WHAT ARE THE BERNOULLI NUMBERS?

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Abstract. For the "What is?" seminar today we will be investigating the Bernoulli numbers. This surprising sequence of numbers has many applications including summing powers of integers, evaluating the zeta function, finding asymptotics of Stirling's formula, and estimating the harmonic series. We investigate properties of these numbers and introduce Bernoulli polynomials, a closely related topic. Then, we establish the Euler summation formula and use the formula to provide the aforementioned applications. Furthermore, a study of the Fourier coefficients, yields surprising results on the zeta function.

1. Introduction

Bernoulli numbers first appeared in a post humorous publication of Jakob Bernoulli (1654-1705) in 1713, and were independently discovered by Japanese mathematician Seki Kōwa in 1712. Bernoulli observed these numbers in the course of his investigations of sums of powers of integers. That is sums of squares, cubes, or higher powers. Let

$$S_p(n) = \sum_{k=1}^{n-1} k^p.$$

We have the following closed forms of $S_p(n)$ for small p:

$$\begin{array}{lcllllllllllllll}
S_0(n) & = & n \\
S_1(n) & = & \frac{1}{2}n^2 & - & \frac{1}{2}n \\
S_2(n) & = & \frac{1}{3}n^3 & - & \frac{1}{2}n^2 & + & \frac{1}{6}n \\
S_3(n) & = & \frac{1}{4}n^4 & - & \frac{1}{2}n^3 & + & \frac{1}{4}n^2 \\
S_4(n) & = & \frac{1}{5}n^5 & - & \frac{1}{2}n^4 & + & \frac{1}{3}n^3 & - & \frac{1}{30}n \\
S_5(n) & = & \frac{1}{6}n^6 & - & \frac{1}{2}n^5 & + & \frac{5}{12}n^4 & - & \frac{1}{12}n^2 \\
S_6(n) & = & \frac{1}{7}n^7 & - & \frac{1}{2}n^6 & + & \frac{1}{2}n^5 & - & \frac{1}{6}n^3 & + & \frac{1}{42}n \\
S_7(n) & = & \frac{1}{8}n^8 & - & \frac{1}{2}n^7 & + & \frac{7}{12}n^6 & - & \frac{7}{24}n^4 & + & \frac{1}{12}n^2 \\
S_8(n) & = & \frac{1}{9}n^9 & - & \frac{1}{2}n^8 & + & \frac{2}{3}n^7 & - & \frac{7}{15}n^5 & + & \frac{2}{9}n^3 & - & \frac{1}{30}n \\
S_9(n) & = & \frac{1}{10}n^{10} & - & \frac{1}{2}n^9 & + & \frac{3}{4}n^8 & - & \frac{7}{10}n^6 & + & \frac{1}{2}n^4 & - & \frac{3}{20}n^2 \\
S_{10}(n) & = & \frac{1}{11}n^{11} & - & \frac{1}{2}n^{10} & + & \frac{5}{6}n^9 & - & n^7 & + & n^5 & - & \frac{1}{2}n^3 & + & \frac{5}{66}n
\end{array}$$

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Through slight manipulations Bernoulli arrived at the following reformulation of these sums:

