DIRECT SUMS OF QUASI-BAER MODULES

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Abstract. While the research work on quasi-Baer rings has been quite extensive in existing literature, the study of quasi-Baer modules introduced in 2004 remains quite limited with only little work available on the notion. In this paper, we extend the study of quasi-Baer modules. A characterization of a quasi-Baer module in terms of its endomorphism ring is obtained. We provide a complete characterization of arbitrary direct sums of any quasi-Baer modules to be quasi-Baer. We also show characterizations of some important properties of quasi-Baer modules. Examples which delineate the concepts and the results are provided.

1. Introduction

Quasi-Baer rings have been extensively studied (see for example, [1, 2, 5, 6] among many other works). In a general module theoretic setting, the notion of a quasi-Baer module via the endomorphism ring of the module was introduced, in 2004, by Rizvi and Roman [11]. A module M is said to be *quasi-Baer* if the right annihilator in M of any 2-sided ideal of $\mathrm{End}_R(M)$ is a direct summand of M. The research on quasi-Baer modules has been limited and this notion has received little attention beyond [11] (and [3]). The purpose of this paper is to further the study of quasi-Baer modules. Conditions which allow for the quasi-Baer property of the endomorphism ring of a module to transfer back to the module are discussed. We introduce a new notion of q-local-retractability of a module, which is shown to be inherent in every quasi-Baer module. We use this to obtain a complete characterization of a quasi-Baer module in terms of its endomorphism ring. It was shown by Rizvi and Roman that every direct summand of a quasi-Baer module is quasi-Baer, while a direct sum of quasi-Baer modules is not always a quasi-Baer module. A partial answer was provided by them for direct sums of quasi-Baer modules to be quasi-Baer, more specifically, for the case when the direct sum consists of copies of the same module (see [11]). In this paper, we consider the question for arbitrary direct sums of distinct quasi-Baer modules to be quasi-Baer. In particular, we fully characterize when a direct sum of arbitrary quasi-Baer modules is quasi-Baer by defining a relative p.q.-Baer property inherently satisfied by a direct sum of modules which is quasi-Baer. Earlier results on special direct sums of quasi-Baer modules follow as a consequence of our theorem. Examples illustrating our results and notions are provided throughout the paper.

After the introduction, in Section 2 we provide some basic results and a characterization of a quasi-Baer module in terms of its endomorphism ring by using the q-local-retractability which is an inherent property of the quasi-Baer module. Our focus in Section 3 is on the question of when is a direct sum of quasi-Baer modules quasi-Baer. We prove that $\bigoplus_{i\in\Lambda} M_i$ is quasi-Baer if and only if M_i is quasi-Baer and the class $\{M_i\}_{i\in\Lambda}$ is relatively p.q.-Baer where Λ is any index set. As a consequence, we show that $\mathbb{Z}^{(\mathbb{R})} \oplus \mathbb{Q}^{(\mathbb{R})}$ is a quasi-Baer $\mathbb{Z}$-module, which is not a Baer $\mathbb{Z}$-module. Our result yields that a direct sum of quasi-Baer modules is quasi-Baer if each summand is fully invariant in the direct sum. If M is a direct sum of cyclic modules over a commutative principal ideal domain, we prove that M is quasi-Baer if and only if M is either semisimple or torsion-free. An alternate proof characterizing a right QI-ring in terms of quasi-Baer modules is also included.

Throughout this paper, R is a ring with unity and M is a unital right R-module. For a right R-module M, $S = \text{End}_R(M)$ will denote the endomorphism ring of M; thus M can be viewed as a left S- right R-bimodule. For $\varphi \in S$, $\text{Ker}\varphi$ and $\text{Im}\varphi$ stand for the kernel and the image of φ, respectively. The notations $N \leq M$, $N \trianglelefteq M$, $N \leq^{\text{ess}} M$, $N \leq^{\oplus} M$, $N \trianglelefteq^{\text{ess}} M$, and $N \trianglelefteq^{\oplus} M$ mean that N is a submodule, a fully invariant submodule, an essential submodule, a direct summand, a fully invariant essential submodule, and a fully invariant direct summand of M, respectively. We use $M^{(n)}$ to denote the direct sum of n copies of M. By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of real, rational, integer, and natural numbers, respectively. For $1 < n \in \mathbb{N}$, $\mathbb{Z}_n$ denotes the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ and $E(M)$ denotes the injective hull of M. We also denote $\mathbf{r}_M(X) = \{m \in M \mid Xm = 0\}$, $\mathbf{r}_S(X) = \{\varphi \in S \mid X\varphi = 0\}$, and $\mathbf{l}_S(X) = \{\varphi \in S \mid \varphi X = 0\}$ for $\emptyset \neq X \subseteq S$; $\mathbf{l}_S(N) = \{\varphi \in S \mid \varphi N = 0\}$ for $N \leq M$; $\mathbf{r}_R(I) = \{r \in R \mid Ir = 0\}$ and $\mathbf{l}_R(I) = \{r \in R \mid rI = 0\}$ for $\emptyset \neq I \subseteq R$.

2. Quasi-Baer modules

In 2004, Rizvi and Roman proved that the endomorphism ring of a quasi-Baer module is a quasi-Baer ring, while the converse did not hold true in general. To obtain a complete converse, in this section we introduce the notion of the q-local-retractability. It is shown that this is an inherent property of every quasi-Baer module. Using this notion, we obtain a complete characterization of a quasi-Baer module in terms of its endomorphism ring. We also provide a characterization of the closely connected notion of FI-$\mathcal{K}$-nonsingularity satisfied by every quasi-Baer module. It is shown that for an FI-$\mathcal{K}$-nonsingular module, essential closures of a fully invariant submodule, which happen to be direct summands, are equal.

We begin with the following definition due to Rizvi and Roman [11].

Definition 2.1. Let M be a right R-module and $S = \text{End}_R(M)$. Then M is said to be a *quasi-Baer module* if for any 2-sided ideal J of S, $\mathbf{r}_M(J) = eM$ for some $e^2 = e \in S$.

Remark 2.2. A module M is quasi-Baer if and only if for any $N \trianglelefteq M, \mathbf{l}_S(N) = Sf$ for some $f^2 = f \in S = \text{End}_R(M)$.

Example 2.3. (i) R_R is a quasi-Baer module if and only if R is a quasi-Baer ring.

(ii) Every semisimple module is a quasi-Baer module.

(iii) Any Baer module is quasi-Baer. Recall from [11] that a module M is said to be *Baer* if for each $\emptyset \neq X \subseteq \text{End}_R(M)$, $\mathbf{r}_M(X) = eM$ for some $e^2 = e \in \text{End}_R(M)$.

(iv) Every endoprime module is quasi-Baer. Recall that a module M is said to be *endoprime* if $\mathbf{l}_S(N) = 0$ for all $0 \neq N \trianglelefteq M$ and $S = \text{End}_R(M)$ ([4]).

(v) Any free (projective) module over a quasi-Baer ring is quasi-Baer.

(vi) $\mathbb{Z}^{(\mathbb{R})}$ is a quasi-Baer $\mathbb{Z}$-module, but it is not Baer.

Recall from [11] that a module M is said to be *FI-$\mathcal{K}$-nonsingular* if, for all 2-sided ideals J of $\text{End}_R(M)$, $\mathbf{r}_M(J) \leq^{\text{ess}} eM$ with $e^2 = e \in \text{End}_R(M)$ implies that $Je = 0$. It is known that every quasi-Baer module is FI-$\mathcal{K}$-nonsingular (see Theorem 2.6). Recall that in every nonsingular module, essential closures of a submodule are unique. Our next result shows that for an FI-$\mathcal{K}$-nonsingular module M, direct summand essential closures of a fully invariant submodule of M are unique. Further, we obtain a characterization of an FI-$\mathcal{K}$-nonsingular module.

