DESARGUES' THEOREM

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Two triangles ABC and $A'B'C'$ are said to be in perspective *axially* when no two vertices are equal and when the three intersection points $AC\cap A'C'$, $AB\cap A'B'$ and $BC\cap B'C'$ are collinear. We say ABC and $A'B'C'$ are in perspective *centrally* when no two vertices are equal and when the intersection points $A'A\cap B'B$, $A'A\cap C'C$ and $B'B\cap C'C$ are all equal. Girard Desargues, the father of projective geometry, proved the following theorem in the 17th century. It was first published in 1648 by Abraham Bosse.

Theorem 1. *Two triangles in the real projective plane are in perspective centrally if and only if they are in perspective axially.*

These properties of perspective only make sense when we know that any two lines intersect at exactly one point. Thus the natural setting for studying these properties is projective space, rather than the more familiar affine space. We restrict ourselves to projective planes because in higher dimensions Desargues' theorem follows from the fact that any two planes intersect in a line.

1. Affine and Projective Planes

An *affine plane* consists of a set $\mathbf{P}$ of points and a set $\mathbf{L}$ of subsets of $\mathbf{P}$ satisfying the following three axioms.

AP1 Any two distinct points lie on one and only one line.
AP2 Given a line l and a point P not contained in l there is one and only one line which contains P and is disjoint from l.
AP3 There are three points not all contained in a single line.

Members of $\mathbf{L}$ are called *lines*. Two lines are called *parallel* if they are equal or disjoint. A set of points is called *collinear* if there is a line containing the set. In these terms **AP2** states that through any point off a given line there passes exactly one parallel line, and **AP3** postulates the existence of three non-collinear points.

Example 1. Our first example is the Euclidean plane $\mathbb{R}^2$. In this setting the axioms are familiar facts about points and lines.

Example 2. One can check that the familiar points and lines of $\mathbb{R}^3$ do not form an affine space. To talk about the affine geometry of $\mathbb{R}^3$ one would need to modify and extend the axioms to take planes into account.

Example 3. What is the smallest affine plane? By **AP3** we have three distinct points P, Q and R. Since any two points lie on exactly one line we get three lines PQ, PR and QR. Since the three points are not collinear the three lines are distinct. We do not yet have an affine plane because **AP2** fails. Taking the line QR and the point P we are forced to introduce a line l containing P and disjoint from QR. The line PQ and the point R fail **AP2** so we need a line m through R and disjoint from PQ. But now R and l fail **AP2** unless one of QR and m intersects l. Since l was assumed disjoint from QR we require l and m to intersect at a point S. Check that S cannot belong to $\{P, Q, R\}$. Adding the line SQ as we must we arrive at an affine plane.

A *projective plane* consists of a set **P** of *points*, a set **L** of *lines*, and an incidence relation telling us which points lie on which lines, satisfying the following axioms.

- **PP1** There is a unique line containing any two distinct points.
- **PP2** There is a unique point lying on any two distinct lines.
- **PP3** There exist three points not all on the same line.
- **PP4** There exist three points on any given line.

Again a set of points is called *collinear* if there is a line containing the set. We call a set of lines *concurrent* if there is a point common to all lines in the set. We call the unique line defined by two points the *join* of the points and the unique point lying on two lines the *intersection* of the two lines. Since no two lines are disjoint, we no longer have a notion of parallelism. We call any collection of three non-collinear points a *triangle*.

Example 4. The Fano plane is a projective plane.

Example 5. Here we extend the affine plane $\mathbb{R}^2$ to a projective plane $\mathrm{E}\mathbb{R}^2$. Let **P** be the set of points in $\mathbb{R}^2$ and let **L** be the set of lines in $\mathbb{R}^2$. For any line ℓ in $\mathbb{R}^2$ define the pencil $\mathcal{P}_\ell$ to be the set of all lines parallel to ℓ. For each pencil append to **P** a point P_ℓ, called a *point at infinity.* Add to each line its corresponding point at infinity. Finally, append to **L** the set of all points at infinity. This line is called the line at infinity.

