x)$. This holds for each $x \in (-r, r)$. This shows the assertion. □

Remark. The above result for term-by-term differentiation of power series is a special case of the following general theorem. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If each f_n' is continuous and $\{f_n'\}$ converges uniformly on $[a, b]$, then f_n converges uniformly on $[a, b]$ to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$.*

Proof. Let g be the function on $\mathbb{R}$ defined by $g(x) = \lim_{n\to\infty} f_n'(x)$. We first show $\{f_n\}$ converges uniformly on $[a, b]$. Since $\mathbb{R}$ is complete, it suffices to show that $\{f_n\}$ is uniformly Cauchy. Fix $x \in [a, b]$ and $\epsilon > 0$. Then for any $n, m \in \mathbb{N}$, the triangle inequality yields

$$|f_n(x) - f_m(x)| \leq |[f_n(x) - f_m(x)] + [f_n(x_0) - f_n(x_0)]| + |f_n(x) - f_m(x_0)|. \tag{1}$$

By the mean value theorem applied to the differentiable function $f_n - f_m$, we get

$$|[f_n(x) - f_m(x)] + [f_n(x_0) - f_n(x_0)]| = (x - x_0)(f_n'(\xi) - f_m'(\xi))$$

for some ξ between x and x_0. Since $\{f_n'\}$ converges uniformly, it is uniformly Cauchy, so there exists $N_1 \in \mathbb{N}$ such that $\sup |f_n' - f_m'| < \frac{\epsilon}{2(x-x_0)}$ provided $n, m > N_1$. On the other hand, since $\{f_n(x_0)\}$ converges, there exists $N_2 \in \mathbb{N}$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ for all $n, m > N_2$. Let $N = \max(N_1, N_2)$. Then by (1), we have $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$. Hence $\{f_n\}$ is uniformly Cauchy, as desired. Therefore (f_n) converges to a function f uniformly on $[a, b]$.

It remains to show f is differentiable on (a, b) and $f'(x) = \lim_{n\to\infty} f_n'(x)$. Since each f_n' is continuous on $[a, b]$ and $f_n' \to g$ uniformly on $[a, b]$, so by uniform convergence and integration theorem and the fundamental theorem of calculus, we have

$$\int_{x_0}^x g(t)\, dt = \int_{x_0}^x \lim_{n\to\infty} f_n'(t)\, dt = \lim_{n\to\infty} \int_{x_0}^x f_n'(t)\, dt = \lim_{n\to\infty} (f_n(x) - f_n(x_0)) = f(x) - f(x_0).$$

From this we get
$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\int_x^{x+h} g(t)\,dt.$$
Hence to show f is differentiable at x and $f'(x) = \lim_{n\to\infty} f_n'(x)$, it suffices to show that the right hand side converges to $g(x)$. Let $G(x) = \int_0^x g(t)\,dt$. Then G is differentiable with derivative g, so by the mean value theorem, for each small enough $h \in \mathbb{R}$, $G(x+h)-G(x) = hg(\xi_h)$ for some ξ_h between x and $x+h$. Hence

$$\frac{1}{h}\int_x^{x+h} g(t)\,dt = \frac{G(x+h)-G(x)}{h} = g(\xi_h).$$

Now g is continuous on $[a,b]$ being the uniform limit of continuous functions $\{f_n'\}$, and as $h \to 0$, $\xi_h \to x$, so by the continuity of g, we get $g(\xi_h) \to g(x)$. Hence f is differentiable at x and $f'(x) = \lim_{n\to\infty} f_n'(x)$, which holds for each $x \in (a,b)$. This shows the assertion. $\square$

Exercise 6. *Let a function f be uniformly continuous on $[1,\infty)$. Prove that the function $F(x) = \dfrac{f(x)}{x}$ is bounded on $[1,\infty)$.*

