WHAT IS THE CROFTON FORMULA?

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ABSTRACT. The Crofton formula (sometimes called the Cauchy-Crofton formula) is an amazing formula giving a surprising characterization of the length of a rectifiable curve in the plane. For a rectifiable curve γ and an (oriented) line ℓ in the plane, let $n_\gamma(\ell)$ be the number of intersections of ℓ with γ.[1] Using a convenient parameterization of the space of oriented lines as the elements of a cylinder, with the direction in radians being called φ and the position relative to the origin being p, and letting $n_\gamma(\varphi, p)$ equal $n_\gamma(\ell)$ if (φ, p) corresponds to ℓ, the Crofton formula states that

$$\text{length}(\gamma) = \frac{1}{4} \iint n_\gamma(\varphi, p)\, d\varphi\, dp, \tag{1}$$

i.e., that the length of the curve is a fixed multiple of the "average intersectiveness" of the curve with an oriented line.

We will discuss the background and proof of the formula, then discuss as many applications as we have time to discuss, possibly including some minimum-total-curvature results, Buffon's needle and noodle problems, and a special case of Hilbert's Fourth Problem.

[1]For curves with straight segments, a finite number of oriented lines will have infinite-cardinality interesection with the curve, but this is a "sparse" set that does not end up affecting the integrals.