WHAT IS... A CAYLEY GRAPH?

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Cayley graphs give a way of encoding information about group in a graph. Given a group with a, typically finite, generating set, we can form a Cayley Graph for that group with respect to that generating set. Most of the information found here is taken from John Meier's, *Groups Graphs and Trees*, [1].

1. Cayley Graphs

Before we define Cayley graphs, we will give some basic definitions.

Definition 1. *A graph, Γ, is a set of vertices and edges, where each edge e is associated with a pair of vertices via the function "Ends", where Ends$(e) = \{v, w\}$ for v, w not necessarily distinct vertices of Γ. Some examples can be seen in figure 1.*

Figure 1. Two examples of graphs.

A path is an ordered set of vertices and edges, $\{v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n\}$ such that Ends$(e_k) = \{v_{k-1}, v_k\}$ for any integer k, $1 \leq k \leq n$. A path without backtracking is a path such that $e_k \neq e_{k+1}^{-1}$, that is, if an edge is in a path, the reverse direction of that same edge does not immediately follow. A circuit is a path such that $v_0 = v_n$. A tree is a graph where no paths without backtracking are circuits.

Given a graph Γ, a symmetry, $\alpha : \Gamma \to \Gamma$, is a bijection, sending vertices to vertices and edges to edges, such that if Ends$(e) = \{v, w\}$ for some edge e and vertices v, w, then Ends$(\alpha(e)) = \{\alpha(v), \alpha(w)\}$. The set of all symmetries of a graph forms a group under the operation of composition.

Let us look back at figure 1 and determine the symmetry groups of the two graphs. For the figure on the left, the top left vertex can either stay where it is or switch places with the bottom left vertex. If the top left vertex is not moved, then all of the vertices must remain fixed. If the top left and bottom left vertices are interchanged, then the right vertices must also be switched, and what we get is a reflection. This contributes a factor of $\mathbb{Z}_2$ to the symmetry group. Next, we can interchange the 3 edges on the left however we want. This contributes a factor of S_3. Since reflecting the graph and interchanging the three edges on the left do not affect each other, we get $Sym(\Gamma) = \mathbb{Z}_2 \oplus S_3$.

Similarly, it is a good exercise to see why the symmetry group of the graph on the right in figure 1 is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus S_2 \oplus S_2$.

Meier presents the following two theorems and elaborates on the proofs:

Theorem 2 (Cayley's Basic Theorem)**.** *Every group can be faithfully represented as a group of permutations.*

Proof. Consider the group S_G. □

This theorem can be improved.

Theorem 3 (Cayley's Better Theorem)**.** *Every finitely generated group can be faithfully represented as a symmetry group of a connected, directed, locally finite graph.*

Here, *locally finite* means that for any vertex e, there are finitely many edges $v_1, \dots, v_n$ with $v_k \in Ends(e)$.

Proof. Let G be a finitely generated group with generating set $S = \{s_1, \dots, s_n\}$. We prove this theorem by constructing a graph, $\Gamma_{G,S}$ on which G acts.

The vertices of $\Gamma_{G,S}$ will be the elements of G. For each $g \in G$, $s \in S$, make an edge labeled s from the vertex labeled g to the vertex labeled gs. Since G is finitely generated, this graph is locally finite. Since S generates G, this graph is connected. By construction, G is directed. Let G act on the graph by left multiplication. That is, for any $g \in G$, g will send the vertex labeled h to the vertex labeled gh. It is easily verified that this graph satisfies the desired properties. □

Note that the Cayley graph for a group is not unique, since it depends on the generating set. We now look at some examples to help illustrate this theorem.

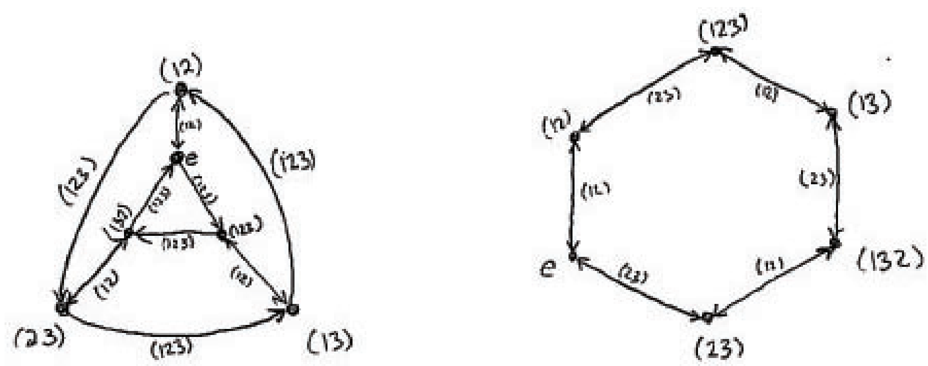

Figure 2. Two Cayley graphs for S_3. The Cayley graph on the left is with respect to generating set $S = \{(12), (123)\}$, while the Cayley graph on the right is with respect to generating set $S' = \{(12), (23)\}$. This helps illustrate how the Cayley graph depends on the generating set.

