BAER MODULE HULLS OF CERTAIN MODULES OVER A DEDEKIND DOMAIN

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Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. The notion of a Baer ring, introduced by Kaplansky, has been extended to that of a Baer module using the endomorphism ring of a module in recent years. There do exist some results in the literature on Baer ring hulls of given rings. In contrast, the study of a Baer module hull of a given module remains wide open. In this paper we initiate this study. For a given module M, a Baer module hull, $\mathfrak{B}(M)$, is the smallest Baer overmodule contained in a fixed injective hull $E(M)$ of M.

For a certain class of modules N over a commutative noetherian domain, we characterize all essential overmodules of N which are Baer. As a consequence, it is shown that Baer module hulls exist for such modules over a Dedekind domain. A precise description of such hulls is obtained. It is proved that a finitely generated module N over a Dedekind domain has a Baer module hull if and only if the torsion submodule $t(N)$ of N is semisimple. Further, in this case, the Baer module hull of N is explicitly described.

As applications, various properties and examples of Baer hulls are exhibited. It is shown that if N_1, N_2 are two modules with Baer hulls, $N_1 \oplus N_2$ may not have a Baer hull. On the other hand, the Baer module hull of $M = \mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) is precisely given by $\mathfrak{B}(M) = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$. It is shown that infinitely generated modules over a Dedekind domain may not have Baer hulls.

1. Introduction

Shoda in 1952 [31] and Eckmann and Schopf in 1953 [7] independently proved that for every module M_R over an arbitrary ring R there exists a unique (up to isomorphism over M)

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minimal injective overmodule $E(M)$ called its injective hull. The study of a more general type of "hull" of M or the unique smallest essential overmodule of M in a fixed injective hull $E(M)$ of M having some special property has been of interest since then. This includes, for a given module M, the study of hulls of M having properties which generalize injectivity (for example, quasi-injective, continuous, quasi-continuous hulls) or properties which are otherwise connected to injectivity. An important focus of investigations has been to prove the existence and explicit descriptions of various types of module hulls.

Recall that a module M is called *quasi-injective* if $f(M) \subseteq M$ for each $f \in \mathrm{End}(E(M))$. Among other well-known generalizations of injectivity, the study of the continuous, quasi-continuous, and extending properties has been extensive in the literature (see for example [6], [18], [19], [20], and [23]). A module M is said to be *extending* if, for each $V \leq M$, there exists a direct summand $W \leq^{\oplus} M$ such that $V \leq^{\mathrm{ess}} W$. And an extending module M is called *quasi-continuous* if for all direct summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a direct summand of M. Furthermore, an extending module M is said to be a *continuous* if every submodule N of M which is isomorphic to a direct summand is also a direct summand of M. The well-known hierarchy of these properties is as follows:

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous} \Rightarrow \text{quasi-continuous} \Rightarrow \text{extending}$$

(while none of the reverse implications hold).

For a given module M, let $H = \mathrm{End}_R(E(M))$ denote the endomorphism ring of its injective hull $E(M)$. Then by Johnson and Wong [11], the unique quasi-injective hull of the module M is precisely HM. Goel and Jain [9] showed that there always exists a unique quasi-continuous hull of every module within a fixed injective hull $E(M)$. The quasi-continuous hull of M is exactly given by ΩM, where Ω is the subring generated by all idempotents of $H = \mathrm{End}(E(M))$. In contrast to this, it was shown by Müller and Rizvi in [19] that continuous module hulls do not always exist in general. However, they did show the existence of continuous hulls of certain classes of modules over a commutative ring (such as nonsingular cyclic ones) and provided a description of these continuous hulls (see [19, Theorem 8]). Similarly, it is also known that extending module hulls do not always exist (for example, see [4, Example 8.4.13, p.319]). Closely linked to these notions, are the notions of Baer and Rickart modules. In particular, every nonsingular extending module is a Baer module while the converse holds under a certain dual condition (see Lemma 3.5). In this paper, we introduce and study Baer (module) hulls of certain modules over a Dedekind domain.

Kaplansky introduced the notions of Baer and Rickart rings in [13]. He and many others obtained a number of interesting results on these two classes of rings which have their roots in Functional Analysis (for example, [1], [2], [4], [5], [10], [13], and [17]). More recently, the notions of a Baer ring and a Rickart ring were extended to analogous module theoretic notions using the endomorphism ring S of the module by Rizvi, Roman, and Lee ([24] and [15]). A module M is called a *Baer module* if, for any $N_R \leq M_R$, there exists $e^2 = e \in S$ such that $\ell_S(N) = Se$, where $S = \mathrm{End}(M_R)$ and $\ell_S(N) = \{f \in S \mid f(N) = 0\}$. Equivalently, a module M is Baer if and only if for any left ideal I of S, $r_M(I) = fM$ with $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$.

Recall from [4, Chapter 8] that the *Baer ring hull of a ring R is the smallest Baer right essential overring of R in $E(R_R)$*. While some work has been done on the existence of a quasi-Baer ring hull of a ring R for some special classes of rings ([2], [3], and [4]), there is almost nothing known about the existence or description of Baer module hulls. To the best of our knowledge, the only explicit results about *Baer ring hulls* in existing literature have been due to Mewborn, Oshiro, and Hirano, Hongan and Ohori. However, these results about ring hulls are only for the case of *commutative semiprime rings* or of *reduced right Utumi rings*. Further, in each of these cases the Baer ring hull is not distinct from the quasi-Baer ring hull of R. Mewborn [17, Proposition 2.5], showed the existence of a unique Baer ring hull of a commutative semiprime ring R and showed that it is exactly the subring of the maximal ring of quotients $Q(R)$ generated by R and the idempotents of $Q(R)$. That is, the Baer ring hull of a commutative semiprime ring is given by $R\mathcal{B}(Q(R))$, where $\mathcal{B}(Q(R))$ is the set of all (central) idempotents of $Q(R)$. As a direct consequence, Oshiro [21, Proposition 3.3] showed that the Baer ring hull of a commutative von Neumann regular ring is a continuous regular ring. Oshiro [22] then extended his work and constructed the Baer ring hull of a commutative von Neumann regular ring by using sheaf representations. Hirano, Hongan, and Ohori [10,Theorem 4] proved the existence of a Baer ring hull for a reduced right Utumi ring. These results on the existence of Baer ring hulls of a commutative semiprime ring or a reduced Utumi ring were recently extended and a unified result was obtained for the case of arbitrary semiprime rings by Birkenemier, Park, and Rizvi [2, Theorem 3.3].

In contrast to the ring hull of a given ring R, the study of a module hull of a given module M_R appears to be more natural because the injective hull $E(M_R)$ as an overmodule of M_R always exists, while $E(R_R)$ does not have a ring structure in general. So the ring hulls of R are more useful when these are either contained in the maximal right ring of quotients $Q(R)$ of R or the injective hull $E(R_R)$ is endowed with a compatible ring structure (see [4, Chapter 7]).

For a given module M, the smallest Baer overmodule of M in $E(M)$ is called the *Baer module hull* (in short, the Baer hull) of M and we denote it by $\mathfrak{B}(M)$. One of the difficulties in dealing with the Baer module hull of a module M_R is the interplay of the scalar multiplication of M with R on one side of M and with the endomorphism ring $S = \text{End}(M_R)$ on the other side of M. Such an overmodule of M not only has to satisfy the conditions for being Baer but also being the smallest such overmodule of M to be a hull. The Baer property of the hull thus necessitates a consideration of endomorphism rings of all overmodules of M in $E(M)$ before we can locate the smallest Baer overmodule of M.

In this paper, we initiate the study of the Baer module hull of a given module M_R. In particular, for a module M and a fixed injective hull $E(M)$ of M we study the existence of the Baer *module* hull of M of certain types of modules over a Dedekind domain. It is shown that a Baer hull, may not always exist for infinitely generated modules over Dedekind domains. We obtain explicit description of Baer hulls of a certain class of modules over a Dedekind domain.

From [24] it is known that $M = \mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) is not a Baer $\mathbb{Z}$-module, while $\mathbb{Z}_p$ and $\mathbb{Z}$ are. We first characterize when intermediate modules between an analogous direct sum as a module over a commutative noetherian domain and its injective hull are Baer. This result is then used to explicitly construct and characterize the Baer hull of an module N over a Dedekind domain R, when $\text{Ann}_R(t(N)) \neq 0$ and $N/t(N)$ is finitely generated, where $t(N)$ denotes the torsion submodule of N. Consequently, we prove that every finitely generated module over a Dedekind domain, has a unique Baer hull precisely when its torsion submodule is semisimple. This unique hull is explicitly described.

All rings are assumed to have identity and all modules are assumed to be unitary. For right

R-modules M_R and N_R, we use $\mathrm{Hom}(M_R, N_R)$, $\mathrm{Hom}_R(M, N)$, or $\mathrm{Hom}(M, N)$ to denote the set of all R-module homomorphisms from M_R to N_R. Likewise, $\mathrm{End}(M_R), \mathrm{End}_R(M)$, or $\mathrm{End}(M)$ denote the endomorphism ring of an R-module M. For an R-module homomorphism $f \in \mathrm{Hom}_R(M, N)$, $\mathrm{Ker}(f)$ is used for the kernel of f.

We use $E(M_R)$ or $E(M)$ to denote an injective hull of a module M_R. For a module M, $L \leq M$, $N \leq^{\mathrm{ess}} M$, and $U \leq^{\oplus} M$ denote that L is a submodule of M, N is an essential submodule of M, and U is a direct summand of M, respectively. If M is an R-module, $\mathrm{Ann}_R(M)$ stands for the annihilator of M in R.

For a module M and a set Λ, let $M^{(\Lambda)}$ denote the direct sum of $|\Lambda|$ copies of M, where $|\Lambda|$ is the cardinality of Λ. When Λ is finite with $|\Lambda| = n$, then $M^{(n)}$ stands for $M^{(\Lambda)}$. $\mathrm{Mat}_n(R)$ denotes the $n \times n$ matrix ring over a ring R. The symbols $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z}_n (n > 1)$ stand for the field of rational numbers, the ring of integers, and the ring of integers modulo n, respectively.

As mentioned, we will use the term *Baer hull* for Baer module hull in this paper.

2. Baer Module Hulls

In [24], it was shown that a finitely generated module N, over a commutative PID, is a Baer module if and only if N is either semisimple or torsion-free. In particular, $\mathbb{Z}_p \oplus \mathbb{Z}$ is not a Baer $\mathbb{Z}$-module, where p is a prime integer. Motivated by the preceding result and example, for a given finitely generated module N over a commutative domain R, it is of interest to study intermediate Baer modules between N and $E(N)$, and investigate possible existence of the Baer hull of N. The investigations on Baer hulls are even more relevant since nothing is known about the Baer hulls. The only information that exists is that of a couple of special cases of Baer ring hulls, which in fact, were shown to be precisely quasi-Baer ring hulls [4] as discussed earlier.

Let $G = \mathbb{Z}_{p_1^{k_1}} \oplus \mathbb{Z}_{p_2^{k_2}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}} \oplus \mathbb{Z}^{(m)}$, where $p_1, p_2, \ldots, p_n$ are prime integers (not necessarily distinct) and $k_1, k_2, \ldots, k_n$, and m are nonnegative integers. If one of k_is is greater than 1, then there is no intermediate Baer $\mathbb{Z}$-module between G and $E(G_{\mathbb{Z}})$. In fact, say V is an intermediate abelian group between G and $E(G_{\mathbb{Z}})$, and assume $k_1 \geq 2$. Then by Kulikov [29, 5.2.10, p.98], either $\mathbb{Z}_{p_1^{\ell_1}}$ with $\ell_1 \geq k_1$ or $\mathbb{Z}_{p_1^\infty}$ is a direct summand of V. Because neither $\mathbb{Z}_{p_1^{\ell_1}}$ nor $\mathbb{Z}_{p_1^\infty}$ is a Baer $\mathbb{Z}$-module, V is not a Baer $\mathbb{Z}$-module (see Lemma 2.4(ii)). So it is absurd to study the Baer hull for G when one of $k_1, k_2, \ldots, k_n$ is greater than 1. Thus an investigation of Baer hulls of finitely generated abelian groups G makes sense only when the torsion subgroup of G is a semisimple $\mathbb{Z}$-module.

