To pass this exam, it is sufficient to completely solve four problems.

1. Suppose $E_n \subset X$, $n = 1, 2, \ldots$, are measurable subsets of a measure space $(X, \mu)$ and
$$\lim_{n \to \infty} \mu(E_n) = \mu(X).$$
If $\mu(X) < \infty$, show there is a subsequence $n_1 < n_2 < \ldots$ such that
$$\mu(\cap_{k=1}^{\infty} E_{n_k}) > 0.$$ 
Does it matter whether $\mu(X) < \infty$ or not? Is it necessarily true that $\mu(\cap_{n=1}^{\infty} E_n) > 0$?

2. Let $f \in L^1(A)$ be a nonnegative function, where $A \subset \mathbb{R}^n$ is a Lebesgue measurable set. Let $\mu$ denote Lebesgue measure. Show the following: given $\epsilon > 0$, there exists $\delta > 0$ such that for every subset $B \subset A$ with $\mu(B) < \delta$, it holds that
$$\int_B f < \epsilon.$$ 

3. Let $f \in C_0^\infty(\mathbb{R})$. Define
$$I_f(x) = \int_\mathbb{R} e^{ixt} f(t) \, dt,$$
and show that $I_f$ has faster than polynomial decrease as $x \to \infty$. That is, for each $N \in \mathbb{Z}^+$, show that there exists a constant $C > 0$ such that for $x \in \mathbb{R}^+$
$$|I_f(x)| \leq C x^{-N}.$$ 

4. Let $N_1 = C^1([0,1])$ and $N_2 = C([0,1])$, both equipped with the sup norm. Let $D : N_1 \to N_2$ be the linear map defined $D(f) = f'$. Show the following:
(a) $D$ is not bounded.
(b) The graph of $D$ is closed.
(c) Conclusion (b) does not violate the Closed Graph Theorem.

5. Let $H$ be a Hilbert space and $S \subset H$ a closed, linear subspace. Show that $S$ is a Hilbert space with operations inherited from $H$. Let $\{e_k\}_{k \in I}$ be an orthonormal basis for $S$. If $h \in H$, show that the unique element of $S$ nearest $h$ is the element
$$\tilde{h} =: \sum_{k \in I} (h, e_k) e_k.$$ 
(First show that $\tilde{h}$ is well-defined.)

6. Let $(X, \mu)$ be a measure space and suppose $\mu(X) < \infty$. Show that $g \in L^1(X, \mu)$ if and only if
$$\sum_{k=1}^{\infty} 3^k \mu \left\{ x : |g(x)| > 3^k \right\} < \infty.$$