$$
\begin{array}{lcl}
S_0(n) & = & \frac{1}{1}\binom{1}{0}n \\
S_1(n) & = & \frac{1}{2}\left[\binom{2}{0}n^2 - \binom{2}{1}\frac{1}{2}n\right] \\
S_2(n) & = & \frac{1}{3}\left[\binom{3}{0}n^3 - \binom{3}{1}\frac{1}{2}n^2 + \binom{3}{2}\frac{1}{6}n\right] \\
S_3(n) & = & \frac{1}{4}\left[\binom{4}{0}n^4 - \binom{4}{1}\frac{1}{2}n^3 + \binom{4}{2}\frac{1}{6}n^2\right] \\
S_4(n) & = & \frac{1}{5}\left[\binom{5}{0}n^5 - \binom{5}{1}\frac{1}{2}n^4 + \binom{5}{2}\frac{1}{6}n^3 - \binom{5}{4}\frac{1}{30}n\right] \\
S_5(n) & = & \frac{1}{6}\left[\binom{6}{0}n^6 - \binom{6}{1}\frac{1}{2}n^5 + \binom{6}{2}\frac{1}{6}n^4 - \binom{6}{4}\frac{1}{30}n^2\right] \\
S_6(n) & = & \frac{1}{7}\left[\binom{7}{0}n^7 - \binom{7}{1}\frac{1}{2}n^6 + \binom{7}{2}\frac{1}{6}n^5 - \binom{7}{4}\frac{1}{30}n^3 + \binom{7}{6}\frac{1}{42}n\right] \\
S_7(n) & = & \frac{1}{8}\left[\binom{8}{0}n^8 - \binom{8}{1}\frac{1}{2}n^7 + \binom{8}{2}\frac{1}{6}12n^6 - \binom{8}{4}\frac{1}{30}n^4 + \binom{8}{6}\frac{1}{42}n^2\right] \\
S_8(n) & = & \frac{1}{9}\left[\binom{9}{0}n^9 - \binom{9}{1}\frac{1}{2}n^8 + \binom{9}{2}\frac{1}{6}3n^7 - \binom{9}{4}\frac{1}{30}n^5 + \binom{9}{6}\frac{1}{42}n^3 - \binom{9}{8}\frac{1}{30}n\right] \\
S_9(n) & = & \frac{1}{10}\left[\binom{10}{0}n^{10} - \binom{10}{1}\frac{1}{2}n^9 + \binom{10}{2}\frac{1}{6}n^8 - \binom{10}{4}\frac{1}{30}n^6 + \binom{10}{6}\frac{1}{42}n^4 - \binom{10}{8}\frac{1}{30}n^2\right] \\
S_{10}(n) & = & \frac{1}{11}\left[\binom{11}{0}n^{11} - \binom{11}{1}\frac{1}{2}n^{10} + \binom{11}{2}\frac{1}{6}n^9 - \binom{11}{4}\frac{1}{30}n^7 + \binom{11}{6}\frac{1}{42}n^5 - \binom{11}{8}\frac{1}{30}n^3 + \binom{11}{10}\frac{5}{66}n\right]
\end{array}
$$

Through this reformulation we notice the repeated occurrence of certain numbers within the closed form sums. These are the Bernoulli numbers. Here are the first few:

$$
B_0 = 1, \quad B_1 = \frac{-1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{-1}{30}, \quad B_5 = 0,
$$

$$
B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = \frac{-1}{30}, \quad B_9 = 0, \quad B_{10} = \frac{5}{66}, \quad B_{11} = 0.
$$

More generally, via the Euler summation formula, we will prove that

(1) $$S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\, n^{m+1-k} \qquad \text{(Here } B_k \text{ is the kth Bernoulli number).}$$

We also achieve the results on the value of the zeta function on even integers and the asymptotics of Stirling's formula and the partial sums of the harmonic series through investigating the Bernoulli numbers and the Bernoulli polynomials.

2. Definition and elementary properties

Bernoulli first discovered through studying sums of integers raised to fixed powers. This approach hinted at above properly defines the Bernoulli numbers, but may present difficulties when trying to calculate larger numbers in the sequence since we would first need closed forms of $S_p(n)$. Additionally, to take this as the definition we would need to prove that the consistency of equation (1). The modern approach is to define the Bernoulli numbers through the use of the generating function $\frac{x}{e^x-1}$ and then prove formula (1).

Definition 1. *The Bernoulli numbers* $\{B_k\}_{k=0}^{\infty}$ *are defined as the constants in the power series expansion of the analytic function* $\frac{x}{e^x-1}$

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Regarding the above equation as a formal power series, the equation

$$1 = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$$

produces the equation

$$\sum_{i+j=k} \frac{B_i}{i!(j+i)!} = \begin{cases} 1 & k=0 \\ 0 & k \geq 1. \end{cases}$$

From observing the above chart one sees that

$$B_3 = B_5 = B_7 = B_9 = B_{11} = 0.$$

Indeed this is true for all odd numbers larger than 2

Lemma 2.1. *Let n an odd number larger than* 2*. Then* $B_n = 0$*.*

Proof.