Proof. (O. Khalil) Since f is uniformly continuous, then $\exists\delta > 0$ such that $\forall x, y \in [1,\infty)$, whenever $|x-y| < 2\delta$, we have that $|f(x)-f(y)| < 1$. Now, let $x \in [1,\infty)$ be arbitrary. Let $n \in \mathbb{N}\cup 0$ be the largest integer so that $1+n\delta < x$. Hence, we have the following

$$\begin{aligned}
|f(x)| - |f(1)| &\le |f(x)-f(1)| \\
&= |f(x) - f(n\delta+1) + f(n\delta+1) - f((n-1)\delta+1) + \cdots + f(1+\delta) - f(1)| \\
&\le |f(x) - f(n\delta+1)| + |f(n\delta+1) - f((n-1)\delta+1)| + \cdots + |f(1+\delta) - f(1)| \\
&< n+1
\end{aligned}$$

where by the choice of n, we have that $x - (n\delta+1) < \delta$. Now, observe that

$$n\delta + 1 < x \Rightarrow n+1 < \frac{x-1}{\delta} + 1$$

Hence, we get that

$$|F(x)| = \left|\frac{f(x)}{x}\right| < \frac{x-1}{\delta x} + \frac{1+|f(1)|}{x} \le \frac{1}{\delta} + 1 + |f(1)|$$

where we used the fact that $x \ge 1$. Hence, $F(x)$ is bounded. $\square$

2001 - Spring

Exercise 1. *State and prove Cauchy's inequality for real sequences* $(a_k)_{k=1}^{\infty}$ *and* $(b_k)_{k=1}^{\infty}$*, and obtain necessary and sufficient conditions for when it is an equality.*

Proof. (K. Nowland) The statment is what follows: Suppose $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ are both convergent series. Then

$$\sum_{k=1}^{\infty} a_k b_k \leq \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2}.$$

Equality holds if and only if $a_k = \lambda b_k$ for some constant λ. The proof is the same as for the finite and integral cases. Consider the sequence $(a_k - \lambda b_k)^2$. We calculate that

$$0 \leq \sum_k (a_k - \lambda b_k)^2 = \sum_k a_k^2 - 2\lambda \sum_k a_k b_k + \sum_k b_k^2.$$

Note that the above holds if we instead take finite sums and replace a_k, b_k with $|a_k|$ and $|b_k|$, which is what implies that in the limit, the sum $\sum_k a_k b_k$ converges absolultely and therefore converges generally. Thus the above is a quadratic polynomial in λ which is nonnegative. This implies that the discriminant is nonpositive, i.e.,

$$\left(2\sum_k a_k b_k\right)^2 \leq 4 \sum_k a_k \sum_k b_k,$$

which is the desired inequality. □

Exercise 2. *How many positive solutions does the equation* $e^x = 4\cos x$ *have? Prove your answer.*

Proof. (K. Nowland) (less-rigorous) We claim that there is only one positive solution to the equation. Note that $e^0 = 1$ and $4\cos 0 = 4$. As $x \to \pi/2$, e^x increases and is always positive, while $4\cos x$ decreases down to zero. This monotonicity of the dervative in the region $[0, \pi/2]$ implies that there is at most one solution in this interval, but by continuity and the intermediate valuetheorem, there is at least one solution to the equation. We want to show that there is no other solution.

Note that $4\cos x \leq 0$ on $[0, 3\pi/2]$. At $3\pi/2$, the derivative of $4\cos x$ is $-4\sin x$ such that the value of the derivative of $4\cos x$ is 4. If e^x, which is strictly positive has derivative larger than 4 at this point, $4\cos x$ will not catch up to e^x, as 4 is the maximum value of the derivative of x. In other words, we would have for $x > 0$, by the fundamental theorem of calculus,

$$e^{3\pi/2+x} = \int_{3\pi/2}^{x} e^x dx + ef^{3\pi/2} \geq \int_{3\pi/2} (4\cos x)' dx = 4\cos(3\pi/2 + x).$$

Note that $\pi/2 > 1.5$ because $\pi > 3$. Thus $3\pi/2 > 4$. Since $e^x > x$ for all $x \in \mathbb{R}$, (look at the power series form or make a geometric argument with derivatives), we have $e^{3\pi/2} > 4$, as required. This comple □

Exercise 3. *If f is a convex function on $[0,1]$, prove that $f(x)+f(1-x)$ is decreasing on $[0,\frac{1}{2}]$. (A function f is* convex *on $[a,b]$ if for any $x,y \in [a,b]$ and any $0 \le t \le 1$ it satisfies $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$. Note that f is not assumed to be differentiable.)*