Theorem 2.4. *The following statements hold true.*

(i) *Let M be a FI-$\mathcal{K}$-nonsingular module and $N \trianglelefteq M$. If $N \leq^{\text{ess}} N_i \leq^{\oplus} M$ for $i = 1, 2$ then $N_1 = N_2$.*

(ii) *A module M is FI-$\mathcal{K}$-nonsingular if and only if any $N \trianglelefteq M$ with $N \leq^{\text{ess}} eM$ implies that $\mathbf{r}_M(\mathbf{l}_S(N)) = eM$ where $e^2 = e \in S = \text{End}_R(M)$.*

Proof. (i) Let M be FI-$\mathcal{K}$-nonsingular and $N \trianglelefteq M$ such that $N \leq^{\text{ess}} N_i = e_i M \leq^{\oplus} M$ for $e_i^2 = e_i \in S$ where $i = 1, 2$. Then $N \leq \mathbf{r}_M(\mathbf{l}_S(N)) \leq^{\text{ess}} e_i M$. Hence $\mathbf{l}_S(N)e_i = 0$ as the FI-$\mathcal{K}$-nonsingularity. Thus, $\mathbf{l}_S(N) \leq S(1-e_i)$ implies that $\mathbf{r}_M(\mathbf{l}_S(N)) \geq e_i M$. So, $\mathbf{r}_M(\mathbf{l}_S(N)) = e_i M$ for $i = 1, 2$. Therefore $N_1 = N_2$.

(ii) Let M be FI-$\mathcal{K}$-nonsingular and N be a fully invariant submodule of M such that $N \leq^{\text{ess}} eM$ for some $e^2 = e \in S$. In the proof of part (i), $\mathbf{r}_M(\mathbf{l}_S(N)) = eM$. Conversely, take J a 2-sided ideal of S such that $\mathbf{r}_M(J) \leq^{\text{ess}} eM$ for some $e^2 = e \in S$. As $\mathbf{r}_M(J) \trianglelefteq M$ and $\mathbf{r}_M(J) = \mathbf{r}_M(\mathbf{l}_S(\mathbf{r}_M(J)))$, by hypothesis $\mathbf{r}_M(J) = eM$ implies that $Je = 0$. Thus, M is FI-$\mathcal{K}$-nonsingular. □

A module M is called *$\mathcal{K}$-nonsingular* if, for any $0 \neq \varphi \in \text{End}_R(M)$, $\text{Ker}\varphi \leq^{\text{ess}} M$ implies that $\varphi = 0$ ([11, Definition 2.5]). Note that every $\mathcal{K}$-nonsingular module is FI-$\mathcal{K}$-nonsingular, but the converse does not hold.

Remark 2.5. (i) It is of interest to compare Theorem 2.4(i) with [12, Proposition 2.8], which states: If M is $\mathcal{K}$-nonsingular, $N \leq M$, and $N \leq^{\text{ess}} N_i \leq^{\oplus} M$ for $i = 1, 2$, then $N_1 = N_2$.

(ii) From Theorem 2.4(ii), it is easy to check that for an FI-$\mathcal{K}$-nonsingular module M, $N \trianglelefteq^{\text{ess}} M$ yields $\mathbf{l}_S(N) = 0$. Also, it is interesting to compare this with the fact that for a $\mathcal{K}$-nonsingular module M, $N \leq^{\text{ess}} M$ implies that $\mathbf{l}_S(N) = 0$.

Recall that a module M is said to be *FI-extending* if, for any $N \trianglelefteq M$, there exists $e^2 = e \in \text{End}_R(M)$ such that $N \leq^{\text{ess}} eM$ (see [1, Section 2.3] for more details). A module M is called *FI-$\mathcal{K}$-cononsingular* if, for all $L \trianglelefteq^{\oplus} M$ and $N \trianglelefteq L$ such that $\varphi N \neq 0$ for all $0 \neq \varphi \in \text{End}_R(L)$, we get that $N \leq^{\text{ess}} L$ ([11, Definition 3.7]), equivalently, if for any $N \trianglelefteq M$, $\mathbf{r}_M(\mathbf{l}_S(N)) \leq^{\oplus} M$ implies that $N \leq^{\text{ess}} \mathbf{r}_M(\mathbf{l}_S(N))$ (see [11, Proposition 3.8]). It is shown that every FI-extending module is *FI-$\mathcal{K}$-*cononsingular.

Theorem 2.6. ([11, Theorem 3.10]) *A module M is FI-$\mathcal{K}$-cononsingular quasi-Baer if and only if M is an FI-$\mathcal{K}$-nonsingular FI-extending module.*

Next, we recall [1, Theorem 3.2.38]: Let R be a right nonsingular ring. Then R_R is FI-extending if and only if R is quasi-Baer and $J \leq^{\text{ess}} \mathbf{r}_R(\mathbf{l}_R(J))$ for all $J \trianglelefteq R$.

As a consequence of Theorem 2.4, our next two corollaries improve and extend this result.

Corollary 2.7. *A module M is quasi-Baer and $N \leq^{\text{ess}} \mathbf{r}_M(\mathbf{l}_S(N))$ for all $N \trianglelefteq M$ where $S = \text{End}_R(M)$ if and only if M is an FI-$\mathcal{K}$-nonsingular FI-extending module.*

Proof. It directly follows from Theorems 2.4(ii) and 2.6. □

For an alternate look at connections between a quasi-Baer ring and a right FI-extending ring, we define the following two notions: A ring R is called *right FI-nonsingular* if for any 2-sided ideal J of R, $J \leq^{\text{ess}} eR$ for some $e^2 = e \in R$ implies $\mathbf{r}_R(\mathbf{l}_R(J)) = eR$. A ring R is called *right FI-cononsingular* if for any 2-sided ideal J of R, $\mathbf{r}_R(\mathbf{l}_R(J)) = eR$ for some $e^2 = e \in R$ implies $J \leq^{\text{ess}} eR$.

It is easy to see that, when $M = R_R$, the notions of the FI-$\mathcal{K}$-nonsingularity (resp., FI-$\mathcal{K}$-cononsingularity) for a module and the right FI-nonsingularity (resp., right FI-cononsingularity) for a ring coincide. We remark that the proof of Theorem 2.6 depends on four Lemmas. The next result also follows from Theorem 2.6, however, we provide an alternate short proof for the case of rings.

Corollary 2.8. *A ring R is right FI-cononsingular quasi-Baer if and only if R is a right FI-nonsingular, right FI-extending ring.*

Proof. Let R be a right FI-cononsingular quasi-Baer ring and J be a 2-sided ideal of R. Then $\mathbf{l}_R(J) = Re$ for some $e^2 = e \in R$ as R is quasi-Baer. Since $\mathbf{r}_R(\mathbf{l}_R(J)) = (1-e)R$, by the right FI-cononsingularity of R, $J \leq^{\text{ess}} (1-e)R$. Thus, R is a right FI-extending ring. Next, suppose $J \leq^{\text{ess}} fR$. Since $J \leq^{\text{ess}} \mathbf{r}_R(\mathbf{l}_R(J)) \leq^{\text{ess}} fR$ and $\mathbf{r}_R(\mathbf{l}_R(J)) = gR$ for some $g^2 = g \in R$ as R is quasi-Baer, $fR = gR$. Therefore R is right FI-nonsingular.

