Egyptian fractions, Sylvester's sequence, and the Erdős-Straus conjecture

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1 Egyptian fractions

Many of these ideas are from the Wikipedia entry "Egyptian fractions."

1.1 Introduction

An Egyptian fraction is a number of distinct unit fractions with positive denominators added together.

Example. $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$
$\frac{1}{5} = \frac{1}{5}$ $(= \frac{1}{6} + \frac{1}{30})$
$\frac{3}{179} = \frac{1}{60} + \frac{1}{10740}$

The restriction on distinct unit fractions is needed, because otherwise all fractions can easily be made into Egyptian fractions simply by using $\frac{a}{b} = \frac{1}{b} + \cdots + \frac{1}{b}$. The Egyptians did not represent $\frac{2}{3}$ or $\frac{3}{4}$ by unit fractions. (It can be though, $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.)

1.2 History

1.2.1 Hieroglyphs

Ancient Egyptians represented a reciprocal of a number by placing a hieroglyph above the number. The fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ had their own distinct symbols. (Clagett)

1.2.2 The Rhind Mathematical Papyrus

The Rhind Mathematical Papyrus is a document made around 1650 BC (during the Second Intermediate Period) in the Egyptian Middle Kingdom. It was later bought by Henry Rhind, and the papyrus was named after him. It was written by Ahmes (Ahmose). The papyrus is divided into three parts. The first part contains a list of Egyptian fraction representations of $\frac{2}{n}$ for odd n from $n = 3$ to $n = 101$, as well as 40 problems. The second part is about geometry, specifically volumes, areas, and pyramids. The third book contains 24 additional problems. (Spalinger)

1.2.3 Other ancient texts

Several other Egyptian papyri from earlier times have tables of Egyptian fractions. The Lahun Mathematical Papyri (around 1850 BC) also had Egyptian fraction decompositions for $\frac{2}{n}$, the Egyptian Mathematical Leather Roll (around 1900 BC) had Egyptian fraction decompositions for $\frac{1}{n}$ fractions, and the Akhmin wooden tablet (around 1950 BC) also contains Egyptian fraction decompositions for $\frac{1}{n}$ fractions.

1.3 Methods to convert a non-Egyptian fraction into an Egyptian fraction

1.3.1 Fractions in the form $\frac{1}{n}$

- $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$

1.3.2 Fractions in the form $\frac{2}{n}$

- $\frac{2}{n} = \frac{2}{n+1} + \frac{2}{n(n+1)}$
- $\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}$
- $\frac{2}{n} = \frac{1}{2}\frac{1}{n} + \frac{3}{2}\frac{1}{n}$ (n is a multiple of 3)
- $\frac{2}{n} = \frac{1}{3}\frac{1}{n} + \frac{5}{3}\frac{1}{n}$ (n is a multiple of 5)
- $\frac{2}{mn} = \frac{1}{kn} + \frac{1}{kmn}$ ($k = \frac{m+1}{2}$) (Gardner 2002)
- $\frac{2}{mn} = \frac{1}{m}\frac{1}{k} + \frac{1}{n}\frac{1}{k}$ ($k = \frac{m+n}{2}$) (Eves 1953)

1.3.3 Fractions in the form $\frac{m}{n}$

- $\frac{a}{mn} = \frac{1}{m}\frac{1}{k} + \frac{1}{n}\frac{1}{k}$ ($k = \frac{m+n}{a}$) (Eves 1953)
- The greedy algorithm

1.4 The greedy algorithm

Some of these ideas are from the Wikipedia entry "Greedy algorithm for Egyptian fractions."