Proof. (R. Garrett) First, we claim for any convex function $f:[0,1]\to\mathbb{R}$ and triple of points $x_1 < x_2 < x_3$ in $[0,1]$ we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_1)}{x_3-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

To show this, first set $\lambda = \frac{x_2-x_1}{x_3-x_1}$ and notice $1-\lambda = \frac{x_3-x_2}{x_3-x_1}$. By convexity, $f(x_2) = f((1-\lambda)x_1+\lambda x_3) \le \frac{x_3-x_2}{x_3-x_1}f(x_1)+\frac{x_2-x_1}{x_3-x_1}f(x_3)$. Subtracting $f(x_1)$ from both sides, we get $f(x_2)-f(x_1) \le \frac{x_2-x_1}{x_3-x_1}[f(x_3)-f(x_1)]$, which is clearly equivalent to the first inequality we wanted to show. By the first string of inequalities we obtained from convexity, we also have $-f(x_2) \ge \frac{x_3-x_2}{x_3-x_1}(-f(x_1))-\frac{x_2-x_1}{x_3-x_1}f(x_3)$, and by adding $f(x_3)$ to both sides, we get by easy simplification $f(x_3)-f(x_2) \ge \frac{x_3-x_2}{x_3-x_1}[f(x_3)-f(x_1)]$. The second inequality we wanted to prove immediately follows. Our claim is proved.

Now, let $x<y$ lie in the interval $[0,\frac{1}{2}]$. First, suppose $y \ne \frac{1}{2}$. Then, we have $x<y<1-y<1-x$, and we apply our claim to get

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(1-y)-f(x)}{1-y-x} \le \frac{f(1-y)-f(y)}{1-2y} \le \frac{f(1-x)-f(1-y)}{y-x}$$

Since $y-x>0$, we may clear the denominators of the first and last terms to obtain $f(y)-f(x) \le f(1-x)-f(1-y)$, which implies $f(y)+f(1-y) \le f(x)+f(1-x)$. Now, suppose $y=\frac{1}{2}$, then by the claim,

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(1-x)-f(y)}{1-x-y}$$

Notice both denominators are $1/2-x>0$, so we clear them to obtain $f(y)-f(x) = f(1-y)-f(x) \le f(1-x)-f(y)$, which gets us $f(y)+f(1-y) \le f(x)+f(1-x)$. Thus, in all cases, $f(x)+f(1-x)$ is decreasing on $[0,\frac{1}{2}]$. ☐

Exercise 4. *Let $a>1$. Find $\lim\limits_{n\to\infty}(a-\sqrt[n]{a})^n$.*

Proof. (K. Nowland) Note that if $a>2$, then for n large enough, $a-\sqrt[n]{a}>1$, since $\sqrt[n]{a}$ decreases to 1 as $n\to\infty$. Let N be any such n, then for all $n \ge N$, we have

$$(a-\sqrt[n]{a})^n \ge (a-\sqrt[N]{a})^n.$$

Since $a-\sqrt[N]{a}$ is a constant greater than 1, as $n\to\infty$, this goes to infinity, such that the limit is $+\infty$ (or does not exist).

Now suppose $1<a<2$. Then $a-\sqrt[n]{a} \le a-1$, since $\sqrt[n]{a} \ge 1$ for all $n\in\mathbb{N}$. But By assumption we have $0<a-1<1$, such that the limit is zero, since $(a-1)^n \to 0$ as $n\to\infty$.

The only case that remains to examine is $a = 2$. In this case we write

$$\begin{aligned}(2 - \sqrt[n]{2})^n &= (1 + (1 - \sqrt[n]{2}))^n \\ &= (1 + (1 - \sqrt[n]{2}))^{\frac{n}{1 - \sqrt[n]{2}}}\end{aligned}$$

Note that

$$\lim_{n\to\infty} (1 + (1 - \sqrt[n]{2}))^{\frac{1}{(1+\sqrt[n]{2}}} = e,$$

such that the limit of $(2 - \sqrt[n]{2})^n = +\infty$, as in the $a > 2$ case. To be precise, there exists $N \in \mathbb{N}$ such that for $n \geq N$, the typical term is bounded below by 2^n which tends to infinity as n does. $\square$

Exercise 5. *Consider the series*

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n} + \cdots .$$

(i) Prove that the series converges for every real x.

