What is the Bellows Conjecture?

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Abstract

Cauchy's rigidity theorem states:

> If P and P' are combinatorially equivalent convex polyhedra such that the corresponding facets of P and P' are congruent, then P and P' are congruent polyhedra.

For many years it was unknown whether the same theorem was true in general for non-convex polyhedron.
In 1977, more than 160 years after the work of Cauchy, Robert Connelly discovered a polyhedron P (without self-intersections) that allowed for a continuous deformation keeping the facets of P flat and congruent. It was soon noticed that the volume of Connelly's polyhedron remained constant under the flexing motion, and the same fact was found to be true of all later-discovered flexible polyhedra. It was conjectured that this fact would hold in general, and it came to be known as the bellows conjecture.
In 1995, Idjad Sabitov proved the bellows conjecture by showing that for any (oriented) polyhedron P, the volume of P is a root of a polynomial depending only on the combinatorial structure and edge lengths of P. Moreover, the coefficients of the polynomial are themselves polynomials of the squares of the lengths of the edges of P with rational coefficients, with the coefficient polynomials depending only on the combinatorial structure of P. Hence, the volume of a polyhedron P with combinatorial structure K is a finitely valued function of the edge lengths of P, and in this way the theorem may be viewed as a generalization of Heron's formula for the area of a triangle.

1 Formulas from Antiquity

Often attributed to Heron of Alexandria (though perhaps known already to Archimedes), Heron's formula expresses the area of a triangle as a function of the side lengths.

Heron's Formula. *Let T be a triangle with side lengths (a, b, c).*

$$Area(T) = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

We all know from elementary geometry that a triangle is determined by its side lengths. With this knowledge, Heron's formula is, perhaps, unsurprising. For quadrilaterals, we quickly observe that the side lengths do not uniquely determine the area, so there is no hope of finding a similar formula that holds in the same generality. However, if we restrict to the case of cylic quadrilaterals, i.e., quadrilaterals whose vertices all lie on a common circle, then an analogous formula does exist.

Brahmagupta's Formula. *Let Q be a cyclic quadrilateral with side lengths (a, b, c, d).*

$$Area(Q) = \frac{1}{4}\sqrt{(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)}$$

If we let an edge of Q have length 0, we see that Heron's Formula appears as a special case of Brahmahgupta.
It is natural to ask whether or not a similar formula exists for cyclic pentagons, hexagons, etc. Now Heron's formula may be easily derived using nothing more than the Pythagorean theorem, and in [3], we see that Brahmagupta's formula follows easily from Heron. Despite the relative ease of these first two cases, the formula for the cylic pentagon went unknown for over 1300 years after Brahmagupta recorded his formula for the quadrilateral. In 1994 D. P. Robbins discovered the formulas for both the cyclic pentagon and the cyclic hexagon and proved the following more general result.

Theorem (Robbins [6])**.** *For any natural number n, there exists a unique (up to sign) irreducible homogenous polynomial f (where we regard the first argument as degree 4, the rest as degree 2) with integer coefficients such that for any cyclic n-gon P, the edge lengths $(a_1, a_2, \dots, a_n)$ and area K of P satisfy*

$$f(16K^2, a_1^2, \dots, a_n^2) = 0.$$

2 Simplices and Complexes [5] [8]

We will always assume $k, n \in \mathbb{N}$.

Definition (k-Simplex). *A* k-simplex *is the convex hull of $k+1$ points (vertices) in general position in $\mathbb{R}^n (n \geq k)$.*

More explicitly, given $k+1$ vertices $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$ in general position (i.e., the set of k vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ is linearly independent over $\mathbb{R}$), the k-simplex determined by $\{v_0, \ldots, v_k\}$ is $\{\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k | \lambda_i \geq 0, i = 0, \ldots, k, \sum_{i=0}^{k} \lambda_i = 1\}$. A face of a simplex is the convex hull of any subset of its vertices. An m-face is a face of dimension m, and a facet is a $(k-1)$-face of a k-simplex.

Definition (Simplicial Complex). *A set of simplices $\mathcal{K}$ is a* simplicial complex *if:*

1. *Every face of every simplex of $\mathcal{K}$ is in $\mathcal{K}$.*
2. *The intersection of any two simplices of $\mathcal{K}$ is a face of both simplices.*

The body of $\mathcal{K}$ is $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$

It will be necessary to generalize the notion of the volume of a simplicial complex, and for this we will need oriented complexes.

Definition (Oriented Simplex). *Let σ be a k-simplex with a fixed ordering of its vertex set $(v_0, \ldots, v_k)$. The equivalence class of orderings $\{(v_{\tau(0)}, \ldots, v_{\tau(k)}) \mid \tau$ is an even permutation$\}$ is the orientation of σ, and σ with its class of orderings we call an* oriented simplex. *The equivalence class of orderings $\{(v_{\tau(0)}, \ldots, v_{\tau(k)}) \mid \tau$ is an odd permutation$\}$ is the reverse orientation.*

Given an oriented simplex σ, there is an induced orientation on the facets of σ given by the following construction. Identify each facet with the vertex of σ that it does not contain. Select an ordering from the orientation of σ and delete the vertex corresponding to the facet on which we wish to induce an orientaion. This ordering defines an orientation of the facet, but we take the reverse orientation precisely when the parity of the position of the removed vertex was odd.