In general, if R is a Dedekind domain and N is a finitely generated R-module, we need to assume that $t(N)$ is semisimple for investigation of intermediate Baer modules between N and $E(N)$, where $t(N)$ is the torsion submodule of N (cf. Theorems 2.6 and 2.13). In this case, $t(N) \oplus E(N/t(N))$ is the 'largest' intermediate Baer module between N and $E(N)$ (see Theorem 2.6). Hence it is of interest to investigate the existence of the 'smallest' Baer intermediate module between N and $E(N)$ (i.e., Baer hull of N). We explicitly construct and characterize the Baer hull of a module N over a Dedekind domain R when $\mathrm{Ann}_R(t(N)) \neq 0$ and $N/t(N)$ is finitely generated (thus, $N \cong t(N) \oplus N/t(N)$). In particular, it is shown that if M is a module over a Dedekind domain R whose annihilator in R is nonzero, then $M \oplus (\oplus_{i=1}^m K_i)$ has a Baer hull if and only if M is semisimple, where $\{K_i \mid 1 \leq i \leq m\}$ are fractional ideals of R (Theorem 2.13). As a consequence, in this paper every finitely generated module N over a Dedekind domain has a Baer hull precisely when $t(N)$ is semismple (Theorem 2.18). The Baer module hull is explicitly described in these cases. Some application of our results and explicit examples that illustrate and delimit our results are provided in Section 3.

Among explicit constructions of Baer hulls, for given nonempty sets $\Gamma_1, \Gamma_2, \Gamma_3$ and a positive integer m, we show in Example 3.2 that the Baer hull of $\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}^{(m)}$ is precisely given by $\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}[1/30]^{(m)}$. Thus $M = \mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) has a unique Baer hull given by $V = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$. We note that M and V are not extending. In contrast, $W = \mathbb{Z}_{p^2}$ is the extending (quasi-injective) hull of itself, but W has no Baer hull.

We start with the following definition.

Definition 2.1. ([24, Definition 2.2]) A right R-module M is called a *Baer module* if, for any $N_R \leq M_R$, there exists $e^2 = e \in S$ such that $\ell_S(N) = Se$, where $S = \mathrm{End}(M_R)$ and $\ell_S(N) = \{f \in S \mid f(N) = 0\}$. Thus a right R-module M is Baer if and only if for any left ideal I of S, $r_M(I) = fM$ with $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$.

We recall that a ring R is called a *Baer ring* if the right annihilator of any nonempty subset of R is generated, as a right ideal, by an idempotent of R. Thus a ring R is a Baer ring if and only if R_R is a Baer module.

Definition 2.2. ([26, Definition 2.3]) Let M_R be an R-module and $S = \mathrm{End}_R(M)$. Then M_R is called *quasi-retractable* if $\mathrm{Hom}_R(M, r_M(I)) \neq 0$ for every left ideal I of S with $r_M(I) \neq 0$ (or, equivalently, if $r_S(I) \neq 0$ for every left ideal I with $r_M(I) \neq 0$).

Lemma 2.3. ([26, Theorem 2.5]) A module M_R is Baer if and only if $\mathrm{End}_R(M)$ is a Baer ring and M_R is quasi-retractable.

A module M is said to have the *strong summand intersection property* (SSIP) if the intersection of any family of direct summands of M is a direct summand.

Lemma 2.4. (i) ([24, Proposition 2.2]) A module M is Baer if and only if $\mathrm{Ker}(f) \leq^{\oplus} M$ for each $f \in \mathrm{End}(M)$ and M has the SSIP.

(ii) ([24, Theorem 2.17]) Any direct summand of a Baer module is a Baer module.

Say M and N are R-modules. Then M is said to be N-*injective* if, for any $W \leq N$ and $f \in \mathrm{Hom}(W, M)$, there exists $\varphi \in \mathrm{Hom}(N, M)$ such that $\varphi|_W = f$. Recall from [15] that M is called N-*Rickart* if $\mathrm{Ker}(f) \leq^{\oplus} M$ for each $f \in \mathrm{Hom}(M, N)$. Also recall from [26] that M and N are said to be *relatively Rickart* if M is N-Rickart and N is M-Rickart.

The following lemma is [26, Theorem 3.6] (also see [4, Theorem 4.2.17, p.105]).

Lemma 2.5. Let $\{M_i \mid 1 \leq i \leq n\}$ be a finite set of Baer modules. Assume that M_i and M_j are relatively Rickart for $i \neq j$, and M_i is M_j-injective for $i < j$. Then $\oplus_{i=1}^n M_i$ is a Baer module.

Let R be a commutative noetherian domain and F be its field of fractions. Assume that $N = M_R \oplus (\oplus_{i \in \Lambda} K_i)$, where M is semisimple with a finite number of homogeneous components, and $\{K_i\}_{i\in\Lambda}$ is a set of nonzero submodules of F_R. Our first focus is to study intermediate modules between N and $E(N)$ which happen to be Baer modules.

Theorem 2.6. Let R be a commutative noetherian domain, which is not a field. Assume that M is a nonzero semisimple R-module with only a finite number of homogeneous components, and $\{K_i \mid i \in \Lambda\}$ is a nonempty set of nonzero submodules of F_R, where F is the field of fractions of R. Let V_R be an essential extension of $M_R \oplus (\oplus_{i\in\Lambda} K_i)_R$. Then the following are equivalent.

(i) V is a Baer module.

(ii) (1) $V = M \oplus W$ for some Baer essential extension W of $(\oplus_{i\in\Lambda} K_i)_R$.

(2) $\mathrm{Hom}_R(W, M) = 0$.

Proof. (i)$\Rightarrow$(ii) We assume that V is a Baer module. Let $\{H_k \mid 1 \le k \le s\}$ be the set of all homogeneous components of M. Then $M = \oplus_{k=1}^{s} H_k$. For each k, $1 \le k \le s$, there exists a nonempty set Γ_k such that $H_k = \oplus_{\alpha\in\Gamma_k} M_{(k,\alpha)}$, where each $M_{(k,\alpha)}$ is a simple module.

For k, $1 \le k \le s$, we put $P_k = \mathrm{Ann}_R(M_{(k,\alpha)})$, where $\alpha \in \Gamma_k$, and recall that $\mathrm{Ann}_R(-)$ denotes the annihilator of a module in R. Then $I := \cap_{k=1}^{s} P_k = \mathrm{Ann}_R(M)$. So $I \ne 0$ because each $P_k \ne 0$. As R is noetherian, $I = a_1R + \cdots + a_mR$, for some $a_1, \ldots, a_m \in I$. For i, $1 \le i \le m$, define $\varphi_{a_i} : V \to V$ by $\varphi_{a_i}(v) = va_i$, where $v \in V$. Then $\varphi_{a_i} \in \mathrm{End}_R(V)$.

Since V is a Baer module, Lemma 2.4(i) yields that $\mathrm{Ker}(\varphi_{a_i}) = \ell_V(a_iR)$ is a direct summand of V, for each i, where $\ell_V(-)$ is the annihilator in V. So $\ell_V(I) = \cap_{i=1}^{m} \ell_V(a_iR)$ is a direct summand of V by Lemma 2.4(i). Hence $V = \ell_V(I) \oplus W$ for some $W \le V$. As V is a Baer module, W is a Baer module by Lemma 2.4(ii).

We claim that $\ell_V(I) = M$. For this, note that $M \subseteq \ell_V(I)$. Let $Y = E(M) \oplus E(R_R^{(\Lambda)})$. As R is noetherian, $E(R_R^{(\Lambda)}) = E(R_R)^{(\Lambda)}$, and so $E(\oplus_{i\in\Lambda} K_{iR}) = \oplus_{i\in\Lambda} E(K_{iR}) = E(R_R)^{(\Lambda)}$. Therefore

$$V = \ell_V(I) \oplus W \le Y = E(M) \oplus E(R_R)^{(\Lambda)} = E(M) \oplus (\oplus_{i\in\Lambda} E(K_{iR})).$$

Let $t(Y)$ be the torsion submodule of Y. Say $0 \ne x \in E(M)$. Then there exists $r \in R$ such that $0 \ne xr \in M \subseteq \ell_V(I)$ since $M \le^{\mathrm{ess}} E(M)$. Take $0 \ne b \in I$. Then $xrb = 0$ with $0 \ne rb \in R$. Thus $E(M) \subseteq t(Y)$.

Next, let $y \in t(Y)$. Then $y = \mu + \nu \in E(M) \oplus E(R_R)^{(\Lambda)}$, where $\mu \in E(M)$ and $\nu \in E(R_R)^{(\Lambda)}$. So there exists $0 \ne c \in R$ such that $0 = yc = \mu c + \nu c$. Hence $\nu c = 0$. Since $E(R_R)$ is the field of fractions of R, we see that $\nu = 0$. Therefore $y = \mu \in E(M)$. Thus $t(Y) \subseteq E(M)$. Consequently, $t(Y) = E(M)$.

Say $v \in \ell_V(I)$. Then v is a torsion element in V, so it is a torsion element of Y. Thus $v \in t(Y) = E(M)$. Since R is noetherian,

$$E(M) = \left(\oplus_{\alpha\in\Gamma_1} E(M_{(1,\alpha)})\right) \oplus \left(\oplus_{\beta\in\Gamma_2} E(M_{(2,\beta)})\right) \oplus \cdots \oplus \left(\oplus_{\gamma\in\Gamma_s} E(M_{(s,\gamma)})\right).$$

So there exist $M_1, \ldots, M_n \in \{M_{(1,\alpha)}, M_{(2,\beta)}, \ldots, M_{(s,\gamma)} \mid \alpha \in \Gamma_1,\ \beta \in \Gamma_2,\ \ldots,\ \gamma \in \Gamma_s\}$ such that $v \in E(M_1) \oplus \cdots \oplus E(M_n)$. Put $v = y_1 + \cdots + y_n \in E(M_1) \oplus \cdots \oplus E(M_n)$, where $y_i \in E(M_i)$ for i, $1 \le i \le n$. For any $a \in I$, we have that $0 = va = y_1a + \cdots + y_na$, hence $y_1a = 0, \ldots, y_na = 0$. Since $y_1a = 0$ for all $a \in I$, $y_1I = 0$. Similarly, $y_2I = 0, \ldots, y_nI = 0$.

Now $M_1 \subseteq y_1R + M_1 \subseteq E(M_1)$, and hence $y_1R + M_1$ is a uniform module. Because $(y_1R + M_1)I = 0$, we see that $y_1R + M_1$ is an R/I-module induced by the R-module structure of $y_1R + M_1$. Hence $y_1R + M_1$ is a uniform R/I-module.

Further, since $I = \cap_{i=1}^{s} P_i$ and P_i are distinct maximal ideals of R,

$$R/I \cong R/P_1 \oplus \cdots \oplus R/P_s$$

by the Chinese Remainder Theorem. So R/I is a finite direct sum of fields R/P_i (hence R/I is a semisimple artinian ring). Therefore $y_1R + M_1$ is a simple R/I-module because $y_1R + M_1$ is uniform as an R/I-module. Since $M_1I = 0$, M_1 is also an R/I-module which is induced from the R-module structure of M_1. Thus $M_1 = y_1R + M_1$ because $y_1R + M_1$ is a simple R/I-module. Hence $y_1 \in M_1$.