(ii) Does the series converge uniformly on $\mathbb{R}$? Prove your answer.

Proof. (K. Nowland)

(i) Note that for $x = 0$, the series is identically zero, such that the series converges there. Now suppose $x \neq 0$ is fixed. We write a partial sum of the series as

$$x^2 \sum_{i=0}^{n} \frac{1}{(1+x^2)^i} = x^2 \frac{1 + \frac{1}{(1+x^2)^{n+1}}}{1 - \frac{1}{1+x^2}}.$$

As $n \to \infty$, the term on top goes to 1, such that the series converges, for $x \neq 0$ to

$$\frac{x^2}{1 - \frac{1}{1+x^2}} = \frac{x^4 + x^2}{x^2} = x^2 + 1.$$

(ii) Note that each term in the series is continuous for all $x \in \mathbb{R}$. The uniform limit of continuous functions is again continuous. But the above is not continuous at zero, since $\lim_{x\to 0} x^2 + 1 = 1$, while the series converges to 0 at zero. Since the limit is not continuous but the partial sums are, the limit must not be uniform.

$\square$

Exercise 6. *Let $f : [0, \infty) \to \mathbb{R}$ be non-negative and let $\int_0^\infty f(t)dt$ converge.*

1. *Show that these conditions do not imply that $f(x) \to 0$ as $x \to \infty$.*
2. *Show that under the additional condition that f is uniformly continuous on $[0, \infty)$, $f(x) \to 0$ as $x \to \infty$.*

Proof. (K. Nowland)

1. For each $n \in N$, let $f(x)$ consist of a line from $(n, 0)$ to $(n + 1/2n^3, 2n)$ and then from $(n + 1/2n^3, 2n)$ to $(0, 1/n^2)$. Let f be zero away from the intervals $[n, n + 1/n^3]$. The itnegral over each interval $[n, n + 1]$ is the area of the isoceles triangle of height $2n$ and base n^{-3}, such that the area over each interval is n^{-2}. Since the series $\sum n^{-2}$ converges, the integral $\int_0^\infty f(x)dx$ converges. Clealry $f(x) \not\to 0$ as $n \to \infty$, because there is in fact a sequence of points that tends to infinity.

2. Suppose $f(x)$ does not tend to zero but that f is uniformly continuous. Thus there is some $\varepsilon > 0$ such that there exists a sequence $\{x_n\}$ of strictly increasing points tending to infinity such that $f(x_n) > \varepsilon$ for each n. Since the points are tending to infinity, we may suppose that $x_{n+1} - x_n > 1$ for all n. By uniform continuity there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$. We may suppose that $\delta < 1/2$. In particular, there is a δ ball around each x_n such that $f(x) > \varepsilon/2$ in each ball. Thus $\int_{x_n-\delta}^{x_n+\delta} f(x)dx > \varepsilon$ for every $n \in \mathbb{N}$. Since $x_n \to \infty$, the integral failes the Cauchy convergence criterion and thus cannot converges. The contradiction proves the claim that $f(x) \to 0$ as $x \to \infty$.

$\square$

2000 - Autumn

Exercise 1. *Let $a_n > 0$ for all $n \geq 1$. Suppose that there exists*

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Prove that $\lim_{n\to\infty} a_n^{\frac{1}{n}}$ exists too, and

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = q. \tag{65}$$

Proof. (K. Nowland) See 09A1 and 05A6 for the one sided version of d'Alembert's ratio test. For a quick solution, let $\varepsilon > 0$ be fixed and let N be so large that

$$q - \varepsilon < \frac{a_{n+1}}{a_n} < q + \varepsilon$$

for all $n \geq N$. Write

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_N + 1}{a_N} a_N.$$

Thus we see that

$$a_N(q-\varepsilon)^{n-N} < a_n < a_N(q+\varepsilon)^{n-N}.$$

If we take the nth root, we see that

$$\limsup_{n\to\infty} a_n^{\frac{1}{n}} \leq q + \varepsilon, \quad \liminf_{n\to\infty} a_n^{\frac{1}{n}} \geq q - \varepsilon.$$