$$\begin{aligned} \frac{x}{e^x-1} - B_1 x &= \frac{x}{e^x-1} + \frac{x}{2} \\ &= \frac{2x + x(e^x-1)}{2(e^x-1)} \\ &= \frac{x(e^x+1))}{2(e^x-1)} \\ &= \frac{x(e^{\frac{x}{2}} + e^{\frac{-x}{2}}))}{2(e^{\frac{x}{2}} - e^{\frac{-x}{2}})} \end{aligned}$$

$e^{\frac{x}{2}} - e^{\frac{-x}{2}}$ is odd, $e^{\frac{x}{2}} + e^{\frac{-x}{2}}$ is even, and x is odd. Thus $\frac{x}{e^x-1} - B_1 x$ is an even function. Thus the power series expansion of $\frac{x}{e^x-1} - B_1 x$ has no nontrivial odd terms. □

The Bernoulli numbers grow quite quickly. Indeed, we will show in section 5 that

$$B_k \sim \frac{-2k!}{(2\pi i)^k} \qquad (\text{as } k \to \infty).$$

For now let us be satisfied with the fact that

$$B_{20} = \frac{-174611}{330}.$$

In order to achieve the results mentioned in the introduction, we will need to define the Bernoulli polynomials.

Definition 2. *The Bernoulli polynomials are a sequence of polynomials* $\{B_k(y)\}_{k=0}^{\infty}$ *are defined through the power series expansion of* $\frac{xe^{xy}}{e^x-1}$

$$\frac{xe^{xy}}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k(y)x^k}{k!}.$$

As above we can derive a closed form for these polynomials by taking the product of the power series for $\frac{x}{e^x-1}$ and e^{xy}.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{B_k(y)x^k}{k!} &= \frac{xe^{xy}}{e^x-1} \\ &= \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{(xy)^k}{k!} \\ &= \sum_{k=0}^{\infty} x^k \sum_{i+j=k} \frac{B_i y^j}{i!j!} \end{aligned}$$

Thus

$$B_k(y) = \sum_{i=0}^{k} \binom{k}{i} B_i y^{k-i}$$

Here are the first few Bernoulli polynomials:

$$\begin{aligned} B_0(y) &= 1 \\ B_1(y) &= y - \frac{1}{2} \\ B_2(y) &= y^2 - y + \frac{1}{6} \\ B_3(y) &= y^3 - \frac{3}{2}y^2 + \frac{1}{2}y \\ B_4(y) &= y^4 - 2y^3 + y^2 - \frac{1}{30} \end{aligned}$$

One may notice that for the listed polynomials $B_k'(y) = kB_{k-1}(y)$. This holds in general!

Lemma 2.2.

$$B_k'(y) = kB_{k-1}(y)$$

Proof. Let us differentiate the defining relation for the Bernoulli polynomials with respect to y.

$$\frac{x^2e^{xy}}{e^x-1} = \sum_{k=1}^{\infty} \frac{B_k'(y)x^k}{k!}.$$

Divide through by x and reindex

$$\begin{aligned} \frac{xe^{xy}}{e^x-1} &= \sum_{k=1}^{\infty} \frac{B_k'(y)x^{k-1}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{B_{k-1}'(y)x^k}{(k+1)!} \end{aligned}$$

Then equating the powers of x gives the desired relation. $\square$

Additionally there are many other interesting facts concerning these polynomials. I encourage you to try to prove the following relations:

$$\begin{array}{rcll} B_k(1-y) & = & (-1)^k B_k(y) & \\ B_k\left(\frac{1}{2}\right) & = & (2^{1-k}-1)B_k & \\ B_k\left(\frac{1}{4}\right) & = & 2^{-k}B_k\left(\frac{1}{2}\right) & \text{(for even k)} \\ \int_0^1 B_k(y)dy & = & 0 & (\text{if } k \geq 1) \\ B_k(y+1) & = & B_k(y) + ky^{k-1} & \\ \int_0^1 B_k(y)B_l(y)dy & = & (-1)^{l-1}\frac{B_{k+l}}{\binom{k+l}{k}} & (\text{if } k, l \geq 1) \end{array}$$

Finally let us make the Bernoulli polynomials restricted to $[0,1]$ into periodic function defined on $\mathbb{R}$ with period 1. That is

Definition 3. *For $k \in \mathbb{N}$, let*

$$B_k^*(y) := B(y - \lfloor y \rfloor).$$

3. Euler-Maclaurin summation formula

Here we aim to prove the following summation formula which will be critical in the course of our analysis.