Conversely, let R be a right FI-nonsingular, right FI-extending ring and J be a 2-sided ideal of R. Then there exists $e^2 = e \in R$ such that $J \leq^{\text{ess}} eR$ because R is right FI-extending. By the right FI-nonsingularity of R, $\mathbf{r}_R(\mathbf{l}_R(J)) = eR$. Thus, $\mathbf{l}_R(J) = \mathbf{l}_R(\mathbf{r}_R(\mathbf{l}_R(J))) = R(1-e)$. Hence R is a quasi-Baer ring. Next, suppose $\mathbf{r}_R(\mathbf{l}_R(J)) = eR$ for some $e^2 = e \in R$. Since R is right FI-extending, there exists $f^2 = f \in R$ such that $J \leq^{\text{ess}} fR$. As $J \leq^{\text{ess}} \mathbf{r}_R(\mathbf{l}_R(J)) \leq^{\text{ess}} fR$, $eR = fR$. So R is right FI-cononsingular. □

While a Baer module has been characterized in terms of its endomorphism ring (see Theorem 2.19), there is no characterization of a quasi-Baer module connecting to its endomorphism ring until now. The only known results in this regard for quasi-Baer modules are shown in the following proposition.

Recall that a module M is said to be *retractable* if $\text{Hom}_R(M, N) \neq 0$ for all $0 \neq N \leq M$ (see [7]).

Proposition 2.9. *The following statements hold true.*

(i) ([11, Theorem 4.1]) *The endomorphism ring of every quasi-Baer module is a quasi-Baer ring.*

(ii) ([11, Proposition 4.7]) *Let M be retractable. Then M is a quasi-Baer module if and only if* $\text{End}_R(M)$ *is a quasi-Baer ring.*

Our next example illustrates that the converse of Proposition 2.9(i) is not true in general.

Example 2.10. (i) Let $M = \mathbb{Z}_{p^\infty}$ be a $\mathbb{Z}$-module. Then while M is not a quasi-Baer module, $\text{End}_{\mathbb{Z}}(M)$ is a domain, hence a quasi-Baer ring.

(ii) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and $e = \begin{pmatrix} 1 & \bar{0} \\ 0 & \bar{0} \end{pmatrix}$ be an idempotent of the ring R. Consider $M = eR = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ as a right R-module. Since $S = \text{End}_R(M) = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$, it is a quasi-Baer ring. However, M is not quasi-Baer: For $\varphi = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in S$, $\mathbf{r}_M(\varphi S) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, which is not a direct summand of M.

Recall that a module M is called *quasi-retractable* if $\text{Hom}_R(M, \mathbf{r}_M(I)) \neq 0$ for all $0 \neq \mathbf{r}_M(I)$ where I is any left ideal of $S = \text{End}_R(M)$, i.e., $\mathbf{r}_M(I) \neq 0$ implies that $\mathbf{r}_S(I) \neq 0$ for any left ideal I of S ([13, Definition 2.3]). Clearly, every retractable module is quasi-retractable.

Now, we extend Proposition 2.9(ii) to quasi-retractable modules in the following result.

Proposition 2.11. *Let M be quasi-retractable. Then M is a quasi-Baer module if and only if* $\mathrm{End}_R(M)$ *is a quasi-Baer ring.*

Proof. From Proposition 2.9(i), it remains to show the sufficient condition. Let $S = \mathrm{End}_R(M)$ be a quasi-Baer ring. Then for a given 2-sided ideal J, there exists $e \in S$ such that $\mathbf{r}_S(J) = eS$. Thus, $Je = 0$. Hence, $eM \subseteq \mathbf{r}_M(J)$. Assume that $0 \neq m \in \mathbf{r}_M(J) \setminus eM$. We may assume $0 \neq m = (1-e)m \in \mathbf{r}_M(J)$. Hence, $0 \neq m \in \mathbf{r}_M(J) \cap (1-e)M = \mathbf{r}_M(J) \cap \mathbf{r}_M(Se) = \mathbf{r}_M(J + Se)$. Since M is quasi-retractable, $\mathbf{r}_S(J + Se) \neq 0$, which contradicts that $\mathbf{r}_S(J + Se) = \mathbf{r}_S(J) \cap \mathbf{r}_S(Se) = eS \cap (1-e)S = 0$. Thus, $\mathbf{r}_M(J) = eM$. Therefore M is a quasi-Baer module. □

The next example exhibits that Proposition 2.11 sharpens Proposition 2.9(ii).

Example 2.12. (i) Let R be a commutative domain which is not a field and F be the field of fractions of R. Then F_R is quasi-Baer and quasi-retractable. But F_R is not retractable. For, since $\varphi F = F$ for every $0 \neq \varphi \in \mathrm{End}_R(F) \cong F$, there does not exist $0 \neq \psi \in \mathrm{End}_R(F)$ such that $\psi F \subseteq R$.

(ii) Let $M = \mathbb{Q} \oplus \mathbb{Z}_2$ be a $\mathbb{Z}$-module. Then M is quasi-Baer and quasi-retractable. However, M is not retractable. (See [12, Example 2.5].)

From Proposition 2.9 and Example 2.10, it is clear that the endomorphism ring of a quasi-Baer module inherits the quasi-Baer property without any additional conditions, while the converse does not hold true. Proposition 2.9, Proposition 2.11, and the preceding examples motivate us to find a suitable retractability condition which can help provide a full characterization of a quasi-Baer module via its endomorphism ring. We introduce the notion of q-local-retractability and show that every quasi-Baer module satisfies this condition. This notion is shown to be useful in obtaining the required characterization.

Recall that a module M is called *local-retractable* if $\mathbf{r}_M(I) = \mathbf{r}_S(I)(M)$ for any left ideal I of $S = \mathrm{End}_R(M)$ ([10, Definition 2.18]). To obtain a characterization of a quasi-Baer module via its endomorphism ring, we define the following notion which is the 2-sided ideal version of the local-retractability.

Definition 2.13. A module M is called *q-local-retractable* if $\mathbf{r}_M(J) = \mathbf{r}_S(J)(M)$ for any 2-sided ideal J of $S = \mathrm{End}_R(M)$.

Example 2.14. (i) Any free module is q-local-retractable.

(ii) Every local-retractable module is q-local-retractable.

(iii) Any $\mathfrak{L}$-Rickart module is q-local-retractable (see [10, Lemma 2.21] and (ii)). Recall that a module M is called *$\mathfrak{L}$-Rickart* if for any $m \in M$, $\mathbf{l}_S(m) = Se$ for some $e^2 = e \in S = \mathrm{End}_R(M)$.

(iv) Baer modules and (Zelmanowitz) regular modules are q-local-retractable (see [10, Corollary 2.12] and (iii)).

For our next characterization, see also [1, Corollary 5.6.6].

Theorem 2.15. *A module M is quasi-Baer if and only if* $\mathrm{End}_R(M)$ *is a quasi-Baer ring and M is q-local-retractable.*

Proof. We first show that every quasi-Baer module is q-local-retractable: Let M be a quasi-Baer module. For any 2-sided ideal J of $S = \mathrm{End}_R(M)$, there exists $e^2 = e \in S$ with $\mathbf{r}_M(J) = eM$ and $\mathbf{r}_S(J) = eS$. So $\mathbf{r}_S(J)(M) = eS(M) = eM = \mathbf{r}_M(J)$. Thus M is q-local-retractable. The necessary condition now follows from Proposition 2.9(i).

Conversely, let J be any 2-sided ideal of S. Then there exists $e^2 = e \in S$ such that $\mathbf{r}_S(J) = eS$. Since M is q-local-retractable $\mathbf{r}_M(J) = \mathbf{r}_S(J)(M) = eS(M) = eM$. Therefore, M is a quasi-Baer module. □

We remark that the two conditions "$\mathrm{End}_R(M)$ is a quasi-Baer ring" and "M is q-local-retractable" are independent. Indeed, $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ is a quasi-Baer ring while $\mathbb{Z}_{p^\infty}$ is not a q-local-retractable $\mathbb{Z}$-module where p is a prime number. On the other hand, $\mathbb{Z}_4$ and $\mathbb{Z} \oplus \mathbb{Z}_2$ are q-local-retractable $\mathbb{Z}$-modules, but the endomorphism ring of each is not quasi-Baer.

