Algebra Qualifying Exam I
2019

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. For a group H that has a Jordan-Hölder series, let $\ell(H)$ denote the number of terms in such series. Now, let G be a group and N be a normal subgroup of G. Prove that G has a Jordan-Hölder series if, and only if both N and G/N do. In this case, prove that $\ell(G) = \ell(N) + \ell(G/N)$.

2. Let G be a finite group, and H a proper subgroup of G. Prove G is not the union of all conjugates of H in G.

3. Let R be a ring (with unit, but possibly noncommutative) and consider elements $a, b \in R$. Prove that if $1 - ab$ is invertible, then so is $1 - ba$.

4. Let R be a commutative ring with 1, and let M be an R-module. For a prime ideal $\mathfrak{p} \subset R$, let $M_{\mathfrak{p}}$ be the localization at the multiplicatively closed set $S = R \setminus \mathfrak{p}$. Prove that
$$M = 0 \text{ if and only if } M_{\mathfrak{p}} = 0 \text{ for every prime ideal } \mathfrak{p}$$

5. Let V be a finite dimensional vector space over a field of characteristic 0. Consider the canonical map $\pi\colon T^n(V) \to S^n(V)$ from the n-fold tensor product of V onto the n-fold symmetric product of V. Prove that π has a section.