Similarly, $y_2 \in M_2, \ldots, y_n \in M_n$. Thereby $v = y_1 + \cdots + y_n \in M_1 \oplus \cdots \oplus M_n$, and thus $v \in M$. Hence $\ell_V(I) \subseteq M$. Therefore $\ell_V(I) = M$. So $V = M \oplus W$.

Next, we show that

$$(\oplus_{i\in\Lambda} K_i)_R \leq W_R \leq E(R_R)^{(\Lambda)}.$$

For this, we first prove that W is torsion-free. Say $w \in W \leq E(M) \oplus E(R_R)^{(\Lambda)}$. Then $w = x_w + y_w \in E(M) \oplus E(R_R)^{(\Lambda)}$ such that $x_w \in E(M)$ and $y_w \in E(R_R)^{(\Lambda)}$. Let $f : W_R \to E(R_R)^{(\Lambda)}$ be defined by $f(w) = y_w$ for $w \in W$. Since $M \cap W = 0$, $E(M) \cap W = 0$. So f is an R-module monomorphism. Thus W is torsion-free.

Now we take $0 \neq a \in I$. Because $M \oplus (\oplus_{i\in\Lambda} K_i) \subseteq M \oplus W \subseteq E(M) \oplus E(R_R)^{(\Lambda)}$, it follows that $(M \oplus (\oplus_{i\in\Lambda} K_i))a \subseteq (M \oplus W)a$, and thus $(\oplus_{i\in\Lambda} K_i)a \subseteq Wa$ since $\ell_V(I) = M$. First for $((\oplus_{i\in\Lambda} K_i)a)_R \leq^{\text{ess}} (Wa)_R$, let $0 \neq wa \in Wa$. Then, as $M \oplus (\oplus_{i\in\Lambda} K_i) \leq^{\text{ess}} M \oplus W$ and $wa \in Wa \subseteq W$, there is $b \in R$ such that $0 \neq wab \in M \oplus (\oplus_{i\in\Lambda} K_i)$.

Say $wab = u + \eta \in M \oplus (\oplus_{i\in\Lambda} K_i)$ with $u \in M$ and $\eta \in \oplus_{i\in\Lambda} K_i$. Note that $0 \neq wab \in W$, W is torsion-free, and $a \neq 0$. Hence $0 \neq (wab)a = \eta a \in (\oplus_{i\in\Lambda} K_i)a$. Therefore we have $((\oplus_{i\in\Lambda} K_i)a)_R \leq^{\text{ess}} (Wa)_R$.

We notice that $0 \neq ((\oplus_{i\in\Lambda} K_i)a)_R \leq (\oplus_{i\in\Lambda} K_i)_R$ and $0 \neq (Wa)_R \leq W_R$, obviously $((\oplus_{i\in\Lambda} K_i)a)_R \leq^{\text{ess}} (\oplus_{i\in\Lambda} K_i)_R$ and $(Wa)_R \leq^{\text{ess}} W_R$. Hence it follows that

$$E(R_R)^{(\Lambda)} = E[(\oplus_{i\in\Lambda} K_i)_R] = E[((\oplus_{i\in\Lambda} K_i)a)_R] = E[(Wa)_R] = E(W_R).$$

Therefore $W \leq E(R_R)^{(\Lambda)}$.

To verify $(\oplus_{i\in\Lambda} K_i)_R \leq W_R$, let $\xi \in \oplus_{i\in\Lambda} K_i \subseteq M \oplus (\oplus_{i\in\Lambda} K_i) \subseteq M \oplus W$. Now we write $\xi = \tau + \omega \in M \oplus W$ with $\tau \in M$ and $\omega \in W$. Take $0 \neq a \in I$. Then $\xi a = \tau a + \omega a = \omega a$ because $M = \ell_V(I)$. Therefore $(\xi - \omega)a = 0$.

Note that $\xi \in \oplus_{i\in\Lambda} K_i \leq E(R_R)^{(\Lambda)}$, $\omega \in W \leq E(R_R)^{(\Lambda)}$, and $E(R_R)$ is the field of fractions of R. Thus we obtain $\xi - \omega = 0$, and hence $\xi = \omega \in W$. Therefore $\oplus_{i\in\Lambda} K_i \leq W$. As a consequence, $(\oplus_{i\in\Lambda} K_i) \leq W \leq E(R_R)^{(\Lambda)} = E[(\oplus_{i\in\Lambda} K_i)_R]$.

Let $N \in \{M_{(1,\alpha)}, M_{(2,\beta)}, \ldots, M_{(s,\gamma)} \mid \alpha \in \Gamma_1, \beta \in \Gamma_2, \ldots, \gamma \in \Gamma_s\}$. Assume on the contrary that $\text{Hom}_R(W, N) \neq 0$. Take $0 \neq f \in \text{Hom}_R(W, N)$. Since N is simple, f is onto. Let $\phi : N \oplus W \to N \oplus W$ be defined by $\phi(n, w) = (f(w), 0)$ for $(n, w) \in N \oplus W$. Then $\text{Ker}(\phi) = N \oplus \text{Ker}(f) \subseteq N \oplus W$.

As $\text{Ker}(f) \neq W$, $\text{Ker}(\phi) = N \oplus \text{Ker}(f) \subsetneq N \oplus W$. Since $N \oplus W \leq^{\oplus} V = M \oplus W$ and V is Baer, $N \oplus W$ is Baer by Lemma 2.4(ii). Thus $N \oplus W = \text{Ker}(\phi) \oplus U = N \oplus \text{Ker}(f) \oplus U$ for some $U \leq N \oplus W$.

Define $h : W \to \text{Ker}(f) \oplus U$ by $h(w) = x + u$, where $w \in W$ and $w = n + x + u$ with $n \in N, x \in \text{Ker}(f)$, and $u \in U$. Then h is an R-module isomorphism. Next, we let $\pi : \text{Ker}(f) \oplus U \to U$ be $\pi(x + u) = u$, where $x \in \text{Ker}(f)$ and $u \in U$. Put $g = \pi \circ h$.

Say $w \in \mathrm{Ker}(g)$ with $w = n + x + u \in N \oplus \mathrm{Ker}(f) \oplus U$, where $n \in N$, $x \in \mathrm{Ker}(f)$, and $u \in U$. Then

$$0 = \pi(h(w)) = \pi(x + u) = u, \quad \text{so} \quad w = n + x + u = n + x.$$

Hence $w - x = n \in W \cap N = 0$, thus $w = x \in \mathrm{Ker}(f)$. Conversely, suppose that $x \in \mathrm{Ker}(f)$. Then $g(x) = \pi(h(x)) = \pi(x) = 0$, so $x \in \mathrm{Ker}(g)$. Therefore $\mathrm{Ker}(f) = \mathrm{Ker}(g)$.

Since $g : W \to U$ is onto, $W/\mathrm{Ker}(g) \cong U$. On the other hand, $W/\mathrm{Ker}(f) \cong N$ because $f \neq 0$ is onto. Therefore

$$N \cong W/\mathrm{Ker}(f) = W/\mathrm{Ker}(g) \cong U.$$

Note that $UI = 0$ as $NI = 0$. Recall that $h : W \to \mathrm{Ker}(f) \oplus U$ is an R-module isomorphism. So $W \cong \mathrm{Ker}(f) \oplus U$. Thus U is torsion-free, hence $U = 0$, and so $W/\mathrm{Ker}(f) \cong U = 0$. Therefore $W = \mathrm{Ker}(f)$, a contradiction. Thus $\mathrm{Hom}_R(W, N) = 0$.

Consequently, for any $N \in \{M_{(1,\alpha)}, M_{(2,\beta)}, \dots, M_{(s,\gamma)} \mid \alpha \in \Gamma_1, \beta \in \Gamma_2, \dots, \gamma \in \Gamma_s\}$, we have that $\mathrm{Hom}_R(W, N) = 0$. For $k, 1 \leq k \leq s$, we note by [27, Theorem 2.6, p.31] that $(\mathrm{Hom}_R(W, \prod_{\alpha\in\Gamma_k} M_{(k,\alpha)}), +) \cong (\prod_{\alpha\in\Gamma_k} \mathrm{Hom}_R(W, M_{(k,\alpha)}), +)$, and by the preceding argument $\mathrm{Hom}_R(W, M_{(k,\alpha)}) = 0$ for each $\alpha \in \Gamma_k$. Therefore $\mathrm{Hom}_R(W, \prod_{\alpha\in\Gamma_k} M_{(k,\alpha)}) = 0$. Because $\mathrm{Hom}_R(W, H_k) = \mathrm{Hom}_R(W, \oplus_{\alpha\in\Gamma_k} M_{k,\alpha)}) \subseteq \mathrm{Hom}_R(W, \prod_{\alpha\in\Gamma_k} M_{(k,\alpha)})$, we obtain $\mathrm{Hom}_R(W, H_k) = 0$ for each k, $1 \leq k \leq s$. Hence $\mathrm{Hom}_R(W, M) = \mathrm{Hom}_R(W, \oplus_{k=1}^s H_k) = 0$ from [27, Theorem 2.6, p.31]. This proves that (ii) holds.

(ii)$\Rightarrow$(i) For any $N \in \{M_{(1,\alpha)}, M_{(2,\beta)}, \dots, M_{(s,\gamma)} \mid \alpha \in \Gamma_1, \beta \in \Gamma_2, \dots, \gamma \in \Gamma_s\}$, we can check that $\mathrm{Hom}_R(N, W) = 0$. Indeed, say $N = M_{(1,\alpha)} \cong R/P_1$ as R-modules. Let $h \in \mathrm{Hom}_R(R/P_1, W)$ and say $h(1 + P_1) = w \in W$. Take $0 \neq a \in I$. Then we see that

$$0 = h(a + P_1) = h(1 + P_1)a = wa.$$

Since W is torsion-free, $w = 0$ and thus $h = 0$. Therefore $\mathrm{Hom}_R(R/P_1, W) = 0$, hence $\mathrm{Hom}_R(N, W) = 0$. So $\mathrm{Hom}_R(M, W) = 0$ from [27, Theorem 2.4, p.30]. By hypothesis, $\mathrm{Hom}_R(W, M) = 0$. Thus M and W are relatively Rickart.

Now W is a Baer module by assumption. Because M is semisimple, M is a Baer module and W is M-injective. Therefore $V = M \oplus W$ is a Baer module by Lemma 2.5. $\square$

Recall from [8, p.112] that a module M is said to be Σ-*injective* if $M^{(\Lambda)}$ is injective for any set Λ. By [8, Exercise 20.4C(d), pp.114–115], a ring R is right Goldie if and only if $E(R_R)$ is Σ-injective.

Corollary 2.7. Let R be a commutative domain, which is not a field. Let M be a nonzero semisimple artinian R-module. Say $\{K_i \mid i \in \Lambda\}$ is a nonempty set of nonzero submodules of F_R, where F is the field of fractions of R. Assume that V_R is an essential extension of $M_R \oplus (\oplus_{i\in\Lambda} K_i)_R$. Then the following are equivalent.

(i) V is a Baer module.

(ii) (1) $V = M \oplus W$ for some Baer essential extension W of $(\oplus_{i\in\Lambda} K_i)_R$.

(2) $\mathrm{Hom}_R(W, M) = 0$.