Let M and N be right R-modules. By $\mathrm{Tr}_R(M, N) = \sum \{\mathrm{Im}\varphi \mid \varphi \in \mathrm{Hom}_R(M, N)\}$, we denote the *trace* of M in N. The following lemma is due to N. Tung [14].

Lemma 2.16. $\mathrm{Tr}_R(M, \mathbf{r}_M(J)) = \mathbf{r}_S(J)(M)$ *for any 2-sided ideal J of $S = \mathrm{End}_R(M)$.*

Proof. Let $n \in \mathrm{Tr}_R(M, \mathbf{r}_M(J))$. Then there exists $0 \neq \varphi \in \mathrm{Hom}_R(M, \mathbf{r}_M(J)) \subseteq S$ such that $n \in \varphi M \subseteq \mathbf{r}_M(J)$. Since $J\varphi M = 0$, $\varphi \in \mathbf{r}_S(J)$. Therefore $n \in \mathbf{r}_S(J)(M)$. For the reverse inclusion, let $n \in \mathbf{r}_S(J)(M)$ be arbitrary. Then there exists $\varphi \in \mathbf{r}_S(J)$ such that $n \in \varphi M$. Since $J\varphi = 0$, $\varphi M \subseteq \mathbf{r}_M(J)$. So $n \in \mathrm{Tr}_R(M, \mathbf{r}_M(J))$. □

Next, the q-local-retractability of a module is characterized as follows.

Proposition 2.17. *The following conditions are equivalent for a module M:*

(a) *M is q-local-retractable;*
(b) $\mathrm{Tr}_R(M, \mathbf{r}_M(J)) = \mathbf{r}_M(J)$ *for any 2-sided ideal J of* $\mathrm{End}_R(M)$*;*
(c) *for any 2-sided ideal J of* $\mathrm{End}_R(M)$ *and any $0 \neq m \in \mathbf{r}_M(J)$, there exists $\psi_m \in \mathrm{Hom}_R(M, \mathbf{r}_M(J))$ such that $m \in \psi_m(M)$.*

Proof. (a)⇔(b) It directly follows from Lemma 2.16.

(a)⇒(c) Consider any 2-sided ideal J of $S = \mathrm{End}_R(M)$ such that $\mathbf{r}_M(J) \neq 0$. Let any $0 \neq m \in \mathbf{r}_M(J)$. Since $\mathbf{r}_S(J)(M) = \mathbf{r}_M(J)$ as M is q-local-retractable, $m \in \psi(M) \subseteq \mathbf{r}_M(J)$ for some $0 \neq \psi \in \mathbf{r}_S(J)$. Take $\psi = \psi_m$ as desired.

(c)⇒(a) For any 2-sided ideal J of S, $\mathbf{r}_M(J) \supseteq \mathbf{r}_S(J)(M)$ as $J(\mathbf{r}_S(J)(M)) = 0$. For the reverse inclusion, let $0 \neq m \in \mathbf{r}_M(J)$ be arbitrary. Then there exists $0 \neq \psi_m \in S$ such that $m \in \psi_m(M) \subseteq \mathbf{r}_M(J)$ by hypothesis. Since $\psi_m \in \mathbf{r}_S(J)$, $m \in \psi_m(M) \subseteq \mathbf{r}_S(J)(M)$. Therefore $\mathbf{r}_M(J) = \mathbf{r}_S(J)(M)$. □

Remark 2.18. (i) If M is local-retractable then for any element $m \in M$, there exists $\psi \in S$ such that $\mathbf{l}_S(m) = \mathbf{l}_S(\psi)$.
(ii) If M is q-local-retractable then for any element $m \in M$, there exists $\psi \in S$ such that $\mathbf{l}_S(Sm) = \mathbf{l}_S(S\psi)$.

Recall that a Baer module is characterized as follows:

Theorem 2.19. ([13, Theorem 2.5]) *A module M is Baer if and only if* $\mathrm{End}_R(M)$ *is a Baer ring and M is quasi-retractable.*

In view of Theorem 2.15, we provide an analogous characterization of a Baer module via its endomorphism ring and the local-retractability.

Theorem 2.20. *A module M is Baer if and only if* $\mathrm{End}_R(M)$ *is a Baer ring and M is local-retractable.*

Proof. Let M be a Baer module. In view of Theorem 2.19, it is enough to show that M is local-retractable. For any left ideal I of $S = \mathrm{End}_R(M)$, there exists $e^2 = e \in S$ such that $\mathbf{r}_M(I) = eM$ and $\mathbf{r}_S(I) = eS$. So, $\mathbf{r}_S(I)(M) = eS(M) = eM = \mathbf{r}_M(I)$. Therefore M is local-retractable.

Conversely, for any left ideal I of S, $\mathbf{r}_S(I) = eS$ for some $e^2 = e \in S$ as S is Baer. Since M is local-retractable, $\mathbf{r}_M(I) = \mathbf{r}_S(I)(M) = eS(M) = eM$. Therefore, M is a Baer module. □

Corollary 2.21. *The endomorphism ring of a free module F_R is a (quasi-)Baer ring if and only if F_R is a (quasi-)Baer module.*

We next provide an example of a quasi-retractable module which is not local-retractable.

Example 2.22. ([8, Example 3.13]) For a ring A, let $\text{Aut}(A)$ denote the *group of ring automorphisms of* A. Let G be a subgroup of $\text{Aut}(A)$. For $a \in A$ and $g \in G$, we let a^g denote the image of a under g. We use A^G to denote the *fixed ring* of A under G, i.e., $A^G = \{a \in A \mid a^g = a \text{ for every } g \in G\}$.

The *skew group ring*, $A * G$, is denoted to be $A * G = \bigoplus \sum_{g \in G} Ag$ with addition given componentwise and multiplication given as follows: For $a, b \in A$ and $g, h \in G$, $(ag)(bh) = ab^{g^{-1}}gh \in Agh$. Also, say $G = \{g_1, g_2, \cdots, g_n\}$. For $a \in A$ and $\beta = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in A * G$ with $a_i \in A$, define

$$a \cdot \beta = a^{g_1}a_1^{g_1} + a^{g_2}a_2^{g_2} + \cdots + a^{g_n}a_n^{g_n}.$$

Then A is a right $A * G$-module.

Let $A = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, the 2×2 upper triangular matrix ring over $\mathbb{Z}_2$. Note that A is a quasi-Baer ring. Let $g \in \text{Aut}(A)$ be the conjugation by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, i.e., $a^g = g^{-1}ag$ for $a \in A$. Then $g^2 = 1$ since the characteristic of $\mathbb{Z}_2$ is 2. Let $G = \{1, g\}$ be a subgroup of $\text{Aut}(A)$, $R = A * G$ be the skew group ring of G over A, and $M_R = A_{A*G}$ be a right $A * G$-module. Then

$$S := \text{End}_R(M) = A^G = \left\{\varphi_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \varphi_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \varphi_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \varphi_3 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right\}.$$

Since $\mathbf{r}_M(\varphi_0) = M$, $\mathbf{r}_M(\varphi_1) = \left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$, and $\mathbf{r}_M(\varphi_2) = \mathbf{r}_M(\varphi_3) = 0$, it is easy to see that M is quasi-retractable. Next, note that $\mathbf{r}_S(\varphi_1) = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$ and $\mathbf{r}_M(\varphi_1) = \left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\mathbf{r}_S(\varphi_1)(M) = \left(\begin{smallmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$, $\mathbf{r}_S(\varphi_1)(M) \neq \mathbf{r}_M(\varphi_1)$. Thus, M is not local-retractable.

The notion of retractability plays an important role in our study as seen from the previous results. We conclude this section with some properties of the relevant weaker notions of retractability.