Since ε was arbitrary, the lim sup and lim inf must agree such that the limit exists and is q, as claimed. □

Exercise 2. *Determine all real numbers α such that the sequence $g_n(x) = x^\alpha e^{-nx}$ is uniformly convergent on $(0, \infty)$ as $n \to \infty$.*

Proof. (K. Nowland) The sequence converges uniformly on $(0, \infty)$ for $\alpha > 0$. Note that for any $\alpha \in \mathbb{R}$ and fixed $x \in (0, \infty)$, $x^\alpha e^{-nx} \to 0$ as $n \to \infty$ such that the limit function in any case will be zero. If $\alpha < 0$, then $x^\alpha e^{-nx}$ goes to infinity as x goes to zero from the right. If $\alpha = 0$, then $x^\alpha e^{-nx}$ goes to 1 as x tends to zero from the right. Thus for $\alpha \leq 0$, $\sup_{x\in(0,\infty)} |g_n(x)| \geq 1$. It follows that g_n cannot converge to zero uniformly on $(0, \infty)$.

Now suppose $\alpha > 0$. In this case, $g_n(0) = 0$ and also $\lim_{x\to\infty} g_n(0) = 0$. Since g_n is continuous for all n, $|g_n(x)|$ is bounded for all n. Fixing n,

$$g_n'(x) = \alpha x^{\alpha-1} e^{-nx} - n x^\alpha e^{-nx} = e^{-nx} x^{\alpha-1}(\alpha - nx).$$

The only critical point for the function is $x = \alpha/n$. Since $g_n(x) > 0$ for every n and x but tends to zero at 0 and $+\infty$, it must be that α/n is a global maximum for $g_n(x)$. Therefore

$$\sup_{x>0} |g_n(x)| = g_n(\alpha/n) = \left(\frac{\alpha}{n}\right)^\alpha e^{-\alpha}.$$

Since $\alpha > 0$, this tends to zero as n tends to infinity. Since $\sup_{x>0} |g_n(x)| \to 0$ as $n \to \infty$, it follows that the sequence of functions converges uniformly to the zero function on $(0, \infty)$. □

Exercise 3. *Prove that the theta-function*

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

is well-defined and infinitely differentiable for $x > 0$.

Proof. (K. Nowland) To show that the function is well-defined and infinitely differentiable for $x > 0$, it suffices to show this for any compact interval $[a, b]$ with $0 < a < b$. In particular, it suffices to show that for any interval, the sequence of partial sums of of $f^{(k+1)}(x)$ converges uniformly on $[a, b]$ and $f^{(k)}(x)$ converges on the interval for all $n \in \mathbb{N} \cap \{0\}$, where $f^{(k)}$ is the kth derivative of f and $f^{(0)} = f$. In other words, it suffices to show that every derivative (and $k = 0$) gives a function which convergs uniformly on any fixed interval. We can of course rewrite θ as

$$\theta(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

Let $\theta_N(x)$ be the partial sum:

$$\theta_N(x) = 1 + 2\sum_{n=1}^{N} e^{-\pi n^2 x}.$$

The kth derivative of th_N is

$$\theta_N^{(k)}(x) = 2(-\pi)^k \sum_{n=1}^{N} n^{2k} e^{-\pi n^2 x}.$$

If $[a, b]$ is fixed, then $|\theta_N^{(k)}(x)| \leq |\theta_N^{(k)}(a)|$. It therefore suffices to prove that for any integral $k \geq 0$, $\theta_N^{(k)}(x)$ converges for any $x > 0$. Since the constant out front is unimportant, we just need to show that

$$\sum_{n=1}^{\infty} n^{2k} e^{-\pi n^2 x}$$

converges. Note that for n large enough, $e^{\pi n^2 x} \geq n^{2k+2}$ for any fixed k and x. I.e., there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $n^{2k} e^{-\pi n^2 x} \leq n^{-2}$. Since the series $\sum n^{-2}$ converges, so will the above series, as desired.