Proof. (i)$\Rightarrow$(ii) Assume that V is a Baer module. Let $M = \oplus_{i=1}^n M_i$, where each M_i is a simple module. Let $P_i = \mathrm{Ann}_R(M_i)$ and $I = \cap_{i=1}^n P_i$. Then $I = \mathrm{Ann}_R(M)$ and $I \neq 0$. For each $a \in I$, we define $\varphi_a : V \to V$ by $\varphi_a(v) = va$ for $v \in V$. Then $\varphi_a \in \mathrm{End}_R(V)$. Since V is a Baer module, $\mathrm{Ker}(\varphi_a) \leq^{\oplus} V$ from Lemma 2.4(i). Hence $\ell_V(I) = \cap_{a \in I} \mathrm{Ker}(\varphi_a) \leq^{\oplus} V$ by the SSIP (see Lemma 2.4(i)), and thus $V = \ell_V(I) \oplus W$ for some $W \leq V$. Now as in the proof of Theorem 2.6, $\ell_V(I) = M$. Therefore we obtain $V = M \oplus W$ for some $W \leq V$. Note that $E(M) = \oplus_{i=1}^{n} E(M_i)$.

As $E(R_R) = F$ is a field, so R is right Goldie and thus $E(R_R)$ is Σ-injective. Hence $E(R_R)^{(\Lambda)}$ is injective. Since $(\oplus_{i \in \Lambda} K_i)_R \leq^{\mathrm{ess}} E(R_R)^{(\Lambda)}$ and $E(R_R)^{(\Lambda)}$ is injective, we see that $E(\oplus_{i \in \Lambda} K_{iR}) = E(R_R)^{(\Lambda)}$. The remaining part of the proof follows from the proof of Theorem 2.6.

(ii)$\Rightarrow$(i) The proof is similar to the proof of (ii)$\Leftrightarrow$(i) in the proof of Theorem 2.6. $\square$

Let R be a commutative domain with the field of fractions F. A submodule K of F_R is called a *fractional ideal* of R if $rK \subseteq R$ for some $0 \neq r \in R$. Thus $K_R \cong (rK)_R$ and rK is an ideal of R. We note that any ideal of R is a fractional ideal.

For a fractional ideal K of R, we put $K^{-1} = \{q \in F \mid qK \subseteq R\}$, which is called the *inverse* of K. We say that a fractional ideal K is *invertible* if $KK^{-1} = R$. It is well-known that for a nonzero ideal I of a commutative domain R, I_R is projective if and only if $II^{-1} = R$. In this case, I_R is finitely generated and I^{-1} is a fractional ideal of R.

Recall that a commutative domain R is a *Dedekind domain* if and only if R is hereditary. Thus for each nonzero ideal I of a Dedekind domain R, it follows that $II^{-1} = R$ because I_R is projective. Furthermore, every nonzero fractional ideal of a Dedekind domain is invertible.

We note that a Dedekind domain is noetherian because every ideal is projective (hence every ideal is finitely generated) (see [14, p.37] and [30, Chapter 6] for more details on Dedekind domains).

Assume that I is an invertible ideal of a commutative domain R. Then we let $I^{-2} = I^{-1}I^{-1}$, $I^{-3} = I^{-1}I^{-1}I^{-1}$, and so on. For convenience, we put $I^0 = R$.

Lemma 2.8. Assume that R is a Dedekind domain. Then:
(i) For nonzero ideals $I_1, I_2 \ldots, I_n$ of R, $(I_1 I_2 \cdots I_n)^{-1} = I_n^{-1} \cdots I_2^{-1} I_1^{-1}$.
(ii) For a nonzero ideal I of R and a positive integer n, $I^{-n} = (I^n)^{-1}$.

Proof. (i) Let I and J be nonzero ideals of R. Then IJ is also a nonzero ideal of R. Thus I, J, and IJ are invertible since R is a Dedekind domain. Say $q \in (IJ)^{-1}$. Then $qIJ \subseteq R$, so $qI \subseteq J^{-1}$. Hence $qII^{-1} \subseteq J^{-1}I^{-1}$. As $II^{-1} = R$, $qR \subseteq J^{-1}I^{-1}$ and thus $q \in J^{-1}I^{-1}$. Conversely, let $k \in J^{-1}I^{-1}$. Then $k = x_1y_1 + \cdots + x_\ell y_\ell$, where $x_1, \ldots, x_\ell \in J^{-1}$ and $y_1, \ldots, y_\ell \in I^{-1}$. Thus $kIJ = (x_1y_1 + \cdots + x_\ell y_\ell)IJ \subseteq x_1Jy_1I + \cdots + x_\ell J y_\ell I \subseteq R$, so $k \in (IJ)^{-1}$. Therefore $(IJ)^{-1} = J^{-1}I^{-1}$. Inductively, $(I_1 I_2 \cdots I_n)^{-1} = I_n^{-1} \cdots I_2^{-1} I_1^{-1}$.

(ii) By (i), this is evident. $\square$

Lemma 2.9. Assume that R is a Dedekind domain and I is a nonzero ideal of R. Let $A = \sum_{\ell \geq 0} I^{-\ell}$. Then we have the following.
(i) $A = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{i=1}^{n} r_i q_i$ for some $r_i \in I$ and $q_i \in I^{-1}$ with $1 \leq i \leq n$.
(ii) A is a Dedekind domain.

Proof. (i) Let F be the field of fractions of R. Say I is a nonzero ideal of R. Then I is invertible, and thus $II^{-1} = R$. Hence there exist $r_1, r_2, \ldots, r_n \in I$ and $q_1, q_2, \ldots, q_n \in I^{-1}$ such that $1 = \sum_{i=1}^{n} r_i q_i$. Further, $\sum_{i=1}^{n} q_i R \subseteq I^{-1}$. Next, take $q \in I^{-1}$. Then $qI \subseteq R$.

Because $1 = \sum_{i=1}^n r_i q_i$ and $qr_i \in qI \subseteq R$ for each i, we obtain that

$$q = q(\sum_{i=1}^n r_i q_i) = \sum_{i=1}^n q_i(qr_i) \in \sum_{i=1}^n q_i R.$$

Hence $I^{-1} \subseteq \sum_{i=1}^n q_i R$. Consequently, $I^{-1} = \sum_{i=1}^n q_i R$. Now we observe that

$$I^{-2} = I^{-1}I^{-1} = (\sum_{i=1}^n q_i R)(\sum_{j=1}^n q_j R) = \sum_{i,j=1}^n q_i q_j R,$$

and so on. Therefore, $A = \sum_{\ell \geq 0} I^{-\ell} = R[q_1, q_2, \ldots, q_n]$ since $I^0 = R$.

(ii) Note that $A = R[q_1, q_2, \ldots, q_n]$ is a noetherian ring since R is noetherian. As R is Dedekind (hence R is Prüfer) and A is an intermediate ring between R and F, the domain A is Prüfer. Therefore A is a Dedekind domain because A is noetherian. $\square$

For a ring R and a nonempty set Λ, we use $\mathrm{CFM}_\Lambda(R)$ to denote the $\Lambda \times \Lambda$ column finite matrix ring over the ring R.

Lemma 2.10. Let R be a commutative domain with the field of fractions F. Assume that A is an intermediate domain between R and F. Then $A_R^{(\Lambda)}$ is quasi-retractable for any nonempty set Λ.

Proof. Assume that Λ is a nonempty set. We put $M = A_R^{(\Lambda)}$. Then it follows that $S := \mathrm{End}_R(M) = \mathrm{End}_R(A^{(\Lambda)})$. We show that $S = \mathrm{End}_A(A^{(\Lambda)}) = \mathrm{CFM}_\Lambda(A)$. For this, first note that $\mathrm{End}_A(A^{(\Lambda)}) \subseteq S$. Next, we let $f \in S$. Assume on the contrary that $f \notin \mathrm{End}_A(A^{(\Lambda)})$. Then there exist $y \in A^{(\Lambda)}$ and $q \in A$ such that $f(yq) - f(y)q \neq 0$.

Put $q = ac^{-1}$, where $a, c \in R$ and $c \neq 0$. Since $A^{(\Lambda)}$ is a torsion-free R-module, so

$$0 \neq (f(yq) - f(y)q)c = f(yq)c - f(y)a = f(yqc) - f(ya) = f(ya) - f(ya) = 0,$$

which is a contradiction. Therefore $f \in \mathrm{End}_A(A^{(\Lambda)})$. Hence $S = \mathrm{End}_A(A^{(\Lambda)}) = \mathrm{CFM}_\Lambda(A)$.

If $|\Lambda| = 1$, clearly A_R is quasi-retractable because $R_R \leq A_R \leq F_R$ and A is an intermediate domain between R and F.

Next consider when $|\Lambda| = 3$. Our method for the case when $|\Lambda| = 3$ can be applied to general case. We show that $M = A_R^{(3)}$ is quasi-retractable. For this, let I be a left ideal of $S = \mathrm{Mat}_3(A)$. Say

$$0 \neq m = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in r_M(I).$$

Let e_{ij} be the matrix in $S = \mathrm{Mat}_3(A)$ with 1 in the (i, j)-position and 0 elsewhere. Put

$$0 \neq s = q_1 e_{11} + q_2 e_{21} + q_3 e_{31} \in \mathrm{Mat}_3(A) = S.$$

Now take $[f_{ij}] \in I$. Then $[f_{ij}]m = 0$. Hence $f_{i1}q_1 + f_{i2}q_2 + f_{i3}q_3 = 0$, for $1 \leq i \leq 3$. So $[f_{ij}]s = 0$ for all $[f_{ij}] \in I$, and thus $0 \neq s \in r_S(I)$. As a consequence, $M = A_R^{(3)}$ is quasi-retractable. $\square$

Definition 2.11. Let M_R be a module. We fix an injective hull $E(M_R)$ of M_R. Let $\mathfrak{M}$ be a class of modules. We call, when it exists, a module H_R the $\mathfrak{M}$ *hull* of M_R if H_R is the smallest extension of M_R in $E(M_R)$ that belongs to $\mathfrak{M}$. In particular, we denote the Baer hull of a module M by $\mathfrak{B}(M)$ when it exists (see also [4, Definition 8.4.1, p.310]).

Lemma 2.12. ([30, Theorem 6.11, p.171]) Let R be a Dedekind domain and M an R-module with nonzero annihilator in R. Then there exists a unique family $\{P_i,\, n_i\}_{i\in\Gamma}$ such that:

(i) The P_i are maximal ideals of R and these are only finitely many distinct ones.
(ii) $\{n_i \mid i \in \Gamma\}$ is a bounded family of positive integers.
(iii) $M \cong \oplus_{i\in\Gamma}(R/P_i^{n_i})$ as R-modules.

Let R be a Dedekind domain and N an R-module. Say $t(N)$ is the torsion submodule of N. Suppose that $N/t(N)$ is finitely generated as an R-module. Since $N/t(N)$ is torsion-free, $N/t(N) \cong (\oplus_{j=1}^{m} K_j)$ (as R-modules) for some fractional ideals $K_j, 1 \le j \le m$, of R from [30, Theorem 6.16, p.177] (see also Lemma 2.16). So $N/t(N)$ is projective, and hence we have that $N \cong t(N) \oplus N/t(N) \cong t(N) \oplus (\oplus_{j=1}^{m} K_j)$ as R-modules.

Our next result provides a complete characterization for the existence of the Baer hull of a module N when $N/t(N)$ is finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$ (see Theorem 2.16). Furthermore, we describe the Baer hull of N explicitly in this case.

Theorem 2.13. Let R be a Dedekind domain. Assume that M is an R-module with nonzero annihilator in R, and $\{K_1, K_2, \ldots, K_m\}$ is a finite set of nonzero fractional ideals of R. Then the following are equivalent.

(i) $M_R \oplus (\oplus_{i=1}^{m} K_i)_R$ has a Baer hull.
(ii) M_R is semisimple.
(iii) $M_R \oplus (\oplus_{i=1}^{m} K_i)_R$ has a Baer essential extension.