Proposition 2.23. *The following hold true for a module M and $S = \text{End}_R(M)$.*

(1) *If M is local-retractable. Then:*
  (1A) *for any left ideal I of S, $\mathbf{r}_M(I) \neq 0$ implies $\mathbf{r}_S(I) \neq 0$.*
  (1B) *for any left ideal I of S such that $Im = 0$ with $0 \neq m \in M$, there exists $\psi \in S$ such that $\mathbf{l}_S(m) \supseteq \mathbf{l}_S(\psi) \supseteq I$.*

(2) *If M is q-local-retractable. Then:*
  (2A) *for any 2-sided ideal J of S, $\mathbf{r}_M(J) \neq 0$ implies $\mathbf{r}_S(J) \neq 0$.*
  (2B) *for any 2-sided ideal J of S such that $Jm = 0$ with $0 \neq m \in M$, there exists $\psi \in S$ such that $\mathbf{l}_S(m) \supseteq \mathbf{l}_S(\psi) \supseteq J$.*

Proof. We only prove (2A) and (2B) because proofs of (1A) and (1B) are similar. Let M be q-local-retractable. (2A) follows directly from the fact that $\mathbf{r}_S(J)(M) = \mathbf{r}_M(J)$ for all 2-sided ideal J of S. Further, from Proposition 2.17, for any 2-sided ideal J of S and any $0 \neq m \in \mathbf{r}_M(J)$, there exists $m \in \psi(M) \subseteq \mathbf{r}_M(J)$. Thus, $\mathbf{l}_S(m) \supseteq \mathbf{l}_S(\psi) \supseteq \mathbf{l}_S(\mathbf{r}_M(J)) \supseteq J$, which proves (2B). □

Note from Proposition 2.23 that (1A) implies (2A) and every local-retractable module is quasi-retractable. The following example illustrates that a quasi-retractable module (hence, satisfying the (1A) condition) may not be local-retractable, further, a module satisfying the (2A) condition may not be q-local-retractable, in general. (See also Example 2.22.)

Example 2.24. ([8, Example 3.14]) Take $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ as a right $\mathbb{Z}$-module where p is a prime number. Since M is retractable, it is easy to see that M satisfies Proposition 2.23(1A) and (2A).

However, M is not local-retractable. (Also, it can be easily checked that M is not q-local-retractable.) For, note that

$$S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p) = \begin{pmatrix} \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_{p^\infty}) \\ 0 & \mathbb{Z}_p \end{pmatrix}.$$

Consider $\varphi = \begin{pmatrix} p^2 & 0 \\ 0 & 0 \end{pmatrix} \in S$. Since $\mathbf{r}_S(\varphi) \cong \begin{pmatrix} 0 & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p \end{pmatrix}$, $\mathbf{r}_S(\varphi)(M) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, but $\mathbf{r}_M(\varphi) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. So, $\mathbf{r}_S(\varphi)(M) \neq \mathbf{r}_M(\varphi)$.

Example 2.25 illustrates that if M is not local-retractable (resp., not q-local-retractable) then M may not satisfy Proposition 2.23(1B) (resp., (2B)).

Example 2.25. Consider $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module and $S = \mathrm{End}_R(M)$ where p is a prime number. Since $\mathbf{l}_S(\varphi) = 0$ for all nonzero $\varphi \in S$, there is no nonzero element $\psi \in S$ such that $\mathbf{l}_S(\psi) \supseteq S\varphi$. Thus, Proposition 2.23(1B) and (2B) do not hold true. Note that M is not q-local-retractable.

3. Direct sums of quasi-Baer modules

Whether or not an algebraic property is inherited by direct summands and direct sums has always been of interest. While the quasi-Baer property is inherited by direct summands, direct sums of quasi-Baer modules are not quasi-Baer, in general. This question about direct sums to inherit respective properties is open for both the case of Baer modules and the case of quasi-Baer modules. We settle this question for direct sums of quasi-Baer modules completely in Theorem 3.6. Rizvi and Roman in [11] provided a partial answer showing that the direct sum of copies of a quasi-Baer module inherits the quasi-Baer property. However, a complete characterization for arbitrary direct sums of distinct quasi-Baer modules to be quasi-Baer remains open. In this section, we fully characterize when an arbitrary direct sum of quasi-Baer modules is quasi-Baer. To do this, we introduce a new notion, which we call the relative p.q.-Baer property for a family of modules.

We start with a result showing the inheritance of the quasi-Baer property by direct summands of quasi-Baer modules. A example which shows that a direct sum of quasi-Baer modules may not inherit the quasi-Baer property, is presented.

Lemma 3.1. ([11, Theorem 3.17]) *Every direct summand of a quasi-Baer module is a quasi-Baer module.*

Example 3.2. Let R be a commutative domain with a nonzero maximal ideal P. Then $R \oplus R/P$ is not quasi-Baer (see Example 3.8), while R and R/P are quasi-Baer R-modules. In particular, $\mathbb{Z} \oplus \mathbb{Z}_p$ is not a quasi-Baer $\mathbb{Z}$-module for any prime number p (while $\mathbb{Z}$ is a domain and $\mathbb{Z}_p$ is simple, hence quasi-Baer $\mathbb{Z}$-modules).

Definition 3.3. Let $\{M_k\}_{k\in\Lambda}$ be a class of right R-modules where Λ is any index set. Set $S_{ij} = \mathrm{Hom}_R(M_j, M_i)$ for all $i, j \in \Lambda$. Then M_i is called *M_j-p.q.-Baer* in $\{M_k\}_{k\in\Lambda}$ if for all $k \in \Lambda$, $\mathbf{r}_{M_i}(\varphi_{jk} S_{ki}) \leq^{\oplus} M_i$ for all $\varphi_{jk} \in S_{jk}$. The class $\{M_k\}_{k\in\Lambda}$ is called *relatively p.q.-Baer* if M_i is M_j-p.q.-Baer in $\{M_k\}_{k\in\Lambda}$ for all $i, j \in \Lambda$.

Note that two modules M_i and M_j are called *relatively p.q.-Baer* in $\{M_k\}_{k\in\Lambda}$ if M_i is M_j-p.q.-Baer and M_j is M_i-p.q.-Baer in $\{M_k\}_{k\in\Lambda}$.

Lemma 3.4. *Let* $M = \bigoplus_{k\in\Lambda} M_k$ *and* $S = \text{End}_R(M)$ *with an index set* Λ. *Set* $S_{ij} = \text{Hom}_R(M_j, M_i)$ *for all* $i, j \in \Lambda$. *Consider* $\varphi = (\varphi_{ij}) \in S$. *Then for each* $k \in \Lambda$, $\mathbf{r}_M(\varphi S) \cap M_k = \bigcap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$ *where* $\varphi_{ij} \in S_{ij}$.