2. The Desargues configuration

When Desargues' theorem holds in a projective plane we get ten points and ten lines with each line containing exactly three of the ten points and any three lines intersecting at exactly one of the ten points. This is the so called *Desargues configuration.* It is self-dual in the sense that the following exchanges

$$\text{points} \leftrightarrow \text{lines}$$
$$\text{collinear} \leftrightarrow \text{concurrent}$$
$$\text{join} \leftrightarrow \text{intersection}$$

result in the same diagram. Via this duality we get the correspondence

$$\text{triangles in perspective axially} \leftrightarrow \text{triangles in perspective centrally}$$

so duality lets us get away with only proving or disproving one of the implications in Desargues' theorem.

3. A non-Desarguesian projective plane

Here we give an example of a projective plane in which Desargues' theorem does not hold. Our example is the projective geometry of one-way-refracted light-rays at an interface. Take points to be the points of the extended Euclidean plane. For lines we take the set of lines in $\mathrm{E}\mathbb{R}^2$, remove the extended Euclidean lines with positive slope and adjoin the refracted Euclidean lines

$$y = \begin{cases} \frac{1}{2}(x-a)\tan\theta & x > a \\ (x-a)\tan\theta & x \le a \end{cases}$$

as a varies over $\mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$, each with the point at infinity corresponding to the upper ray adjoined. We can think of the lines as light rays and the x-axis as a material boundary (say between air and water) where only light rays of positive slope are refracted.

One can check that we do get a projective plane. We now look at two underwater triangles with parallel sides which are in perspective centrally in $\mathrm{E}\mathbb{R}^2$ at a point O above the water. Since the pairs of sides are parallel the triangles are in perspective axially, the three intersection points lying on the line at infinity. However the two triangles are not in perspective centrally because we replaced one of the lines we needed with a refracted line.

4. Projective planes over division rings

A division ring is an algebraic object satisfying all the field axioms except commutativity of multiplication. The quaternions are an example of a division ring which is not a field. In this section we show how to get a projective plane $\mathrm{P}(2,\mathbb{D})$ from a division ring $\mathbb{D}$ and prove that Desargues' theorem is true in such a projective plane.

Let $\mathbb{D}$ be a division ring and define $\mathrm{P}(2,\mathbb{D})$ to consist of

$$\mathbf{P} = \{(x,y,z) \in \mathbb{D}^3 \ : \ \{x,y,z\} \neq \{0\}\} \ / \ (x,y,z) \sim (x\rho, y\rho, z\rho)$$
$$\mathbf{L} = \{(a,b,c) \in \mathbb{D}^3 \ : \ \{a,b,c\} \neq \{0\}\} \ / \ (a,b,c) \sim (\rho a, \rho b, \rho c)$$

where ρ ranges over the non-zero elements of $\mathbb{D}$. We say that a point (x,y,z) lies on a line (a,b,c) if and only if

$$ax + by + cz = 0.$$

This is easily seen to be well-defined. We are careful to place ρ on the left for lines and on the right for points because multiplication need not be commutative. Given a point A of $\mathbf{P}$ we call any representative $a = (x,y,z)$ of A *homogeneous coordinates* for A. For brevity we use the corresponding lower-case letter to denote homogeneous coordinates of a point.

When $\mathbb{D}$ is a field we can identify $\mathbf{P}$ with the one-dimensional subspaces of $\mathbb{D}^3$ and $\mathbf{L}$ with the two-dimensional subspaces of $\mathbb{D}^3$.

Example 6. $\mathrm{P}(2,\mathbb{F}_2)$ is the Fano plane.

Example 7. $\mathrm{P}(2,\mathbb{R})$ is the same as $\mathrm{E}\mathbb{R}^2$.

Lemma 1. *Let A and A' be distinct points in $P(2,\mathbb{D})$. Let $a = (x,y,z)$ and $a' = (r,s,t)$ be homogeneous coordinates for A and A' respectively. Then the unique line containing A and A' contains exactly the points having homogeneous coordinates*

$$a\lambda + a'\lambda' = (x\lambda + r\lambda', y\lambda + s\lambda', z\lambda + t\lambda')$$

for any λ and λ' from $\mathbb{D}$ not both zero.

