OSU Analysis Qualifying Examination 1 **August 2019**

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Solutions to five problems constitute a complete exam.

Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a two hour, closed book, closed notes exam.

(1) (20pt) Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Suppose $\mu(X) < \infty$. Let $f : X \to \mathbb{R}$ be measurable, let (f_n) be a sequence of measurable functions $f_n : X \to \mathbb{R}$, and suppose $f_n \to f$ in μ-measure. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, let $h = g \circ f$ and for each n, let $h_n = g \circ f_n$. Prove that $h_n \to h$ in μ-measure.
   (b) Show by an example that the condition that $\mu(X) < \infty$ in part (a) cannot be dropped.

(2) (20pt) Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a bounded increasing, and right-continuous function. Show that

$$\int_{\mathbb{R}} (\alpha(x + c) - \alpha(x))\, dx = c \int_{\mathbb{R}} d\alpha \qquad \text{for each } c > 0,$$

where $\int f\, dx$ is the Lebesgue integral and $\int f\, d\alpha$ is the Lebesgue-Stieltjes integral.

(3) (20pt) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f and g be real-valued integrable functions such that $\int_X f\, d\mu = \int_X g\, d\mu$. Prove that either (i) $f = g$ a.e., or (ii) there exists $E \in \mathcal{M}$ such that $\int_E f\, d\mu > \int_E g\, d\mu$.

(4) (20pt) Let X be a topological space and let $f : X \to \mathbb{C}$ be a function. Prove that the set of points at which f is continuous is a Borel subset of X.

(5) (20pt) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be an integrable function on X. Show that given $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \in \mathcal{M}$ with $\mu(E) < \delta$, we have $\int_E |f| < \epsilon$.

(6) (20pt) Let $1 < p < +\infty$ and ℓ^p be the space of square-integrable, real-valued sequences, i.e. $\ell^p := \left\{ (a_j)_{j=1}^{\infty} \,:\, a_j \in \mathbb{R},\, \sum_{j=1}^{\infty} |a_j|^p < +\infty \right\}$. Recall that ℓ^p is endowed with the norm

$$\|\mathbf{a}\| = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{\frac{1}{p}} \qquad \text{if } \mathbf{a} = (a_j)_{j=1}^{\infty}.$$

Show that ℓ^p is complete.