Proof. Since $\mathbf{r}_M(\varphi S) = \oplus_{k\in\Lambda}[\mathbf{r}_M(\varphi S) \cap M_k]$, it suffices to show that

$$\mathbf{r}_M(\varphi S) = \bigoplus_{k\in\Lambda} \left(\cap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})\right).$$

Let $m \in \mathbf{r}_M(\varphi S)$ where $m = (m_k)_{k\in\Lambda}$ with $m_k \in M_k$. Then for all $j \in \Lambda$ and all $k \in \Lambda$, $m \in \mathbf{r}_M(\varphi e_{jj} S e_{kk})$ where e_{jj} is the matrix with 1 in the (j, j)-position and 0 elsewhere. Hence, $m_k \in \bigcap_{i\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$ for all $j \in \Lambda$ and each $k \in \Lambda$. Therefore, $m_k \in \bigcap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$ for each $k \in \Lambda$. So,

$$\mathbf{r}_M(\varphi S) \subseteq \oplus_{k\in\Lambda} \left(\cap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})\right).$$

For the reverse inclusion, let $m = (m_k)_{k\in\Lambda} \in \oplus_{k\in\Lambda} \left(\cap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})\right)$. Then $m_k \in \bigcap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$ for each $k \in \Lambda$. Since $\varphi_{ij} S_{jk} m_k = 0$ for all $i, j, k \in \Lambda$, $\varphi S m = 0$. Thus, $m \in \mathbf{r}_M(\varphi S)$. Hence $\mathbf{r}_M(\varphi S) = \oplus_{k\in\Lambda} \left(\cap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})\right)$. Therefore, for each $k \in \Lambda$, $\mathbf{r}_M(\varphi S) \cap M_k = \bigcap_{i,j\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$. □

Recall that a module M is said to have the *FI-strong summand intersection property* (FI-SSIP) if the intersection of any family of fully invariant direct summands of M is a direct summand of M. An idempotent e of a ring R is said to be *left* (resp., *right*) *semicentral* if $re = ere$ (resp., $er = ere$) for all $r \in R$.

Lemma 3.5. *Every quasi-Baer module has the FI-SSIP.*

Proof. Let M be a quasi-Baer module and $\{e_i\}_{i\in\Lambda}$ a family of left semicentral idempotents in $\text{End}_R(M)$. Then $\cap_{i\in\Lambda} e_i M = \cap_{i\in\Lambda} \mathbf{r}_M(S(1 - e_i)) = \mathbf{r}_M\left(\sum_{i\in\Lambda} S(1 - e_i)\right)$. $\sum_{i\in\Lambda} S(1 - e_i)$ is a 2-sided ideal of $\text{End}_R(M)$ as $1 - e_i$ is a right semicentral idempotent. As M is quasi-Baer, there exists $e \in S$ with $\mathbf{r}_M\left(\sum_{i\in\Lambda} S(1 - e_i)\right) = eM$. Therefore, M has the FI-SSIP. □

Now, we establish a characterization for a direct sum of quasi-Baer modules to be quasi-Baer as follows.

Theorem 3.6. *Let* $\{M_k\}_{k\in\Lambda}$ *be a class of right* R*-modules where* Λ *is an index set. Then the following conditions are equivalent:*

(a) $M = \bigoplus_{k\in\Lambda} M_k$ *is quasi-Baer;*
(b) *each* M_k *is quasi-Baer and the class* $\{M_k\}_{k\in\Lambda}$ *is relatively p.q.-Baer.*

Proof. (a)⇒(b) Since every direct summand of a quasi-Baer module is quasi-Baer by Lemma 3.1, it suffices to show that the class $\{M_k\}_{k\in\Lambda}$ is relatively p.q.-Baer, that is, we claim that $\mathbf{r}_{M_k}(\varphi_{ij} S_{jk}) \leq^{\oplus} M_k$ for all $\varphi_{ij} \in S_{ij} = \text{Hom}_R(M_j, M_i)$ and all $i, j, k \in \Lambda$: For any given $i, j \in \Lambda$, let $\varphi_{ij} \in S_{ij}$ be arbitrary. Take $\psi = (\psi_{\alpha\beta}) \in S = \text{End}_R(M)$ defined by $\psi_{\alpha\beta} = \varphi_{ij}$ if $(\alpha, \beta) = (i, j)$, and $\psi_{\alpha\beta} = 0$ if $(\alpha, \beta) \neq (i, j)$. In the proof of Lemma 3.4, $\mathbf{r}_M(\psi S) = \oplus_{k\in\Lambda} \mathbf{r}_{M_k}(\varphi_{ij} S_{jk})$. Since $\mathbf{r}_M(\psi S) \trianglelefteq^{\oplus} M$, $\mathbf{r}_{M_k}(\varphi_{ij} S_{jk}) \leq^{\oplus} M_k$ for all $k \in \Lambda$. Therefore $\mathbf{r}_{M_k}(\varphi_{ij} S_{jk}) \leq^{\oplus} M_k$ for all $\varphi_{ij} \in S_{ij}$ and all $i, j, k \in \Lambda$, proving the claim.

(b)⇒(a) Let J be any 2-sided ideal of S. Then $\mathbf{r}_M(J) = \bigcap_{\varphi\in J}\mathbf{r}_M(\varphi S)$. Hence $\mathbf{r}_M(J)\cap M_k = (\cap_{\varphi\in J}\mathbf{r}_M(\varphi S))\bigcap M_k = \bigcap_{\varphi\in J}(\mathbf{r}_M(\varphi S)\cap M_k)$. By Lemma 3.4,

$$\mathbf{r}_M(J) = \bigoplus_{k\in\Lambda}(\mathbf{r}_M(J)\cap M_k) = \bigoplus_{k\in\Lambda}\left[\bigcap_{\varphi\in J}(\cap_{i,j\in\Lambda}\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}))\right]$$

where $\varphi = (\varphi_{ij})$. Since $\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) \leq^{\oplus} M_k$ by hypothesis, there exists an idempotent $e_{\varphi_{ij}} \in S_{kk}$ such that $\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) = e_{\varphi_{ij}}M_k$. Since $\varphi_{ij}S_{jk}\cdot S_{kk}e_{\varphi_{ij}} = 0$, $e_{\varphi_{ij}}$ is a left semicentral idempotent of S_{kk}. Since M_k has FI-SSIP by Lemma 3.5, there exists a left semicentral idempotent $e_{\varphi_k} \in S_{kk}$ such that $\bigcap_{i,j\in\Lambda}\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) = e_{\varphi_k}M_k$. Also, there exists a left semicentral idempotent $e_k \in S_{kk}$ such that

$$\bigcap_{\varphi\in J}(\cap_{i,j\in\Lambda}\mathbf{r}_{M_k}(\varphi_{ij}S_{jk})) = \bigcap_{\varphi\in J}e_{\varphi_k}M_k = e_kM_k \trianglelefteq^{\oplus} M_k$$

Thus, $\mathbf{r}_M(J) = \bigoplus_{k\in\Lambda}\left[\bigcap_{\varphi\in J}(\cap_{i,j\in\Lambda}\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}))\right] = \bigoplus_{k\in\Lambda}e_kM_k \trianglelefteq^{\oplus} M$. Therefore, M is a quasi-Baer module. □

Remark 3.7. The only condition we use in Theorem 3.6 for a direct sum of quasi-Baer modules to be quasi-Baer is that the class $\{M_k\}_{k\in\Lambda}$ is relatively p.q.-Baer. And this condition precisely characterizes arbitrary direct sums of quasi-Baer modules to be quasi-Baer.

In contrast, even for finite direct sums of Baer modules to be Baer, one requires that all summands not only be relatively Rickart to each other but also that some of the direct summands be relatively injective to others as a standing assumption (see [1, Theorem 4.2.17]).

The next example is an illustration of Theorem 3.6.

Example 3.8. (i) Let R and P be as in Example 3.2. Let $\pi : R \to R/P$ be defined by $\pi(r) = r + P$. Then $\mathbf{r}_R(\pi\mathrm{End}_R(R)) = P$, which is not a direct summand of R_R. By Theorem 3.6, $R\oplus R/P$ is not quasi-Baer.