Though correct, I am not sure we need to show that $e^{\pi n^2 x} \geq n^{2k+2}$ for n large enoguh. To show this, note that for any $x > 0$, we have from the power series expansion for the exponential function (valid on all of $\mathbb{R}$),

$$e^x \geq \frac{x^{k+2}}{(k+2)!}.$$

Thus it suffices to find n such that

$$\frac{\pi^{k+1} x^{k+1} n^{2k+4}}{(k+1)!} \geq n^{2k+2}.$$

Rewriting,

$$n^2 \geq \frac{(k+1)!}{\pi^{k+1}x^{k+1}}.$$

Though this may require n to be very large if x is very small, since $x > 0$ is fixed, this is possible. $\square$

Exercise 4. *Suppose* $1 < \alpha < 1 + \beta$*. Prove that the function*

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x^\beta}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable on $[0, 1]$*, but its derivative is unbounded on* $[0, 1]$*.*

Proof. (K. Nowland) I am assuming that $\beta > 0$ such that the inequalities make sense.

First suppose $x > 0$. Then $f'(x)$ exists since f is the product of the differentiable x^α with $\sin(1/x^\beta)$, which is differentiable as the composition of differentiable functions. For $x = 0$, we use the definition of the right handed difference quotient:

$$\frac{f(x) - f(0)}{x} = x^{\alpha-1} \sin \frac{1}{x^\beta}.$$

As $x \to 0$, $\sin x^{-\beta}$ is bounded while $x^{\alpha-1}$ tends to zero since $\alpha > 1$ by hypothesis. Thus $f'(0)$ is zero such that f is differentiable on $[0, 1]$, as claimed.

For $x > 0$, the derivative is given by

$$f'(x) = \alpha x^{\alpha-1} \sin \frac{1}{x^\beta} - \beta x^{\alpha-\beta-1} \cos \frac{1}{x^\beta}.$$

For the same reason as above, the first term in the derivative tends to zero as $x \to 0^+$. The other term however, is unbounded as x decreases to zero. This is because $\alpha < \beta + 1$ by hypothesis such that $x^{\alpha-\beta-1}$ tends to positive infinity, while the cosine keeps oscillating between -1 and 1. $\square$

Exercise 5. *Let* $f : [0, 1] \to [0, 1]$ *be increasing. Using the partition definition of the Riemann integral, prove that* f *is Riemann integrable.*

Proof. (K. Nowland) We use the Darboux integration criterion, which says that $f : [0, 1] \to \mathbb{R}$ is Riemann integrable if and only if f is bounded and for all $e > 0$ there exists a partition P of $[0, 1]$ such that $U(f, P) - L(f, P) < \varepsilon$, where

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad M_i = \sup_{[x_{i-1}, x_i]} f(x),$$

and

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad m_i = \sup_{[x_{i-1}, x_i]} f(x).$$

Since it is given that f takes values in $[0, 1]$, we have that f is bounded. Since the function is increasing, for any given partition, we have that

$$U(f, P) - L(f, p) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i+1}).$$

If we take P to be the the even partition of $[0, 1]$ into n parts, this is

$$U(f, P) - L(f, p) = \frac{1}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})].$$

The sum is therefore telescoping, which gives

$$U(f, P) - L(f, P) = \frac{f(1) - f(0)}{n} \leq \frac{1}{n}.$$

Taking n to be so large that $1/n < \varepsilon$, we have the desired result. □

Exercise 6. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous. Prove that if*

$$\int_0^\infty f(x)dx = \lim_{A \to \infty} \int_0^A f(x)dx$$

exists (in short, $\int_0^\infty f(x)dx$ converges), then $\int_0^\infty e^{-\alpha x} f(x)dx$ converges for every $\alpha > 0$ and

$$\lim_{\alpha \to 0^+} \int_0^\infty e^{-\alpha x} f(x)dx = \int_0^\infty f(x)dx. \tag{66}$$

Proof. (K. Nowland) Since $\alpha > 0$, $e^{-\alpha x} \leq 1$ for all $x \in [0, \infty)$. By comparision, we have that for all $\alpha > 0$, the integral $\int_0^\infty e^{-\alpha x} f(x)dx$ converges. (Split the integral into positive and negative parts, then use the inequality on each part separately)