In this case, $\mathfrak{B}(M_R \oplus (\oplus_{i=1}^{m} K_i)_R) = M_R \oplus (\oplus_{i=1}^{m} K_i A)_R$, where $A = \sum_{\ell\ge 0} I^{-\ell}$ with $I = \mathrm{Ann}_R(M)$. Furthermore, $A = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{i=1}^{n} r_i q_i$ with $r_i \in I$ and $q_i \in I^{-1}$, $1 \le i \le n$.

Proof. Since $\mathrm{Ann}_R(M) \neq 0$, there is a unique family $\{P_i,\, n_i\}_{i\in\Gamma}$ satisfying (i), (ii), and (iii) of Lemma 2.12. So $M \cong \oplus_{i\in\Gamma}(R/P_i^{n_i})$ as R-modules.

(i)$\Rightarrow$(ii) Assume that $M_R \oplus (\oplus_{j=1}^{m} K_j)_R$ has a Baer hull. Put $T = \oplus_{i\in\Gamma}(R/P_i^{n_i})$. Then $M_R \cong T_R$. So $T_R \oplus (\oplus_{j=1}^{m} K_j)_R$ has a Baer hull, say V_R. To show that M is semisimple, we need to prove that T is semisimple.

First, if $T = 0$, then we are done. So assume that $T \neq 0$. We put $I = \mathrm{Ann}_R(M)$. Then $I \neq 0$ by assumption, and $I = \mathrm{Ann}_R(T)$. From the proof of Theorem 2.6, $V = \ell_V(I) \oplus W$ for some $W \le V$ as V is Baer and I is finitely generated. Since V is Baer, so is $\ell_V(I)$ by Lemma 2.4(ii).

Say $\{P_1, P_2, \ldots, P_s\}$ is the set of all distinct maximal ideals in $\{P_i \mid i \in \Gamma\}$. We put $J = P_1 P_2 \cdots P_s$. Since $I \subseteq J$, $\ell_V(J) \subseteq \ell_V(I)$. Put $J = a_1R + a_2R + \cdots + a_nR$. For each i, $1 \le i \le n$, define $f_i : \ell_V(I) \to \ell_V(I)$ by $f_i(v) = va_i$, where $v \in \ell_V(I)$. Then we have that $\ell_V(J) = \cap_{i=1}^{n} \mathrm{Ker}(f_i) \le^{\oplus} \ell_V(I)$ by Lemma 2.4(i).

By the proof of Theorem 2.6, $\ell_V(I) \subseteq E(T_R)$, and $T = \ell_V(I)$. Assume on the contrary that T is not semisimple. Then there exists $n_i\,(i \in \Gamma)$ with $n_i \ge 2$. Let $\Gamma_1 = \{i \in \Gamma \mid n_i \ge 2\}$

and $\Gamma_2 = \{i \in \Gamma \mid n_i = 1\}$. Then $\Gamma_1 \neq \emptyset$.

Now we put
$$U = (\oplus_{i\in\Gamma_1} P_i^{n_i-1}/P_i^{n_i}) \oplus (\oplus_{i\in\Gamma_2} R/P_i^{n_i}) \subseteq \ell_V(J).$$

Then $U \leq^{\text{ess}} T$. For this, say $n_1 \geq 2$. To show that $(P_1^{n_1-1}/P_1^{n_1})_R \leq^{\text{ess}} (R/P^{n_1})_R$, we take $0 \neq (B/P_1^{n_1})_R \leq (R/P^{n_1})_R$. Then $(B/P_1^{n_1})_{R/P_1^{n_1}} \leq (R/P_1^{n_1})_{R/P_1^{n_1}}$. So $B/P_1^{n_1}$ is a nonzero ideal of $R/P_1^{n_1}$, hence B is a nonzero ideal of R containing $P_1^{n_1}$ properly.

Note that as R is a Dedekind domain, every nonzero prime ideal of R is maximal. Also since R is a Dedekind domain, B is a unique finite product of prime ideals. Now we put $B = Q_1 \cdots Q_m$ with each Q_i a nonzero prime ideal. As $P_1^{n_1} \subseteq B = Q_1 \cdots Q_m$, $P_1^{n_1} \subseteq Q_i$ for any i, $1 \leq i \leq m$. So $P_1 \subseteq Q_i$ and hence $P_1 = Q_i$ for any i, $1 \leq i \leq m$. Therefore $B = Q_1 \cdots Q_m = P_1^m$ with $m \leq n_1 - 1$ as B properly contains $P_1^{n_1}$. Consequently, $(P_1^{n_1-1}/P_1^{n_1})_R \leq^{\text{ess}} (R/P_1^{n_1})_R$. Similarly, $(P_i^{n_i-1}/P_i^{n_i})_R \leq^{\text{ess}} (R/P_i^{n_i})_R$ for each $i \in \Gamma_1$. Therefore
$$(\oplus_{i\in\Gamma_1} P_i^{n_i-1}/P_i^{n_i})_R \leq^{\text{ess}} (\oplus_{i\in\Gamma_1} R/P_i^{n_i})_R.$$

So, we see that $U \leq^{\text{ess}} T$.

Hence, by the preceding argument, $U \leq^{\text{ess}} T = \ell_V(I)$ and thus $U \leq^{\text{ess}} \ell_V(I)$. Because $U \leq \ell_V(J) \leq \ell_V(I)$, we have $\ell_V(J) \leq^{\text{ess}} \ell_V(I)$. Hence $\ell_V(J) = \ell_V(I)$ as $\ell_V(J) \leq^{\oplus} \ell_V(I)$.

Let $k \in \Gamma_1$. Then $n_k \geq 2$. Note that $R/P_k^{n_k} \subseteq \ell_V(I) = \ell_V(J)$. So $P_1 \cdots P_k \cdots P_s \subseteq P_k^{n_k}$, thus $P_k^{-1} P_1 \cdots P_k \cdots P_s \subseteq P_k^{-1} P_k^{n_k}$. Hence $P_1 \cdots P_{k-1} P_{k+1} \cdots P_s \subseteq P_k^{n_k-1}$. Since $n_k \geq 2$, $P_i = P_k$ for some $P_i \in \{P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_s\}$, which is a contradiction. Therefore T is semisimple, hence M is semisimple.

(ii)$\Rightarrow$(i) *Case 1.* $M \neq 0$ and $m \geq 1$. Since M is semisimple and $\text{Ann}_R(M) \neq 0$, M has only a finite number of homogeneous components by Lemma 2.12, say $\{H_k \mid 1 \leq k \leq s\}$. For each k, $1 \leq k \leq s$, there exists a nonempty set Γ_k such that $H_k = \oplus_{\alpha\in\Gamma_k} M_{(k,\alpha)}$ with each $M_{(k,\alpha)}$ simple. Thus $P_k := \text{Ann}_R(H_k) = \text{Ann}_R(M_{(k,\alpha)})$ for all $\alpha \in \Gamma_k$. So $P_k, 1 \leq k \leq s$, are distinct maximal ideals of R, and $I := \text{Ann}_R(M) = \cap_{i=1}^s P_i$.

Assume that V is a Baer essential extension of $M_R \oplus (\oplus_{i=1}^m K_i)_R$. From Theorem 2.6, $V = M \oplus W$ with $W \leq V$, $(\oplus_{i=1}^m K_i)_R \leq W_R \leq E(R_R^{(m)})$, and W is Baer. Also, by Theorem 2.6, $\text{Hom}_R(W, M) = 0$, so $\text{Hom}_R(W, M_{(k,\alpha)}) = 0$ for k, $1 \leq k \leq s$ and each $\alpha \in \Gamma_k$.

Since $M_{(1,\alpha)} \cong R/P_1$ as R-modules for each $\alpha \in \Gamma_1$, we have $\text{Hom}_R(W, R/P_1) = 0$. From the exact sequence $0 \to WP_1 \to W \to W/WP_1 \to 0$, we obtain the following exact sequence

$$(*) \qquad 0 \to \text{Hom}_R(W/WP_1, R/P_1) \to \text{Hom}_R(W, R/P_1) \to \text{Hom}_R(WP_1, R/P_1)$$

as $\text{Hom}_R(-, R/P_1)$ is a left exact contravariant functor (see [27, Theorem 2.9, p.35]). Because $\text{Hom}_R(W, R/P_1) = 0$, we have that $\text{Hom}_R(W/WP_1, R/P_1) = 0$ from the preceding exact sequence $(*)$. We notice that W/WP_1 is an R/P_1-module, which is induced from the R-module structure of W/WP_1.

Further, $\text{Hom}_{R/P_1}(W/WP_1, R/P_1) = \text{Hom}_R(W/WP_1, R/P_1) = 0$. Since R/P_1 is a field, W/WP_1 is a vector space over R/P_1. Therefore $W/WP_1 = 0$, and thus $W = WP_1$. Similarly, $W = WP_k$ for k, $2 \leq k \leq s$. Because each $P_i, 1 \leq i \leq s$, is maximal and R is commutative, $I = \cap_{i=1}^s P_i = P_1 P_2 \cdots P_s$ (also, see [30, Lemma 6.12, p.173]). So $WI = WP_1 P_2 \cdots P_s = W$ and hence $WI^\ell = W$ for each nonnegative integer ℓ.

Consider the case when $m = 2$. Then $(K_1 \oplus K_2)_R \leq W_R \leq E(R_R) \oplus E(R_R)$. We let $J = I^{\ell}$, where ℓ is a nonnegative integer. Then $W = WJ$. We now take $k \in K_1$. Then we have that $(k, 0) \in K_1 \oplus K_2 \subseteq W = WJ$. Hence there exists a positive integer n such that $(k, 0) = \sum_{i=1}^{n} w_i a_i$, where $w_i \in W$ and $a_i \in J$ for i, $1 \leq i \leq n$.

Put $w_i = (x_i, y_i) \in E(R_R)^{(2)}$ for i, $1 \leq i \leq n$. Then $(k, 0) = (\sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} y_i a_i)$. So we have $k = \sum_{i=1}^{n} x_i a_i$ and $0 = \sum_{i=1}^{n} y_i a_i$. Take $q \in J^{-1}$. Then $kq = \sum_{i=1}^{n} x_i q a_i$ and $0 = \sum_{i=1}^{n} y_i q a_i$. Since each $a_i \in J$ and $qJ \subseteq R$, each $qa_i \in R$. So

$$(kq, 0) = (\sum_{i=1}^{n} x_i q a_i, \sum_{i=1}^{n} y_i q a_i) = \sum_{i=1}^{n} (x_i, y_i) q a_i = \sum_{i=1}^{n} w_i q a_i.$$

Hence $(kq, 0) \in \sum_{i=1}^{n} w_i R \subseteq W$. Therefore $(K_1 J^{-1}, 0) \subseteq W$. By Lemma 2.8, $J^{-1} = I^{-\ell}$. Thus $(K_1 I^{-\ell}, 0) \subseteq W$ for all nonnegative integers ℓ. Thus $(K_1 A, 0) \subseteq W$.

Similarly, $(0, K_2 A) \subseteq W$. So $K_1 A \oplus K_2 A \subseteq W$. In general, as $(\oplus_{i=1}^{m} K_i)_R \leq W_R$, we obtain $\oplus_{i=1}^{m} K_i A \subseteq W$ by applying the preceding method.

Since K_1 is a fractional ideal, there exists $0 \neq r_1 \in R$ such that $r_1 K_1 \subseteq R$. Put $I_1 = r_1 K_1$. Then $I_1 \cong K_1$ as R-modules. As R is a Dedekind domain, I_1 is a finitely generated projective R-module. So I_{1R} is isomorphic to a direct summand of $R_R^{(h_1)}$ for some positive integer h_1. Hence $I_1 \otimes_R A_R$ is isomorphic to a direct summand of $R^{(h_1)} \otimes_R A_R \cong (R \otimes_R A)_R^{(h_1)} \cong A_R^{(h_1)}$.