(ii) Let $M = \mathbb{Z}^{(\Lambda_1)}\oplus\mathbb{Q}^{(\Lambda_2)}$ be a $\mathbb{Z}$-module where Λ_1 and Λ_2 are any index sets. Then M is quasi-Baer. For, since $\mathbb{Z}$ and $\mathbb{Q}$ are quasi-Baer, we need to show that $\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) \leq^{\oplus} M_k$ for all $M_i, M_j, M_k \in \{\mathbb{Z},\mathbb{Q}\}$ from Theorem 3.6 where $S_{jk} = \mathrm{Hom}_R(M_k, M_j)$. There are exactly eight cases to be checked. Since $\mathbb{Z}$ and $\mathbb{Q}$ are domains and $S_{jk} = 0, \mathbb{Z}$, or $\mathbb{Q}$, it is easy to see that $\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) \leq^{\oplus} M_k$ for all $M_i, M_j, M_k \in \{\mathbb{Z},\mathbb{Q}\}$. Thus, $\mathbb{Z}^{(\Lambda_1)}\oplus\mathbb{Q}^{(\Lambda_2)}$ is a quasi-Baer $\mathbb{Z}$-module.

(iii) From Theorem 3.6, $\mathbb{Z}^{(\mathbb{R})}\oplus\mathbb{Q}^{(\mathbb{R})}$ is a quasi-Baer $\mathbb{Z}$-module as in (ii). However, $\mathbb{Z}^{(\mathbb{R})}\oplus\mathbb{Q}^{(\mathbb{R})}$ is not a Baer $\mathbb{Z}$-module because $\mathbb{Z}^{(\mathbb{R})}$ is not Baer. Note that $\mathbb{Z}^{(n)}\oplus\mathbb{Q}^{(\mathbb{R})}$ is a Baer $\mathbb{Z}$-module ([1, Theorem 4.2.18]).

Theorem 3.6 and the proof of Example 3.8(ii) yield the next result.

Corollary 3.9. *Let $\{M_k\}_{k\in\Lambda}$ be a class of right R-modules. Then $M = \bigoplus_{k\in\Lambda}M_k$ is a quasi-Baer module if and only if $M_i\oplus M_j\oplus M_k$ are quasi-Baer modules for all distinct $i, j, k \in \Lambda$.*

As a consequence of Theorem 3.6, we obtain the following well-known results.

Corollary 3.10. ([11, Proposition 3.19]) *A module M is quasi-Baer if and only if $M^{(\Lambda)}$ is quasi-Baer for any nonempty index set Λ. Hence, a ring R is quasi-Baer if and only if $R_R^{(\Lambda)}$ is a quasi-Baer module.*

Proof. Take $S = \mathrm{End}_R(M^{(\Lambda)}) \subseteq (S_{ij})_{i,j\in\Lambda}$ where $S_{ij} = \mathrm{End}_R(M)$ for all $i, j \in \Lambda$. Thus, the result directly follows from Theorem 3.6. □

Corollary 3.11. *If $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is quasi-Baer, then so is $\bigoplus_{\alpha\in\Lambda} M_\alpha^{(\Lambda_\alpha)}$ where Λ and each Λ_α are index sets.*

Proof. Since $\oplus_{\alpha\in\Lambda} M_\alpha$ is quasi-Baer, so is $\oplus_{\alpha\in\Lambda} M_\alpha^{(3)}$ by Corollary 3.10. From Theorem 3.6 we get that $\mathbf{r}_{M_k}(\varphi_{ij}S_{jk}) \leq^{\oplus} M_k$ for all $M_i, M_j, M_k \in \{M_\alpha\}_{\alpha\in\Lambda}$. Therefore from Corollary 3.9, $\oplus_{\alpha\in\Lambda} M_\alpha^{(\Lambda_\alpha)}$ is quasi-Baer. □

Theorem 3.6 also yields the next characterization for an arbitrary direct sum of quasi-Baer modules to be quasi-Baer, provided that each module is fully invariant in the direct sum.

Corollary 3.12. *Let $M_i \trianglelefteq \bigoplus_{k\in\Lambda} M_k$ for each $i \in \Lambda$ where Λ is any index set. Then $\bigoplus_{k\in\Lambda} M_k$ is a quasi-Baer module if and only if M_i is a quasi-Baer module for all $i \in \Lambda$.*

The next example illustrates Corollaries 3.10, 3.11, and 3.12. For $x \in \mathbb{Z}$, we use $\overline{x} \in \mathbb{Z}_n$ $(n > 1)$ to denote the canonical image of x.

Example 3.13. (i) Let $M = \mathbb{Z}_2^{(\Lambda_1)} \oplus \mathbb{Z}_3^{(\Lambda_2)} \oplus \mathbb{Z}[1/2]^{(\Lambda_3)}$ be as a $\mathbb{Z}$-module where Λ_1, Λ_2, and Λ_3 are any nonempty index sets. Then M is not a quasi-Baer $\mathbb{Z}$-module as a direct summand $\mathbb{Z}_3 \oplus \mathbb{Z}[1/2]$ of M is not quasi-Baer. Now, we consider $\varphi : \mathbb{Z}[1/2] \to \mathbb{Z}_3$ given by $\varphi(1) = \overline{1}$. Note that $\varphi(1/2^n) = \overline{2^n}$, $0 \leq n \in \mathbb{Z}$. Since $\mathrm{Ker}\varphi = 3\mathbb{Z}[1/2]$ and $\varphi\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/2])(3\mathbb{Z}[1/2]) = 0$, $3\mathbb{Z}[1/2] \subseteq \mathbf{r}_{\mathbb{Z}[1/2]}(\varphi\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/2])) \subseteq \mathrm{Ker}\varphi$. Thus,

$$\mathbf{r}_{\mathbb{Z}[1/2]}(\varphi\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/2])) = 3\mathbb{Z}[1/2],$$

which is not a direct summand of $\mathbb{Z}[1/2]$. From Lemma 3.1 and Theorem 3.6, M is not a quasi-Baer $\mathbb{Z}$-module.

(ii) Let $M = \mathbb{Z}_2^{(\Lambda_1)} \oplus \mathbb{Z}_3^{(\Lambda_2)} \oplus \mathbb{Z}[1/6]^{(\Lambda_3)}$ be as a $\mathbb{Z}$-module where Λ_1, Λ_2, and Λ_3 are any index sets. Then M is a quasi-Baer $\mathbb{Z}$-module. For, from Corollary 3.11 we just need to show that $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is a quasi-Baer $\mathbb{Z}$-module. It is clear that $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}[1/6]$ are quasi-Baer. Also, it is easy to see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_3) = 0$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_2) = 0$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}[1/6]) = 0$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}[1/6]) = 0$.

Also, we claim that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/6], \mathbb{Z}_2) = 0$. For, suppose $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/6], \mathbb{Z}_2)$. Since $\varphi(\alpha) = \varphi(6(\alpha/6)) = 6\varphi(\alpha/6) = \overline{0}$ for any $\alpha \in \mathbb{Z}[1/6]$ as $\alpha/6 \in \mathbb{Z}[1/6]$, we get $\varphi = 0$, proving the claim.

Similarly, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/6], \mathbb{Z}_3) = 0$. Thus $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}[1/6]$ are fully invariant submodules of M. From Corollary 3.12, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is a quasi-Baer $\mathbb{Z}$-module.

(iii) Let $M = \mathbb{Z}_2^{(\Lambda_1)} \oplus \mathbb{Z}[1/2]^{(\Lambda_2)} \oplus \mathbb{Q}^{(\Lambda_3)}$ be as a $\mathbb{Z}$-module where Λ_1, Λ_2, and Λ_3 are arbitrary index sets. Then M is a quasi-Baer $\mathbb{Z}$-module. For, it is easy to check that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}[1/2]) = 0$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/2], \mathbb{Z}_2) = 0$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Q}) = 0$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_2) = 0$. Thus, $\mathbb{Z}_2$ and $\mathbb{Z}[1/2] \oplus \mathbb{Q}$ are fully invariant submodules of M.