Theorem 3.1. *Let $a < b \in \mathbb{Z}$ and let f be a smooth function on $[a,b]$. Then for all $m \geq 1$*

$$\sum_{i=a}^{b-1} f(i) = \int_a^b f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\bigg|_a^b + R_m$$

Where

$$R_m = (-1)^{m+1} \int_a^b \frac{B_m^*(x)}{m!} f^{(m)}(x)dx.$$

Proof. By the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t)dt$$

Integrating with respect to x gives us the following:

$$\begin{aligned} \int_0^1 f(x) &= f(0) + \int_0^1 \int_0^x f'(t)dtdx \\ &= f(0) + \int_0^1 f'(t) \int_t^1 dxdt \\ &= f(0) + \int_0^1 f'(t)(1-t)dt \\ &= f(1) + \int_0^1 f'(t)(-t)dt \end{aligned}$$

By adding the last two equations we find that

$$2\int_0^1 f(x)dx = f(0) + f(1) + \int_0^1 (1-2t)f'(t)dt.$$

After dividing by 2

$$\int_0^1 f(x)dx = \frac{f(0)+f(1)}{2} + \int_0^1 (\frac{1}{2} - t)f'(t)dt.$$

Through a quick manipulation we find that

$$f(0) = \int_0^1 f(x)dx + \frac{f(0)-f(1)}{2} + \int_0^1 (x - \frac{1}{2})f'(x)dx.$$

In other words,

$$f(0) = \int_0^1 f(x)dx + B_1 f(x)\Big|_0^1 + \int_0^1 B_1(x)f'(x)dx. \tag{2}$$

Now we prove

$$f(0) = \int_0^1 f(x) + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1 + (-1)^{m+1}\int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)dx. \tag{3}$$

Observe that (2) is the base case of (3). Assume that we have proved equation (3) for all $k \le m$. By the fact that $B_k'(x) = kB_{k-1}(x)$,

$$\int_0^1 B_k(x)f^{(k)}(x)dx \quad = \quad \frac{B_{k+1}(x)}{k+1} f^{(k)}(x)\Big|_0^1 - \frac{1}{k+1}\int_0^1 B_{k+1}(x)f^{(k+1)}(x)dx$$

By the fact that (3) holds for m and the prior calculation,

$$f(0) = \int_0^1 f(x) + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^{m+1}}{m!}\left(\frac{B_{m+1}(x)}{m+1} f^{(m)}(x)\Big|_0^1 - \frac{1}{m+1}\int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx\right).$$

If m is odd, $(-1)^{m+1} = 1$. If m is even $B_{m+1}(0) = B_{m+1}(1) = 0$. Thus

$$\frac{(-1)^{m+1}}{m!}\ \frac{B_{m+1}(x)}{m+1} f^{(m)}(x)\Big|_0^1 = \frac{B_{m+1}}{(m+1)!} f^{(m)}(x)\Big|_0^1, \text{ and}$$

$$f(0) = \int_0^1 f(x) + \sum_{k=1}^{m+1} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_0^1 + \frac{(-1)^{m+2}}{(m+1)!}\int_0^1 B_{m+1}(x)f^{(m+1)}(x)dx.$$

Applying this result to $f(j+x)$ for all $a \le j < b$ and summing gives the desired formula. □

4. Applications of the summation formula

Now we return to our original example, the sum of integers raised to a power.

Corollary 4.1.

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \, n^{m+1-k}.$$

Proof. Fix $p \geq 1$. We will apply the Euler summation formula to the function $f(x) = x^p$, with $a = 0$, $b = n$ and $m = p$. For $l \leq p$

$$f^{(l)}(x) = m(m-1)\dots(m-l+1)x^{m-l}.$$

Thus $f^{(p)} = p!$ and

$$R_m = (-1)^{m+1} \int_a^b B_m^*(x)dx = (-1)^{m+1}(b-a) \int_0^1 B_m(x)dx = 0$$

Thus

$$\begin{aligned}
\sum_{i=0}^{n-1} x^m &= \int_0^n x^m dx + \sum_{k=1}^{m} \frac{B_k}{k!} m(m-1)\dots(m-k+2)x^{m-l+1}\Bigg|_0^n + R_m \\
&= \frac{x^{m+1}}{m+1} + \frac{1}{m+1} \sum_{k=1}^{m} \binom{m+1}{k} B_k n^{m-l+1} \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m-l+1}
\end{aligned}$$