We note that ${}_R A$ is torsion-free. So ${}_R A$ is flat by [27, Theorem 4.33, p.129] since R is a Dedekind domain. Thus $I_1 \otimes_R A$ is embedded in $R \otimes_R A$. By definition of tensor product, the map $f : I_1 \otimes_R A_R \to I_1 A_R$ defined by $f(\sum_{i=1}^{t} x_i \otimes a_i) = \sum_{i=1}^{t} x_i a_i$, for $x_i \in I_1$ and $a_i \in A$, $1 \leq i \leq t$, is well-defined. Further, to show that f is an R-module isomorphism, suppose that $\sum_{i=1}^{t} x_i a_i = 0$ with $x_i \in I_1$ and $a_i \in A$, $1 \leq i \leq t$. Note that $\sum_{i=1}^{t} x_i \otimes a_i$ in $I_1 \otimes_R A$ can be considered as an element in $R \otimes_R A$ because $I_1 \otimes_R A$ is embedded in $R \otimes_R A$. Hence

$$\sum_{i=1}^{t} x_i \otimes a_i = \sum_{i=1}^{t} 1 \otimes x_i a_i = 1 \otimes (\sum_{i=1}^{t} x_i a_i) = 1 \otimes 0 = 0.$$

Thus f is an R-module isomorphism.

Because $I_1 \otimes_R A_R$ is isomorphic to a direct summand of $A_R^{(h_1)}$, $I_1 A_R$ is also isomorphic to a direct summand of $A_R^{(h_1)}$. Similarly, for each i, we see that $I_i A_R$ is isomorphic to a direct summand of $A_R^{(h_i)}$ for some positive integer h_i. We put $h = h_1 + h_2 + \cdots + h_m$. Then $(\oplus_{i=1}^{m} I_i A)_R$ is isomorphic to a direct summand of $A_R^{(h)}$. Furthermore, $(K_i A)_R \cong (I_i A)_R$ for each i. Thus $(\oplus_{i=1}^{m} K_i A)_R$ is isomorphic to a direct summand of $A_R^{(h)}$.

As A is a Dedekind domain by Lemma 2.9, $\mathrm{End}_R(A_R^{(h)}) = \mathrm{Mat}_h(\mathrm{End}_R(A)) = \mathrm{Mat}_h(A)$ is a Baer ring from [16, Corollary 3.7] or [4, Theorem 6.1.4, p.191]. By Lemma 2.10, $A_R^{(h)}$ is a quasi-retractable module. Hence Lemma 2.3 yields that $A_R^{(h)}$ is a Baer module. Consequently, $(\oplus_{i=1}^{m} K_i A)_R$ is a Baer module by Lemma 2.4(ii).

Now we show that $\mathrm{Hom}_R(\oplus_{i=1}^{m} K_i A, M) = 0$. Let $f \in \mathrm{Hom}_R(K_i A, M)$. Since R is a Dedekind domain, $R = I^{-1} I$. Because $MI = 0$, we obtain

$$f(K_i I^{-\ell}) = f(K_i I^{-\ell} I^{-1} I) = f(K_i I^{-\ell} I^{-1}) I = 0.$$

So we obtain $f(K_iA) = 0$. Therefore $f = 0$, and hence $\mathrm{Hom}_R(K_iA, M) = 0$ for each i. We note that $(\mathrm{Hom}_R(\oplus_{i\in\Lambda}K_iA, M), +) \cong (\prod_{i\in\Lambda}\mathrm{Hom}_R(K_iA, M), +)$ from [27, Theorem 2.4, p.30], thus $\mathrm{Hom}_R(\oplus_{i\in\Lambda}K_iA, M) = 0$. Therefore $M_R \oplus (\oplus_{i=1}^m K_iA)_R$ is a Baer module from Theorem 2.6. Furthermore, $M_R \oplus (\oplus_{i=1}^m K_iA)_R \le M_R \oplus W_R = V_R$. Hence $M_R \oplus (\oplus_{i=1}^m K_iA)_R$ is the Baer hull of $M_R \oplus (\oplus_{i=1}^m K_i)_R$.

Case 2. $M \neq 0$ and $m = 0$. Since M is semisimple, M is Baer and so M itself is the Baer hull of M. In this case, $M \oplus (\oplus_{i=1}^m K_iA) = M$.

Case 3. $M = 0$ and $m \ge 1$. By the preceding argument, each K_i is finitely generated projective as an R-module, hence $\oplus_{i=1}^m K_i$ is a finitely generated projective R-module. So $\oplus_{i=1}^m K_i$ is isomorphic to a direct summand of $R_R^{(h)}$ for some positive integer h. Now $\mathrm{Mat}_h(R) = \mathrm{End}_R(R^{(h)})$ is a Baer ring since R is Dedekind (see [16, Corollary 3.7] or [4, Theorem 6.1.4, p.191]). Thus $R_R^{(h)}$ is a Baer module from Lemma 2.3 because $R_R^{(h)}$ is quasi-retractable (from Lemma 2.10).

As a consequence, $\oplus_{i=1}^m K_i$ is Baer by Lemma 2.4(ii) since it is isomorphic to a direct summand of $R_R^{(h)}$. So $\oplus_{i=1}^m K_i$ itself is the Baer hull of $\oplus_{i=1}^m K_i$. As $\mathrm{Ann}_R(M) = R$, we see that $A = \sum_{\ell\ge 0} R^{-\ell} = R$, and hence $\oplus_{i=1}^m (K_iA) = \oplus_{i=1}^m K_i$.

(i)⇒(iii) is clear. For (iii)⇒(ii), say V is a Baer essential extension of $M_R \oplus (\oplus_{i=1}^m K_i)_R$. Put $I = \mathrm{Ann}_R(M) \neq 0$. Assume that $M \neq 0$. From the proof of (i)⇒(ii) in Theorem 2.6, $\ell_V(I) = M$. So we can verify that M is semisimple as in the proof of (i)⇒(ii). If $M = 0$, then we are done.

Finally, from the proof of Lemma 2.9, $A = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{i=1}^n r_iq_i$ with $r_i \in I$ and $q_i \in I^{-1}$, $1 \le i \le n$. □

Remark 2.14. From the proof of Theorem 2.13, we notice the following does hold true: Let R be a PI-ring and W a right R-module. Assume that M is a simple right R-module. Then $\mathrm{Hom}_R(W, M) = 0$ if and only if $WP = W$, where $P = \mathrm{Ann}_R(M)$.

In fact, since P is a right primitive ideal of R, R/P is a primitive PI-ring. Therefore, due to Kaplansky's result, the ring R/P is simple artinian [28, Theorem 1.5.16, p.36]. If $\mathrm{Hom}_R(W, M) = 0$, then $\mathrm{Hom}_R(W/WP, M) = 0$ and hence $\mathrm{Hom}_{R/P}(W/WP, M) = 0$ as in the proof of Theorem 2.13. Since M is simple and the ring R/P is simple artinian, $W/WP = 0$ and hence $W = WP$. Conversely, if $W = WP$, then $f(W) = f(WP) = f(W)P = 0$, for all $f \in \mathrm{Hom}_R(W, M)$, because $MP = 0$. Hence $f = 0$, so $\mathrm{Hom}_R(W, M) = 0$.

The following is a restatement of Theorem 2.13 for the Baer hull of a module N over a Dedekind domain for the case when $N/t(N)$ is finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$.

Theorem 2.15. Let R be a Dedekind domain. Assume that N is an R-module with $N/t(N)$ finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$. Then the following are equivalent.

(i) N has a Baer hull.
(ii) $t(N)$ is semisimple.
(iii) N has a Baer essential extension.

The next lemma details the structure of finitely generated modules over a Dedekind domain.

Lemma 2.16. ([30, Theorem 6.16, p.177]) Let R be a Dedekind domain and N a finitely generated R-module. Then there exist positive integers $n_1, n_2, \ldots, n_k$ (k is a nonnegative integer), nonzero maximal ideals $P_1, P_2, \ldots, P_k$, and nonzero fractional ideals $K_1, K_2, \ldots, K_m$

(m is a nonnegative integer) of R such that $N \cong (\oplus_{i=1}^{k} R/P_i^{n_i}) \oplus (\oplus_{j=1}^{m} K_j)$ as R-modules.

Our next corollary extends [24, Proposition 2.19 and Remark 2.20] to the case of Dedekind domains.

Corollary 2.17. Let R be a Dedekind domain and N be a finitely generated R-module. Then the following are equivalent.
(i) N is Baer.
(ii) N is semisimple or N is torsion-free.

Proof. (i)⇒(ii) Let N be Baer. From Lemma 2.16, $\mathrm{Ann}_R(t(N)) \neq 0$. Since N itself is the Baer hull of N, $t(N)$ is semisimple by Theorem 2.15. Therefore Lemma 2.16 yields that $N \cong (\oplus_{i=1}^{k} R/P_i) \oplus (\oplus_{j=1}^{m} K_j)$, where k and m are nonnegative integers, $P_1, P_2, \ldots, P_k$ are maximal ideals (may not be distinct), and $K_1, K_2, \ldots, K_m$ are nonzero fractional ideals.

Suppose that $k \neq 0$ and $m \neq 0$. Since N is Baer, $\mathrm{Hom}_R(\oplus_{j=1}^{m} K_j, \oplus_{i=1}^{k} R/P_i) = 0$ from Theorem 2.6. Hence $\mathrm{Hom}_R(K_1, R/P_1) = 0$. As in the proof of Theorem 2.13, $K_1P_1 = K_1$ and so $K_1^{-1}K_1P_1 = K_1^{-1}K_1 = R$. Thus $P_1 = R$, a contradiction. Therefore, either $k = 0$ or $m = 0$. So N is torsion-free or N is semisimple.

(ii)⇒(i) Assume that N is semisimple or N is torsion-free. If N is semisimple, then obviously N is Baer. So we suppose N is torsion-free. Then N is R-module isomorphic to a finite direct sum of nonzero fractional ideals by Lemma 2.16. As in the proof of Case 3 in (ii)⇒(i) of Theorem 2.13, we can show that N is Baer. □

Assume that N is a finitely generated module over a Dedekind domain. If N is neither semisimple nor torsion-free, then N is not Baer by Corollary 2.17. In the following theorem, we characterize the existence of the Baer hull of N and a Baer essential extension of N and describe the Baer hull of N explicitly. Recall that from Lemma 2.16, $N \cong \oplus_{i=1}^{k}(R/P_i^{n_i}) \oplus (\oplus_{j=1}^{m} K_j)$, where P_i are nonzero maximal ideals of R and K_j are nonzero fractional ideals of R (where k and m are nonnegative integers).

Theorem 2.18. Let R be a Dedekind domain, and let N be a finitely generated R-module. Then the following are equivalent.
(i) N has a Baer hull.
(ii) $t(N)$ is semisimple.
(iii) N has a Baer essential extension.

In this case, $\mathfrak{B}(N_R) \cong (\oplus_{i=1}^{k}(R/P_i^{n_i}))_R \oplus (\oplus_{i=1}^{m} K_i A)_R$, where $A = \sum_{\ell \geq 0} I^{-\ell}$ with $I = \mathrm{Ann}_R(M)$. Furthermore, $A = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{i=1}^{n} r_i q_i$ with $r_i \in I$ and $q_i \in I^{-1}$, $1 \leq i \leq n$.

Proof. Note that $t(N) \cong \oplus_{i=1}^{k}(R/P_i^{n_i})$ as R-modules. Hence $\mathrm{Ann}_R(t(N)) \neq 0$, so Theorem 2.15 yields the proof. The explicit description of the Baer hull of N follows from Theorem 2.13. □

In the next example, we notice that conclusion of Theorem 2.13 and Theorem 2.15 do not hold when R is noetherian domain, in general.