Let $\varepsilon > 0$. Consider the convergent integral, $\int_0^\infty (1 - e^{-\alpha x}) f(x)dx$. We want to show that as $\alpha \to 0^+$, this integral tends to zero. Since integral converges, there exists $y > 0$ such that

$$\left| \int_y^\infty (1 - e^{-\alpha x}) f(x)dx \right| < \frac{\varepsilon}{2}.$$

Since $f(x)$ is continuous, there exists $M > 0$ such that $f(x) < M$ for all x on the compact interval $[0, y]$. Let α be so small that $1 - e^{-\alpha y} < \varepsilon/2yM$. We calculate

$$\left| \int_0^y (1 - e^{-\alpha x}) f(x)dx \right| \leq M \int_0^y 1 - e^{-\alpha y} dx < \frac{\varepsilon}{2}.$$

By the triangle inequality, we have

$$\left| \int_0^\infty f(x)dx - \int_0^\infty e^{-\alpha x} f(x)dx \right| \leq \left| \int_0^y (1 - e^{-\alpha x}) f(x)dx \right| + \left| \int_y^\infty (1 - e^{-\alpha x}) f(x)dx \right| < \varepsilon,$$

where the second inequality comes from the above estimates. □

2000 - Spring

Exercise 1. *Determine whether or not the following statement is correct. If it is correct, prove it. If it is not, provide a counterexample. Let $f : [0, \infty) \to \mathbb{R}$ be continuous, and let $\lim_{x\to\infty} f(x) = 0$. Then f is uniformly continuous on $[0, \infty)$.*

Proof. (O. Khalil) f is uniformly continuous. To show that, let $\varepsilon > 0$ be fixed. Since $\lim_{x\to\infty} f(x) = 0$, then there exists $M \in (0, \infty)$ such that for all $x > M$, we have that $|f(x)| < \varepsilon/2$. Moreover, we have that since f is continuous on $[0, M + 1]$ which is compact, then it is uniformly continuous (see proof of this fact in solution to Ex 3, Autumn 02). Hence, there exists $\delta > 0$ such that for all $x, y \in [0, M + 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. We may assume that $\delta < 1$.

Now, let $x, y \in [0, \infty)$ with $|x - y| < \delta$. If $x, y > M$, then we have that $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. If both $x, y \leq M$, then uniform continuity of f on $[0, M + 1]$ applies. If $y > M$ and $x \leq M$, then since $\delta < 1$ and $|x - y| < \delta$, then $y < M + 1$ and again uniform continuity of f on $[0, M + 1]$ applies. Thus, f is uniformly continuous on all $[0, \infty)$ as desired. □

Exercise 2. *Determine all real numbers α for which the improper integral* $\displaystyle\int_1^\infty x^\alpha \sin(x^2)dx$

(i) converges absolutely;

(ii) converges.

Proof. (K. Nowland) Before beginning, we change variables:

$$\int_1^\infty x^\alpha \sin(x^2)dx = \frac{1}{2}\int_1^\infty x^{\frac{\alpha-1}{2}} \sin(x)dx.$$

(a) Note that $\int_1^t |\sin(x)|dx$ is not bounded in t, but that $\sin(x)$ is. This is the main difference between the two cases. Note that in this case, we have $|\sin(x)| \leq 1$, such that the integral will certainly converge absolutely for $(\alpha - 1)/2 < -1$, i.e., for $\alpha < -1$. If $\alpha = -1$, we claim that the integral does not converge absolutely. Consider an interval $[k\pi, (k + 1)\pi]$ with $k \in \mathbb{N}$. Then

$$\int_{k\pi}^{(k+1)\pi} x^{-1}|\sin(x)|dx > \frac{1}{k\pi}\int_0^\pi \sin(x)dx = \frac{2}{k\pi}.$$

Since the series, $\sum k^{-1}$ does not converge, the integral cannot converge for $\alpha = -1$. Thus the integral converges absolutely for $\alpha \in (-\infty, -1)$.