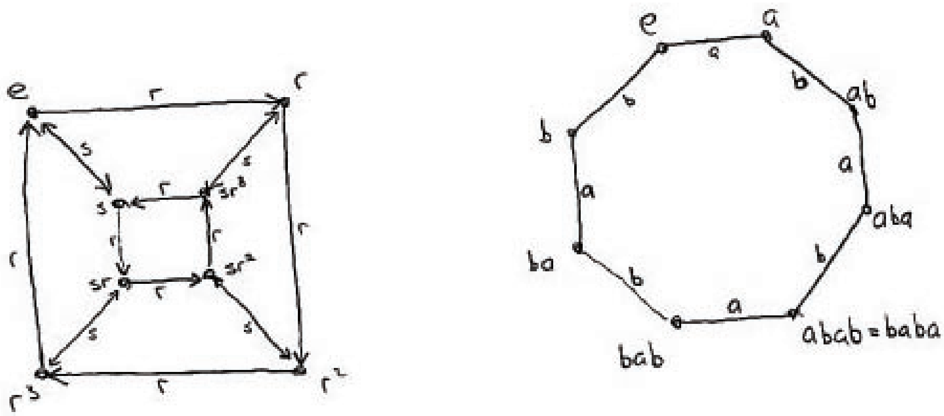

FIGURE 3. Another example of Cayley graphs. This time, we have Cayley graphs for D_4 with generating sets $S = \{r, s\}$, r a rotation and s a symmetry, and $S' = \{a, b\}$, two adjacent reflections

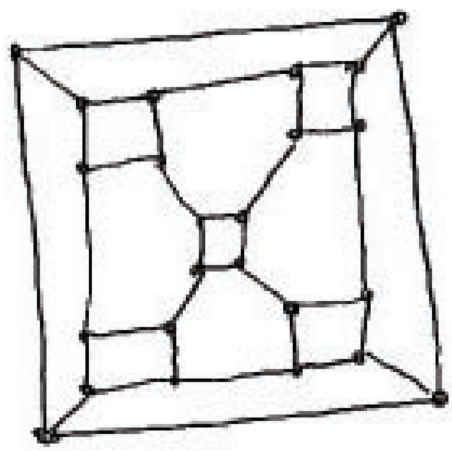

FIGURE 4. This is the Cayley graph for S_4 with respect to generating set $S = \{(12), (23), (34)\}$. This shows that the Cayley graph's complexity increases quickly with the complexity and size of the group.

Notice that generating elements of order 2 result in 2 edges directed in opposite directions with the same label. In these cases it is convenient to connect the vertices with a single undirected edge, as seen in the second graph in figure 3. Figure 4 has no labels at all because it is just supposed to show how complicated the Cayley graphs can be, even for relatively simple groups such as S_4. Figure 5 has just a few lebels written. Hopefully the unwritten labels are clear and can be determined easily.

We now present some results from Meier.

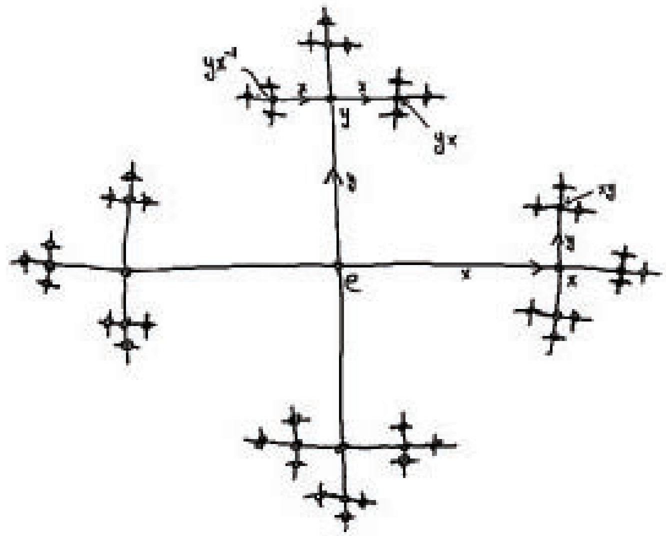

Figure 5. This is the Cayley graph of $\mathbb{F}_2 =< x, y| >$ with respect to generators $S = \{x, y\}$. Each horizontal edge is directed to the right labeled x, and each vertical edge is directed upwards, labeled y. This Cayley graph is a tree.

Theorem 4. *There is a finite index subgroup of $\mathbb{F}_2 =< x, y| >$, the free group with two generators. This subgroup is a free group of rank 3.*

Proof. Consider H, the subgroup of $\mathbb{F}_2$ consisting of all elements of even length, $H =< x^2, xy, xy^{-1} >$. □

Finally, from Meier, some applications to show how studying graphs can help give information about groups.

Theorem 5. *A group G is free if and only if G acts freely on a tree.*

Corollary 6 (Nielsen-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. If G is a free group, then G will act freely on its Cayley graph. If H is a subgroup of G, then H will also act freely on the Cayley graph for G. By the previous theorem, H is free. □

References

[1] J. Meier, *Groups Graphs and Trees: An introduction to the Geometry of infinite groups*, Cambridge University Press, Cambridge, 2008.