Example 2.19. (see also [4, Example 8.4.13, p.319]) Let $R = \mathbb{Z}[x]$, the polynomial ring over $\mathbb{Z}$. Put $N = (R \oplus R)_R$. Then $t(N) = 0$, so $t(N)$ is semisimple. However, N has no Baer hull. For this, note that if N is a Baer module, then $\mathrm{End}_R(N) = \mathrm{Mat}_2(R)$ is a Baer ring from Lemma 2.3. So [16, Corollary 3.7] (or [4, Theorem 6.1.4, p.191]) yields that the ring $R = \mathbb{Z}[x]$ must be Prüfer, which is a contradiction.

Let $F = \mathbb{Q}(x)$, the field of fractions of R. Note that $E(N) = F \oplus F$. Put $U = F \oplus R$. Then by [16, Theorem 2.16] (or by [4, Theorem 4.2.18, p.107]), U_R is a Baer module. Similarly, $V_R := (R \oplus F)_R$ is a Baer module. As $U \cap V = N$ is not Baer, N has no Baer hull. This example exhibits a module N which is the quasi-Baer hull (of itself) but has no Baer hull.

M. Schmidmeier raised the following question recently: *Does $N_1 \oplus N_2$ have a Baer hull when modules N_1 and N_2 have Baer hulls?* We remark that Example 2.19 gives a negative answer to this question (see also Example 3.10).

3. Applications and Examples

In this final section, our focus is on some applications of our results. Properties of Baer hulls are obtained and examples which illustrate our results are provided. In view of Example 3.6, infinitely generated modules over a Dedekind domain may not have Baer hulls. The existence and description of a Baer module hull of a given finitely generated module M over arbitrary commutative rings or domains, remains open.

We start with the following remark.

Remark 3.1. (i) In Theorem 2.13, we put $A = \sum_{\ell \geq 0} I^{-\ell}$, where $I = \mathrm{Ann}_R(M)$. Now we can verify that $A = \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers. In fact, $I \subseteq P_i$ for all i since $I = P_1 P_2 \cdots P_s$. For i, $1 \leq i \leq s$, $P_i^{-1} \subseteq I^{-1}$ and so $P_i^{-\ell} \subseteq I^{-\ell}$ for every nonnegative integer ℓ. Hence, from Lemma 2.8,

$$P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} \subseteq I^{-\ell_1} I^{-\ell_2} \cdots I^{-\ell_s} = I^{-(\ell_1+\ell_2+\cdots+\ell_s)} \subseteq A.$$

Thus $\sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} \subseteq A$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers.

Conversely, from Lemma 2.8, $I^{-1} = (P_1 P_2 \cdots P_s)^{-1} = P_1^{-1} P_2^{-1} \cdots P_s^{-1}$. Therefore it follows that $I^{-\ell} = P_1^{-\ell} P_2^{-\ell} \cdots P_s^{-\ell}$ for any nonnegative integer ℓ. Hence we obtain that $A \subseteq \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers.

Consequently, $A = \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers.

(ii) Let R be a commutative PID. Assume that M is a nonzero semisimple R-module with nonzero annihilator in R. Then from Lemma 2.12, M has only a finite number of homogeneous components. Let $\{H_k \mid 1 \leq k \leq s\}$ be the set of all homogeneous components of M. For k, $1 \leq k \leq s$, we put $H_k = \oplus_\alpha M_{(k,\alpha)}$ with each $M_{(k,\alpha)}$ simple. Therefore $M_{(k,\alpha)} \cong R/p_k R$ for k, $1 \leq k \leq s$, with p_k a nonzero prime.

We put $P_k = \mathrm{Ann}_R(H_k)$ for k, $1 \leq k \leq s$. Then $P_k = p_k R$. For a nonnegative integer ℓ, we can routinely verify that $P_k^{-\ell} = (1/p_k^\ell)R$ for k, $1 \leq k \leq s$. Therefore,

$$P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} = (1/p_1^{\ell_1})(1/p_2^{\ell_2}) \cdots (1/p_s^{\ell_s})R$$

for nonnegative integers $\ell_1, \ell_2, \ldots, \ell_s$.

Let $A = \sum_{\ell \geq 0} I^{-\ell}$, where $I = \mathrm{Ann}_R(M) = P_1 P_2 \cdots P_s = p_1 p_2 \cdots p_s R$. Then from the preceding argument, $A = R[1/p_1, 1/p_2, \ldots, 1/p_s]$. Put $a = p_1 p_2 \cdots p_s$. Thus $A = R[1/a]$ because $I^{-\ell} = (1/a^\ell)R$.

The following example illustrates Theorem 2.13 and Remark 3.1.

Example 3.2. Say Γ_i, $i = 1, 2, 3$, are nonempty sets and m is a positive integer. By Theorem 2.13 and Remark 3.1, the Baer hull of $\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}^{(m)}$ is

$$\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}[1/2,\ 1/3,\ 1/5]^{(m)} = \mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}[1/30]^{(m)}.$$

We recall that a ring R is called *semiprimary* if $R/J(R)$ is artinian and $J(R)$ is nilpotent, where $J(R)$ is the Jacobson radical of R. It is well-known that if R is a semiprimary ring, then R is right hereditary if and only if R is left hereditary.

Lemma 3.3. ([32, Theorem 2] and [26, Theorem 3.3]) Let R be a ring. Then the following are equivalent.

(i) R is a semiprimary right (and left) hereditary ring.

(ii) $\mathrm{CFM}_\Lambda(R)$ is a Baer ring for any nonempty set Λ.

Definition 3.4. (i) ([24, Definition 2.5]) A module M_R is said to be $\mathcal{K}$-*nonsingular* if, for all $\varphi \in \mathrm{End}_R(M)$, $\ \mathrm{Ker}(\varphi)_R \leq^{\mathrm{ess}} M_R$ implies $\varphi = 0$.

(ii) ([24, Definition 2.7]) A module M_R is called $\mathcal{K}$-*cononsingular* if, for any $N_R \leq M_R$, $\ell_S(N) = 0$ implies $N_R \leq^{\mathrm{ess}} M_R$, where $S = \mathrm{End}_R(M)$.

In [24], it is proved that every nonsingular module is $\mathcal{K}$-nonsingular, but converse is not true in general. For more details on $\mathcal{K}$-nonsingular modules, see [24] and [25]. By Chatters and Khuri in [5], a ring R is right extending and right nonsingular if and only if R is Baer and right cononsingular.

The following shows that there are close connections between an extending module and a Baer module.

Lemma 3.5. ([24, Theorem 2.12]) A module M_R is extending and $\mathcal{K}$-nonsingular if and only if M_R is Baer and $\mathcal{K}$-cononsingular.

Let R be a Dedekind domain and let N be an R-module. Assume that $N/t(N)$ is finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$. In this case by Theorem 2.15, N has Baer hull if and only if $t(N)$ is semisimple. The following example exhibits that there exists an R-module N such that $t(N)$ is semisimple and $\mathrm{Ann}_R(t(N)) \neq 0$, but N has no Baer hull. So the assumption "$N/t(N)$ is finitely generated" in Theorem 2.15 is not superfluous.

Example 3.6. Let $M = \oplus_{i=1}^n \mathbb{Z}_{p_i}$, where n is a positive integer, and all p_i are prime integers. Say $p_1, p_2, \ldots, p_s$ are all the distinct prime integers in $\{p_1, p_2, \ldots, p_n\}$. Let $a = p_1 p_2 \cdots p_s$.

Since $\mathbb{Z}[1/a]$ is not a field, $\mathbb{Z}[1/a]$ is not semiprimary because $\mathbb{Z}[1/a]$ is a domain. By Lemma 3.3, there exists a nonempty set Λ such that $\mathrm{CFM}_\Lambda(\mathbb{Z}[1/a])$ is not a Baer ring.

Furthermore, we have the following.

(i) $M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$ is not a Baer $\mathbb{Z}$-module.

(ii) Let $N = M \oplus \mathbb{Z}^{(\Lambda)}$. Then $N/t(N)$ is not finitely generated.

(iii) $N = M \oplus \mathbb{Z}^{(\Lambda)}$ has *no* Baer hull as a $\mathbb{Z}$-module.

To prove (i), first we show that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/a]^{(\Lambda)}) = \mathrm{End}_{\mathbb{Z}[1/a]}(\mathbb{Z}[1/a]^{(\Lambda)})$ by using similar method that used in the proof of Lemma 2.10. Therefore

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/a]^{(\Lambda)}) = \mathrm{End}_{\mathbb{Z}[1/a]}(\mathbb{Z}[1/a]^{(\Lambda)}) = \mathrm{CFM}_\Lambda(\mathbb{Z}[1/a]).$$

Now $\mathbb{Z}[1/a]^{(\Lambda)}$ is not a Baer $\mathbb{Z}$-module from Lemma 2.3 as $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/a]^{(\Lambda)}) = \mathrm{CFM}_{\Lambda}(\mathbb{Z}[1/a])$ is not a Baer ring. By Lemma 2.4(ii), $M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$ is not a Baer $\mathbb{Z}$-module.

For (ii), we prove that Λ is an infinite set. Assume on the contrary that Λ is a finite set, say $|\Lambda| = m$, where m is a positive integer. Note that $\mathbb{Z}[1/a]$ is a Prüfer domain. So $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/a]^{(\Lambda)}) = \mathrm{Mat}_m(\mathbb{Z}[1/a])$ is a Baer ring by [16, Corollary 3.7] (or [4, Theorem 6.1.4, p.191]). Thus we get a contradiction. Therefore Λ is infinite, so $N/t(N) \cong \mathbb{Z}^{(\Lambda)}$ is not finitely generated as a $\mathbb{Z}$-module.

To prove (iii), for each $\alpha \in \Lambda$, let $W_\alpha = \oplus_{i\in\Lambda} U_i$, where $U_\alpha = \mathbb{Z}[1/a]$ and $U_i = \mathbb{Q}$ for $i \neq \alpha$. First we claim that W_α is a Baer $\mathbb{Z}$-module. For this, note that $\mathbb{Z}[1/a]$ is a nonsingular extending $\mathbb{Z}$-module and $\oplus_{i\in\Lambda\setminus\{\alpha\}} U_i = \mathbb{Q}^{(\Lambda\setminus\{\alpha\})}$ is a nonsingular injective $\mathbb{Z}$-module. Due to G. Lee, S.T. Rizvi, and C. Roman (see [4, Theorem 4.2.18, p.107]), W_α is a Baer $\mathbb{Z}$-module for each $\alpha \in \Lambda$.

Assume on the contrary that $M \oplus \mathbb{Z}^{(\Lambda)}$ has a Baer hull, say V. As $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/a], M) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}^{(\Lambda\setminus\{\alpha\})}, M) = 0$ for each $\alpha \in \Lambda$, it follows that $\mathrm{Hom}_{\mathbb{Z}}(W_\alpha, M) = 0$ from [27, Theorem 2.4, p.30]. So $M \oplus W_\alpha$ is a Baer module for each $\alpha \in \Lambda$ by Theorem 2.6. Thus

$$V \subseteq \cap_{\alpha\in\Lambda}(M \oplus W_\alpha) = M \oplus (\cap_{\alpha\in\Lambda} W_\alpha) = M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$$

because $\cap_{\alpha\in\Lambda} W_\alpha = \mathbb{Z}[1/a]^{(\Lambda)}$.