Definition. *Two k-simplices that intersect at a $(k-1)$-simplex σ are called* coherently oriented *if they induce the same orientation on σ. A k-simplicial complex is* orientable *if there is a choice of orientation for each k-simplex such that all pairs are coherently oriented.*

3 Cayley-Menger Determinants [4]

Definition (Cayley-Menger Determinant). *Let $S = \{v_1, \ldots, v_k\}$ be a set of points in $\mathbb{R}^n$. Let $l_{i,j} = \|v_i - v_j\|$, the Euclidean distance between the points v_i and v_j. The Cayley-Menger determinant of S is*

$$\begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & l_{1,1}^2 & l_{1,2}^2 & \ldots & l_{1,k}^2 \\ 1 & l_{2,1}^2 & l_{2,2}^2 & \ldots & l_{2,k}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & l_{k,1}^2 & l_{k,2}^2 & \ldots & l_{k,k}^2 \end{vmatrix}$$

Of course, $l_{i,i} = 0$ for each i, and $l_{i,j} = l_{j,i}$ for each i and j, so the matrix is symmetric with zeroes along the diagonal. We record two important properties of the Cayley-Menger Determinant.

Property 1. *Let σ be a k-simplex with vertex set $S = \{v_0, \ldots, v_k\}$.*

$$V_\sigma^2 = \frac{(-1)^{k+1}}{2^k (k!)^2} det(M_S)$$

where $det(M_S)$ is the Cayley-Menger determinant of the vertex set of σ, and V_σ is the k-dimensional volume of σ.

Notice that this gives the volume of a k-simplex σ as a root of a polynomial in the squares of the edge lengths of σ.

Property 2. *The Cayley-Menger determinant is identically zero whenever $k \geq n+2$, where k is the cardinality of the vertex set, n the dimension of the space.*

An important special case of this is that the 10 distances between 5 points in $\mathbb{R}^3$ are not arbitary, but must satisfy the polynomial relation described by its Cayley-Menger determinant being identically zero.

4 Cauchy's Rigidity Theorem and the Bellows Conjecture

We now focus our attentions on $\mathbb{R}^3$.

Definition. *Given a simplicial 2-complex $\mathcal{K}$, a* polyhedron with combinatorial structure $\mathcal{K}$ *is a continuous map $P : |\mathcal{K}| \to \mathbb{R}^3$ that is linear on the simplices of $\mathcal{K}$.*

We will consider only the case when $|\mathcal{K}|$ is homeomorphic to an orientable 2-manifold of genus $g \geq 0$.

Theorem (Cauchy's Rigidity Theorem [1])**.** *Let P and P' be convex polyhedra such that they are combinatorially equivalent and corresponding facets are congruent. Then P and P' are congruent.*

We may view Cauchy's rigidity theorem as saying that for any convex polyhedron P, there is no continuous deformation of P that leaves the faces of P flat and congruent. It was believed that the same would be true of all polyhedra, so it was a surprise when Connelly presented a (non-convex) polyhdron (with no self-intersections) that allowed such a deformation, called a flexion. As more examples were found it was soon noticed that the volumes of the flexible polyhedra appeared to be preserved under flexions. This became known as the bellows conjecture.

(**The Bellows Conjecture).** *Flexions are volume-preserving. (Proven in 1995 by Idjad Sabitov)*

We require the following notions.
Let σ be an oriented k-simplex in $\mathbb{R}^{k+1}$. Let O be a point of $\mathbb{R}^{k+1} \setminus \sigma$, and form the $(k+1)$-simplex O_σ with vertices from σ and O. Given an orientation of O_σ, we can define the oriented volume of O_σ by taking the negative of the standard volume when O_σ induces the reverse orientation on σ.
Let P be a polyhedron with combinatorial structure $\mathcal{K}$. $|\mathcal{K}|$ is homeomorphic to an orientable 2-manifold of genus $g \geq 0$, and we give it a coherent orientation. This induces an orientation onto P, and we refer to P as an oriented polyhedron.
Let P be an oriented polyhedron with combinatorial structure $\mathcal{K}$. Let O be a point of $\mathbb{R}^3$, and let $\{O_\sigma\}$ be the set of 3-simplices formed by O and the 2-simplices σ of $\mathcal{K}$. The *generalized volume of P* is the sum of the oriented volumes of the $O_\sigma' s$ (where we have coherently oriented the $O_\sigma' s$). The notion does not depend on the choice of O and coincides with the standard notion of volume when such the standard notion exists.
Sabitov proved the bellows conjecture by showing the following more general result.

Theorem (Sabitov [7])**.** *Let P be an oriented surface in $\mathbb{R}^3$ having a given combinatorial structure $\mathcal{K}$ and given edge lengths l_k, $1 \leq k \leq e$ where e is the number of edges of P. Let $\widetilde{P}$ denote the set of all polyhedra in $\mathbb{R}^3$ with the same combinatorial structure and edge lengths as P. Then there exists a polynomial equation*

$$Q(V) = V^{2N} + a_1(l)V^{2N-2} + \ldots + a_N(l) = 0$$

such that the generalized volume of any polyhedron from $\widetilde{P}$ is a root of this equation. Moreover, the coefficients a_i are polynomials in $(l) = (l_1^2, \ldots, l_e^2)$ with rational coefficients depending on $\mathcal{K}$.

In 2011, Alexander Gaifullen proved analogous results in dimension 4 [2].

References

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