(b) The difference in this case, is that $\int_1^t \sin(x)dx$ is bounded in t due to the oscillatory nature of $\sin(x)$. Thus for converge, it is sufficient for $x^{(\alpha-1)/2}$ to be decreasing as $x \to \infty$. For this to occur, we need $(\alpha - 1)/2 < 0$, i.e., we erquire $\alpha < 1$. The series does not converge for $\alpha \geq 1$, since then $x^{(\alpha-1)/2}$ is either constant ($\alpha = 1$) or increasing, in which case in any interval of the form $[2k\pi, (2k + 1)\pi]$ with $k \in \mathbb{N}$, the integral will be bounded below by the constant $\int_0^\pi \sin(x)dx > 0$, such that the integral will fail the Cauchy convergence criterion. The integral therefore converges for $\alpha \in (-\infty, 1)$.

$\square$

Exercise 3. *Let (a_i) be a real sequence and let a be a real number. We say that a_i* converges to a in the sense of Cesàro *if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i = a.$$

Show that if a_i converges to a in the usual sense, then a_i also converges to a in the sense of Cesàro.

Proof. (K. Nowland) Let $\varepsilon > 0$ be given. Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \varepsilon$. Let $A = \sum_{i=1}^{N} a_i$. This is a finite sum. We calculate that for $k \in \mathbb{N}$,

$$\frac{1}{N+k}\sum_{i=1}^{N+k} a_i \leq \frac{1}{N+k}A + \frac{k}{N+k}a + \frac{k}{N+k}\varepsilon$$

Letting $k \to \infty$, we see that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i \leq a + \varepsilon.$$

Similarly,

$$\frac{1}{N+k}A + \frac{k}{N+k}a - \frac{k}{N+k}\varepsilon \leq \frac{1}{N+k}\sum_{i=1}^{N+k} a_i.$$

Again, letting $k \to \infty$ we see that

$$a - \varepsilon \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i.$$

Since ε was arbitrary, this completes the proof. $\square$

Exercise 4. *Let $a > 0$ and $\varepsilon > 0$ be given. Find a positive constant K, depending on a and ε, such that for all $x > K$, the inequality $\log x < \varepsilon x^a$ holds. Verify that your constant works. It is not necessary to find the least possible K.*

Proof. (R. Garrett) Since the question doesn't ask for the minimum K, we may impose the restriction that $x > 0$ and $K > 0$. Then, by the archimedean property, there exists $N \in \mathbb{N}$ such that $an > 1$ for all $n \geq N$. We notice that $\ln(x) < \epsilon x^a$ if and only if $x < e^{(\epsilon x^a)} = \sum_{l=0}^{\infty} \frac{(\epsilon x^a)^l}{l!} > \sum_{l=N}^{\infty} \frac{(\epsilon x^a)^l}{l!} > \frac{\epsilon^n x^{aN}}{N!}$. So, it suffices to find an x such that $\frac{\epsilon^n x^{aN}}{N!} > x$, which is the same as $\frac{\epsilon^n x^{aN-1}}{N!} > 1$ or $x^{aN-1} > \frac{N!}{\epsilon^N}$ or $x > (\frac{N!}{\epsilon^N})^{\frac{1}{aN-1}}$. Now, we may set $K = (\frac{N!}{\epsilon^N})^{\frac{1}{aN-1}}$. $\square$

Exercise 5. *For a bounded interval $[a, b]$, let M be the set of all continuous strictly positive functions on $[a, b]$. For f in M, let $L(f)$ be defined by*

$$L(f) := \left(\int_a^b f(x)dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right).$$

For what function(s) f does $L(f)$ attain its minimum value, and what is that minimum value?

Proof. (O. Khalil) By the Cauchy-Schwarz inequality (see proof in solution of Ex 5, Spring 05), for each $f \in M$, we have that

$$L(f) \geq \left(\int_a^b \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}}dx\right)^2 = (b-a)^2$$

Moreover, the function $g(x) = 1$ belongs to M and has that $L(g) = (b-a)^2$. So, $L(f)$ attains its minimum at g which equals $(b-a)^2$. □

Exercise 6. *Prove the following (Arithmetic-Geometric Mean) inequality: If n is a positive integer, and if $x_1, \ldots, x_n$ are positive numbers, then*

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{67}$$

Proof. See Spring 2006, exercise 5. □