We note that V is a Baer module and $M \oplus \mathbb{Z}^{(\Lambda)} \leq V \leq E(M \oplus \mathbb{Z}^{(\Lambda)})$. Hence from Theorem 2.6, $V = M \oplus W$ such that $\mathbb{Z}^{(\Lambda)} \leq W \leq E(\mathbb{Z}^{(\Lambda)}) = \mathbb{Q}^{(\Lambda)}$, W is a Baer module, and $\mathrm{Hom}_{\mathbb{Z}}(W, M) = 0$. Now put $I = \mathrm{Ann}_{\mathbb{Z}}(M) = a\mathbb{Z}$. As in the proof of (ii)$\Rightarrow$(i) for Theorem 2.13, we have that $WI = W$. Put $A = \sum_{\ell\geq 0} I^{-\ell}$. Then by Remark 3.1, $A = \mathbb{Z}[1/a]$. By the method that used in the proof of (ii)$\Rightarrow$(i) for Theorem 2.13, $A^{(\Lambda)} = (\mathbb{Z}A)^{(\Lambda)} \subseteq W$. So $\mathbb{Z}[1/a]^{(\Lambda)} \subseteq W$. Hence $M \oplus \mathbb{Z}[1/a]^{(\Lambda)} \leq M \oplus W = V$.

Consequently, $V = M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$, and thus $M \oplus \mathbb{Z}[1/a]^{(\Lambda)}$ is a Baer module, which is a contradiction to (i). Hence $M \oplus \mathbb{Z}^{(\Lambda)}$ has no Baer hull.

Example 3.7. (i) Let $V = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$, where p is a prime integer. Then by Theorem 2.13 and Remark 3.1, V is the Baer hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. We note that $\mathbb{Z}_p \oplus \mathbb{Z}$ is not extending by [12, Corollary 2]. Hence in view of Lemma 3.5, one might expect that V is also the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. But this is not true. Furthermore, V is not even extending from [12, Corollary 2].

(ii) We remark that in the chain of $\mathbb{Z}$-submodules $\mathbb{Z}_p \leq \mathbb{Z}_{p^2} \leq \cdots \leq \mathbb{Z}_{p^\infty}$ of $\mathbb{Z}_{p^\infty}$ (p a prime integer), $\mathbb{Z}_p$ is the Baer hull (also the quasi-injective hull) of itself and $\mathbb{Z}_{p^\infty}$ is the injective hull of each of the modules in the chain. However, $\mathbb{Z}_{p^n}$ ($n > 1$) has no Baer hull. by Theorem 2.18. Also note that $\mathbb{Z}_{p^\infty}$ has no Baer hull.

In Proposition 3.8 and Example 3.9, we consider the isomorphism problem for Baer hulls as follows: *Let N_1 and N_2 be modules with Baer hulls $\mathfrak{B}(N_1)$ and $\mathfrak{B}(N_2)$, respectively. Then is it true that $N_1 \cong N_2$ if and only if $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$?*

Proposition 3.8. Let N_1 and N_2 are isomorphic modules. If N_1 has a Baer hull $\mathfrak{B}(N_1)$, then N_2 has a Baer hull $\mathfrak{B}(N_2)$, and $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$ as modules.

Proof. Let $\sigma : N_1 \to N_2$ be a module isomorphism. Then there exists a module isomorphism $\overline{\sigma} : E(N_1) \to E(N_2)$, which is an extension of σ. So $\overline{\sigma}(\mathfrak{B}(N_1)) = \mathfrak{B}(N_2)$. $\square$

The next example shows that the converse of Proposition 3.8 does not hold true. In other words, there exist modules N_1 and N_2 such that $\mathfrak{B}(N_1) = \mathfrak{B}(N_2)$ (hence $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$ as modules), but $N_1 \ncong N_2$. Thus the isomorphism problem does not hold for the case of Baer hulls.

Example 3.9. Let $N_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Then by Theorem 2.13 (also see Remark 3.1), $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is the Baer hull of N_1 as $\mathbb{Z}$-modules.

Next, let $N_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/3]$. Say V is a Baer module such that $N_2 \leq V \leq E(N_2)$. From Theorem 2.6, $V = \mathbb{Z}_2 \oplus Z_3 \oplus W$ for some Baer module W such that $\mathbb{Z}[1/3] \leq W \leq \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}_2 \oplus \mathbb{Z}_3) = 0$. Thus $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}_2) = 0$, and so $2^k W = W$ for any nonnegative integer k as in the proof of Theorem 2.13.

Therefore $1/2^k \in W$ for any positive integer k, and thus $\mathbb{Z}[1/2, 1/3] \leq W$. Hence we obtain

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/2, 1/3] = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6] \leq V.$$

Because $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is Baer as a $\mathbb{Z}$-module, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is the Baer hull of N_2.

Consequently, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is the Baer hull of both N_1 and N_2. However, N_1 is not isomorphic to N_2 as $\mathbb{Z}$-modules.

Indeed, if $N_1 \cong N_2$, then $\mathbb{Z} \cong \mathbb{Z}[1/3]$ as $\mathbb{Z}$-modules. Now say $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}[1/3])$ is a $\mathbb{Z}$-module isomorphism. Let $g(1) = s/3^n \in \mathbb{Z}[1/3]$ with $0 \neq s \in \mathbb{Z}$ and a nonnegative integer n. Then $\mathbb{Z}[1/3] = g(\mathbb{Z}) = g(1)\mathbb{Z} = (s/3^n)\mathbb{Z}$. Thus $1/3^{n+1} = (s/3^n)m$ for some $m \in \mathbb{Z}$. Hence $1 = 3sm$, which is impossible. Therefore $N_1 \ncong N_2$ as $\mathbb{Z}$-modules.

We conclude this paper by an example where we compare the direct sum of Baer hulls with the Baer hull of a direct sum of modules.

Example 3.10. In Example 2.19, we provide two modules U and V such that U and V have Baer hulls, but $U \oplus V$ does not have a Baer hull.

Recall that $\mathfrak{B}(-)$ denotes the Baer hull of a module if it exists. Here, we show that there exist two modules M and N such that M, N, and $M \oplus N$ have Baer hulls $\mathfrak{B}(M)$, $\mathfrak{B}(N)$, and $\mathfrak{B}(M \oplus N)$, respectively. But $\mathfrak{B}(M \oplus N) \ncong \mathfrak{B}(M) \oplus \mathfrak{B}(N)$.

Indeed, let $M = \mathbb{Z}_p$ (p a prime integer) and $N = \mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathfrak{B}(M) = \mathbb{Z}_p$ and $\mathfrak{B}(N) = \mathbb{Z}$ since $\mathbb{Z}_p$ is a semisimple $\mathbb{Z}$-module and $\mathbb{Z}$ is a Baer ring. Therefore we have that $\mathfrak{B}(M) \oplus \mathfrak{B}(N) = \mathbb{Z}_p \oplus \mathbb{Z}$.

On the other hand, $\mathfrak{B}(M \oplus N) = \mathfrak{B}(\mathbb{Z}_p \oplus \mathbb{Z}) = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$ (see Theorem 2.13 and Remark 3.1). Hence $\mathfrak{B}(M \oplus N) \ncong \mathfrak{B}(M) \oplus \mathfrak{B}(N)$ because $\mathbb{Z} \ncong \mathbb{Z}[1/p]$ as $\mathbb{Z}$-modules by the argument that used in Example 3.9.

Question 3.11. Let p be a prime integer. Then is $\mathbb{Z}_p \oplus \mathbb{Z}$ the quasi-Baer module hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module? (See [26] for the definition of quasi-Baer modules and Definition 2.11 for quasi-Baer module hulls.)

In a sequel to this paper, we will study Rickart module hulls and their possible description.

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References

[1] E.P. Armendariz, *A note on extensions of Baer and p.p. rings*, J. Australian Math. Soc. **18** (1974), 470–473.

[2] G.F. Birkenmeier, J.K. Park, and S.T. Rizvi, *Hulls of semiprime rings with applications to C^*-algebras*, J. Algebra **322** (2009), 327–352.

[3] G.F. Birkenmeier, J.K. Park, and S.T. Rizvi, *Modules with FI-extending hulls*, Glasgow Math. J. **51** (2009), 347–357.

[4] G.F. Birkenmeier, J.K. Park, and S.T. Rizvi, *Extensions of Rings and Modules*, Research Monograph, Birkhäuser/Springer, New York-Heidelberg-Dordrecht-London, 2013.

[5] A.W. Chatters and S.M. Khuri, *Endomorphism rings of modules over nonsingular CS-rings*, J. London Math. Soc. **21** (1980), 434–444.

[6] N.V. Dung, D.V. Huynh, P.F. Smith, and R. Wisbauer, *Extending Modules*, Longman, Harlow, 1994.

[7] B. Eckmann and A. Schopf, *Über injektive Moduln*, Arch. Math. **4** (1953), 75–78.

[8] C. Faith, *Algebra II, Ring Theory*, Springer, Berlin-Heidelberg-New York, 1976.

[9] V.K. Goel and S.K. Jain, *π-injective modules and rings whose cyclics are π-injective*, Comm. Algebra **6** (1978), 59–73.

[10] Y. Hirano, M. Hongan, and M. Ohori, *On right P.P. rings*, Math. J. Okayama Univ. **24** (1982), 99–109.

[11] R.E. Johnson and E.T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.

[12] M.A. Kamal and B.J. Müller, *Extending modules over commutative domains*, Osaka J. Math. **25** (1988), 531–538.

[13] I. Kaplansky, *Rings of Operators*, Benjamin, New York, 1968.

[14] I. Kaplansky, *Commutative Rings*, Univ. Chicago Press, Chicago, 1974.

[15] G. Lee, S.T. Rizvi, and C. Roman, *Rickart modules*, Comm. Algebra **38** (2010), 4005–4027.

[16] G. Lee, S.T. Rizvi, and C. Roman, *Direct sums of Rickart modules*, J. Algebra **353** (2012), 62–78.

[17] A.C. Mewborn, *Regular rings and Baer rings*, Math. Z. **121** (1971), 211–219.

[18] S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, Cambridge Univ. Press, Cambridge, 1990.

[19] B.J. Müller and S.T. Rizvi, *On the existence of continuous hulls*, Comm. Algebra **10** (1982), 1819–1838.

[20] B.J. Müller and S.T. Rizvi, *On injective and quasi-continuous modules*, J. Pure and Appl. Algebra **28** (1983), 197–210.

[21] K. Oshiro, *On torsion free modules over regular rings*, Math. J. Okayama Univ. **16** (1973), 107–114.

[22] K. Oshiro, *On torsion free modules over regular rings III*, Math. J. Okayama Univ. **18** (1975/76), 43–56.

[23] S.T. Rizvi, *Commutative rings for which every continuous module is quasi-injective*, Arch. Math. **50** (1988), 435–442.

[24] S.T. Rizvi and C.S. Roman, *Baer and quasi-Baer modules*, Comm. Algebra **32** (2004), 103–123.

[25] S.T. Rizvi and C.S. Roman, *On $\mathcal{K}$-nonsingular modules and applications*, Comm. Algebra **35** (2007), 2960–2982.

[26] S.T. Rizvi and C.S. Roman, *On direct sums of Baer modules*, J. Algebra **321** (2009), 682–696.

[27] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York-London-San Francisco, 1979.

[28] L.H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York-London-San Francisco, 1980.

[29] W.R. Scott, *Group Theory*, Dover, New York, 1964.

[30] D.W. Sharpe and P. Vámos, *Injective Modules*, Cambridge Univ. Press, Cambridge, 1972.

[31] K. Shoda, *Zur Theorie der algebraischen Erweiterungen*, Osaka Math. J. **4** (1952), 133–143.

[32] W. Stephenson and G.M. Tsukerman, *Rings of endomorphisms of projective modules*, Siberian Math. J. **11** (1970), 181–184 (English translation).