As the injective hull of $\mathbb{Z}[1/2]$ is $\mathbb{Q}$ and $\mathbb{Z}[1/2]$ is nonsingular extending, $\mathbb{Z}[1/2] \oplus \mathbb{Q}$ is a Baer module from [9, Theorem 2.16]. So, $\mathbb{Z}[1/2] \oplus \mathbb{Q}$ is quasi-Baer. Also, since $\mathbb{Z}_2$ is quasi-Baer, from Corollary 3.12 $\mathbb{Z}_2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Q}$ is quasi-Baer. Therefore, by Corollary 3.11 M is a quasi-Baer $\mathbb{Z}$-module.

Example 3.14. (i) If $M = \mathbb{Z}_p \oplus L$ is a quasi-Baer $\mathbb{Z}$-module such that $\mathbb{Z} \leq L \leq \mathbb{Q}$ for a $\mathbb{Z}$-module L where p is a prime number. Then $p^k L = L$ for all nonnegative integer k, that is, $\mathbb{Z}[1/p] \leq L$. Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, L) = 0$. Suppose

$\varphi \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}_p)$ defined by $\varphi(1) = \overline{\alpha}$. Then $\mathbf{r}_L(\varphi\mathrm{End}_{\mathbb{Z}}(L)) \supseteq pL$. Since $pL \leq^{\mathrm{ess}} L$ and $\mathbf{r}_L(\varphi\mathrm{End}_{\mathbb{Z}}(L)) \leq^{\oplus} L$ by Theorem 3.6, we get $\mathbf{r}_L(\varphi\mathrm{End}_{\mathbb{Z}}(L)) = L$. Therefore $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}_p) = 0$. Thus, it is easy to see that $p^k L = L$ for all nonnegative integer $k \in \mathbb{Z}$. So, $\mathbb{Z}[1/p] \leq L$.

(ii) If $M = (\oplus_{p\in\mathcal{P}}\mathbb{Z}_p) \bigoplus (\oplus_{i\in\Lambda} L_i)$ is a quasi-Baer $\mathbb{Z}$-module such that $\mathbb{Z} \leq L_i \leq \mathbb{Q}$ where $\mathcal{P}$ is a set of all prime numbers, Λ is any index set, and L_i are $\mathbb{Z}$-modules. Then all L_i are $\mathbb{Q}$. In fact, for any $p \in \mathcal{P}$, from Lemma 3.1 $\mathbb{Z}_p \oplus L_i$ is quasi-Baer for all $i \in \Lambda$. By (i), $pL_i = L_i$. Thus, L_i is p-divisible for all $p \in \mathcal{P}$. Hence L_i is divisible. So, $nL_i = L_i$ and hence $1/n \in L_i$ for all $n \in \mathbb{N}$. Therefore $L_i = \mathbb{Q}$ for all $i \in \Lambda$ as desired.

It is shown in [11] that a finitely generated $\mathbb{Z}$-module is Baer if and only if it is either semisimple or torsion-free. In the next consequence, we obtain the following improved result for a quasi-Baer module over a commutative principal ideal domain.

Corollary 3.15. *Let $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ be a direct sum of cyclic right R-modules M_α (where Λ is any index set) over a commutative principal ideal domain R. Then M is a quasi-Baer module if and only if M is either semisimple or torsion-free.*

Proof. Assume that M is a quasi-Baer module. Since M_α is cyclic quasi-Baer, either $f(M_\alpha) \cong R$ or $t(M_\alpha) \cong \oplus_{P_i\in\mathcal{P}} R/P_i$ where $\mathcal{P}$ is a finite collection of nonzero prime ideals P_i of R, $f(M)$ is the torsion-free submodule of M, and $t(M)$ is the torsion submodule of M. Since $R \oplus R/P_i$ is not quasi-Baer (see Example 3.8(i)), either $t(M) = 0$ or $f(M) = 0$. If $f(M) = 0$, then $t(M) = M$ is semisimple because every nonzero prime ideal of a commutative principal ideal domain is maximal. Finally, if $t(M) = 0$, then $M = f(M)$ is torsion-free. The converse is obvious. □

Remark 3.16. From Corollary 3.15, a finitely generated abelian group is quasi-Baer if and only if it is either semisimple or torsion-free. Thus, for a finitely generated abelian group M, M is Baer if and only if M is quasi-Baer (cf. [11, Proposition 2.19]).

Lemma 3.1 and Theorems 2.15 and 3.6 yield the following well-known results.

Corollary 3.17. ([1, Theorem 3.2.11]) *The following statements hold true.*

(i) *Any matrix ring over a quasi-Baer ring is a quasi-Baer ring.*
(ii) *The quasi-Baer ring property is Morita invariant.*

Proof. (i) This follows from Corollary 3.10 and Theorem 2.15.
(ii) It directly follows from Lemma 3.1 and part (i). □

Corollary 3.18. ([1, Theorem 4.6.19]) *The following are equivalent for a ring R:*

(a) *every projective R-module is a quasi-Baer module;*
(b) *every free R-module is a quasi-Baer module;*
(c) *R is a quasi-Baer ring.*

Recall that a ring R is said to be a *right QI-ring* if every quasi-injective right R-module is injective. If we require that every quasi-injective module is quasi-Baer then we obtain an alternate characterization of a right QI-ring via quasi-Baer modules.

Theorem 3.19. *The following conditions are equivalent for a ring R:*

(a) *every injective right R-module is FI-$\mathcal{K}$-nonsingular;*
(b) *every quasi-injective right R-module is quasi-Baer;*

(c) *R is a right QI-ring.*

Proof. (a)$\Rightarrow$(c) Let M be any quasi-injective right R-module. Consider the module $N := E(M) \oplus E(E(M)/M)$, which is injective. Then N is a quasi-Baer module from Theorem 2.6 as N is FI-$\mathcal{K}$-nonsingular by hypothesis. Let $\varphi : E(M) \to E(E(M)/M)$ be defined by $\varphi(\ell) = \ell + M$ for all $\ell \in E(M)$. Then $\mathrm{Ker}\varphi = M$. Since M is a fully invariant submodule of $E(M)$, $\mathrm{Ker}\varphi = \mathbf{r}_{E(M)}(\varphi T)$, which is a direct summand of $E(M)$ from Theorem 3.6 where $T = \mathrm{End}_R(E(M))$. Since $M \leq^{\text{ess}} E(M)$, $M = E(M)$. Hence, R is a right QI-ring. (See Theorem 5.1 in [3] for an alternate proof.)

(c)$\Rightarrow$(b) Let R be a right QI-ring, M a quasi-injective module, and N a fully invariant submodule of M. Thus, M is an injective module as R is a right QI-ring. Since $SN \subseteq N$ where $S = \mathrm{End}_R(M)$, N is also quasi-injective, hence N is injective because R is a right QI-ring. Thus, $N \leq^{\oplus} M$. So, $\mathbf{l}_S(N) = Se$ for some $e \in S$. Therefore M is quasi-Baer.

(b)$\Rightarrow$(a) follows from the fact that every quasi-Baer module is FI-$\mathcal{K}$-nonsingular (see [11, Lemma 3.13]). □

Open Questions: 1. Is a quasi-Baer module always quasi-retractable?

2. Every local-retractable module is q-local-retractable. Does the converse hold?

3. In contrast to Theorem 2.19, if M is a quasi-Baer module then $S = \mathrm{End}_R(M)$ is a quasi-Baer ring and $\mathbf{r}_M(J) \neq 0$ implies $\mathbf{r}_S(J) \neq 0$ for any 2-sided ideal J of S. Does the converse hold?

Acknowledgments

The authors are thankful to the Math Research Institute, the College of Arts and Sciences, OSU-Columbus and OSU-Lima, for the partial support of this research work. Also, the authors are grateful to Department of Mathematics, Sungkyunkwan University, Republic of Korea, for the support of this research. Gangyong Lee gratefully acknowledges the support of this research work by the Presidential Post-doctoral Fellowship Program through National Research Foundation of Korea funded by the Ministry of Education (NRF-2014R1A6A3A04054622).

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