Theorem 2. *Let $\mathbb{D}$ be a division ring distinct from $\mathbb{F}_2$. Then Desargues' theorem holds in* $\mathrm{P}(2,\mathbb{D})$.

Proof. Let ABC and $A'B'C'$ be triangles in $\mathrm{P}(2,\mathbb{D})$ which are in perspective centrally. Let the points have homogeneous coordinates denoted by a, b, c, a', b', c' respectively. By hypothesis there is a unique point O lying on the three lines $A'A$, $B'B$ and $C'C$. Let o be homogeneous coordinates for O. Applying the lemma to A and A', then B and B' and finally C and C', we can find elements of $\mathbb{D}$ such that

$$o = a\lambda + a'\lambda'$$
$$o = b\mu + b'\mu'$$
$$o = c\nu + c'\nu'$$

Thus we have points

$$n = a\lambda - b\mu = b'\mu' - a'\lambda'$$
$$m = c\nu - a\lambda = a'\lambda' - c'\nu'$$
$$l = b\mu - c\nu = c'\nu' - b'\mu'$$

which lie on $AB \cap A'B'$, $AC \cap A'C'$ and $BC \cap B'C'$ respectively. But

$$l + m + n = 0$$

so the three points are collinear. □

We excluded $\mathbb{F}_2$ because $\mathrm{P}(2, \mathbb{F}_{\not\approx})$, the Fano plane, does not contain enough points for a Desargues configuration.

5. Introducing coordinates

Let π be a projective plane with points $\mathbf{P}$ and lines $\mathbf{L}$. Fix a triangle OXY in π. Think of O as the origin, OX as the x-axis, and OY as the y-axis. Think of the line XY as the line at infinity. Let I be a point on XY distinct from X and from Y. Let U be a point on OI distint from O and from I. Let Δ be a set which is in bijective correspondence with the points on OI which are distinct from I. Think of Δ as the diagonal. We relabel the elements of Δ so that

$$O \leftrightarrow 0 \qquad U \leftrightarrow 1$$

and denote by c the element of Δ corresponding to any point C of OI not belonging to $\{O, U, I\}$. Define the *coordinates* of such a point C to be (c, c). The coordinates of O are $(0, 0)$ and the coordinates of I are $(1, 1)$.

We now coordinatize the rest of the points off the line at infinity. Let P be a point which does not lie on XY. Put $A = YP \cap OI$ and $B = XP \cap OI$. The coordinates of A are (a, a) and the coordinates of B are (b, b). We define the coordinates of P to be (a, b), calling a the x-coordinate and b the y-coordinate. If P happens to lie on OI then $A = B = P$ so the coordinates are consistent with our earlier labeling of points on OI. Note that the x-coordinate is zero if and only if P lies on OY and the y-coordinate is zero if and only if P lies on OX.

Finally we coordinatize the points of XY. Give Y the coordinate (∞). For any point M on XY distinct from Y, let $T = YU \cap OM$. Let T have coordinates $(1, m)$. We define the coordinate of M to be (m).

The idea now is to use properties of projective planes to put structures of addition and multiplication on Δ.

Example 8. To add elements x and y of Δ consider the points $A = (x, 0)$ and $B = (0, y)$ and define $x + y$ to be the y-coordinate of the point $YA \cap IB$.

Example 9. To multiply elements x and y of Δ let M be the point (y) and let A be the point $(x, 0)$. Define $x \cdot y$ to be the y-coordinate of $Z = OM \cap AX$.

A *loop* is a set with a binary operation which has an identity and in which the equations $ax = b$ and $ya = b$ can be solved uniquely for x and y.

Theorem 3. *For any projective space π the systems $(\Delta, +)$ and $(\Delta\backslash\{0\}, \cdot)$ are loops.*

Additional algebraic properties of the two loops $(\Delta, +)$ and $(\Delta\backslash\{0\}, \cdot)$ correspond to additional geometric properties of the plane.

Theorem 4. *Let π be a projective plane. Desargues' theorem is true in π if and only if Δ is a division ring.*

6. Things not covered

- The transformations of projective planes.
- Projective spaces.
- Pappus' theorem.
- Conics in projective planes.

References

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