□

Next we apply the summation formula in the case of the harmonic series. Let $f(x) = \frac{1}{x}$, $a = 1$, $b = n$, and $m \geq 1$. Then

$$f^{(l)}(x) = (-1)^l \frac{l!}{x^{l+1}}.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{1}{i} &= \ln(n) + \sum_{k=1}^{m} \frac{B_k}{k!} (-1)^{k-1} \frac{(k-1)!}{x^k}\Bigg|_1^n + (-1)^{m-1} \int_1^n \frac{B_m^*(x)}{m!} (-1)^m \frac{m!}{x^{m+1}} dx \\
&= \ln(n) + B_1 \left(\frac{1}{n} - 1\right) - \sum_{k=2}^{m} \frac{B_k}{k} \left(\frac{1}{n^k} - 1\right) - \int_1^n \frac{B_m^*(x)}{x^{m+1}} dx
\end{aligned}$$

Let

$$\gamma = -B_1 + \sum_{k=2}^{m} \frac{B_k}{k} - \int_1^\infty \frac{B_m^*(x)}{x^{m+1}}.$$

Thus

$$\sum_{i=1}^{n} \frac{1}{i} = \ln(n) + \gamma + \frac{1}{2n} - \sum_{k=2}^{m} \frac{B_k}{kn^k} - \int_n^\infty \frac{B_m^*(x)}{x^{m+1}} dx.$$

Finally a similar calculation can be made in the case $f(x) = \ln(x)$ to gain results on Stirling's formula. Let $a = 1$ and $b = 2$ and let m be any integer larger than 1. Following the same logic as above, we arrive at the formula:

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln(n) - n + C + \sum_{k=2}^{m} \frac{B_k}{k(k-1)n^{k-1}} - \int_n^{\infty} \frac{B_m^*(x)}{mx^m}dx$$

5. Fourier analysis of Bernoulli polynomials

Let us now calculate the Fourier coefficients of the periodic functions $B_k^*(y)$. We will denote the lth Fourier coefficient of $f(x)$ by $\hat{f}(l)$.

$$\hat{B}_1^*(0) = \int_0^1 B_1(x)dx = 0.$$

For $l \neq 0$,

$$\begin{aligned}\hat{B}_1^*(l) &= \int_0^1 (x - \frac{1}{2})e^{-2\pi i x l}dx \\ &= \frac{-1}{2\pi i l}\end{aligned}$$

Then using the fact that for all smooth functions f we have that $\hat{f'}(l) = 2\pi i l \hat{f}(l)$ and the fact that $B_k'(x) = kB_{k-1}(x)$. We get that

$$2\pi\, i\, l\hat{B}_k^*(l) = k\, \hat{B}_k^*(l)$$

We will take for granted that the Fourier series acts nicely for $k \geq 2$(for $k \geq 2$ our functions are C^1). So for $k \geq 2$

$$B_k^*(x) = \sum_{l \neq 0} \frac{-k!}{(2\pi\, i\, l)^k} e^{2\pi\, i\, l}$$

In particular for $k \geq 2$ and $x = 0$

$$B_k = \sum_{l \neq 0} \frac{-k!}{(2\pi\, i\, l)^k}$$

For even k

$$\begin{aligned}B_k &= \frac{-2k!}{(2\pi\, i)^k} \sum_{l=1}^{\infty} l^k \\ &= \frac{-2k!}{(2\pi\, i)^k}\zeta(k)\end{aligned}$$

Thus we can calculate the values of $\zeta(k)$ for even values of k very quickly. For example

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90} \qquad \zeta(6) = \frac{\pi^6}{945} \qquad \zeta(8) = \frac{\pi^8}{9450.}$$

Furthermore since $\lim_{n\to\infty} \zeta(2n) = 1$, we find the asymptotic relation mentioned in section 2:

$$B_k \sim \frac{-2k!}{(2\pi i)^k} \qquad (\text{as } k \to \infty).$$