1.4.1 Introduction

The greedy algorithm is a method that will convert any fraction into an Egyptian fraction. $\dfrac{m}{n} = \dfrac{1}{\lceil\frac{n}{m}\rceil} + \dfrac{-n \pmod m}{n\lceil\frac{n}{m}\rceil}$. If $-n \bmod m \neq 1$, then repeat the method with $\dfrac{-n \pmod m}{n\lceil\frac{n}{m}\rceil}$ as the new "$\frac{m}{n}$" until all fractions are unit fractions. (Sigler 2002, Sylvester 1880)

$$
\begin{aligned}
\frac{5}{31} &= \frac{1}{7}+\frac{4}{217}\\
&\frac{1}{7}+\frac{1}{55}+\frac{3}{11935}\\
&\frac{1}{7}+\frac{1}{55}+\frac{1}{3979}+\frac{2}{47489365}\\
&\frac{1}{7}+\frac{1}{55}+\frac{1}{3979}+\frac{1}{23744683}+\frac{1}{1127619917796295}
\end{aligned}
$$

$0 < -n \pmod{m} < m$. So, the numerator of the second fraction in the expansion continues to decrease until it reaches 1. $\lceil \frac{n}{m} \rceil < n$ (assuming $n > m$), so the denominator of the first fraction is smaller than the denominator of the original fraction. Also, $n\lceil \frac{n}{m} \rceil > n$ (assuming $n > m$), so the denominators of the fractions increase every step.

No known algorithm gives the most concise Egyptian fraction representations of every fraction (for either meaning of *concise*, least number of terms or smallest denominator). The greedy algorithm may give many terms with large denominators when another method gives fewer terms and smaller denominators. The above fraction can be written more concisely as $\frac{5}{31} = \frac{1}{7} + \frac{1}{62} + \frac{1}{434}$, one of five representations of length 3.

1.4.2 Maximum-length expansions

Depending on the fraction $\frac{m}{n}$, the greedy algorithm may give m terms in the resulting Egyptian fraction, or fewer than x terms. $\frac{1}{n}$ trivially has 1 term. $\frac{2}{n}$ always has 2 terms, as (let $n' = \frac{n-1}{2}$) $\frac{2}{2n'+1} = \frac{1}{n'+1} + \frac{1}{(n'+1)(2n'+1)}$. Freitag and Phillips (1999) give a necessary and sufficient condition for a fraction to have m terms.

Theorem. 1 *For a fraction $\frac{m}{n}$, the greedy algorithm for Egyptian fractions gives m terms if $n = km! + 1$ $(k \in \mathbb{N})$.*

Proof. Induction on m is used. The statement is true for $m = 1$. Consider some $m \geq 1$. Then for $k \in \mathbb{N}$, $\frac{m}{km!+1} = \frac{1}{k(m-1)!+1} + \frac{m-1}{k'(m-1)!+1}$ (where $k' = k\,(km! + m + 1)$). $\square$

Theorem. 2 *For a fraction $\frac{m}{n}$, a necessary condition for its greedy algorithm expansion to have x terms is for $y = km + 1$ $(k \in \mathbb{N})$.*

Proof. $n = km + r$ $(0 \leq r < m)$. Then $\frac{m}{km+r} = \frac{1}{k+1} + \frac{m-r}{(k+1)(km+r)}$. When $r = 1$, $m - r = m - 1$. $\square$

From these two theorems it is possible to deduce the necessary and sufficient case. First of all there is an important definition.

Definition. Let sets S_m $(m \geq 3)$ be defined by the following rule: $S_3 = \{0\}$ and $s \in S_m$ iff $0 \leq s < (m-1)!$ and $ms^2 + (m+1)\,s \equiv t\,(m-1) \pmod{(m-1)!}$ $(t \in S_{m-1})$.

- $S_4 = \{0, 4\}$
- $S_5 = \{0, 6, 12, 18\}$
- $S_6 = \{0, 18, 30, 48, 60, 78, 90, 108\}$

Theorem. 3 *A fraction $\frac{m}{n}$ has m terms in its greedy algorithm expansion iff $n = km! + sm + 1$ ($s \in S_m$, $k \geq 0$, and k and s are not both 0).*

Proof. If $\frac{m}{n}$ has m terms in its greedy algorithm expansion, from Theorem 2, $n = (k\,(m-1)! + s)\,m + 1 = km! + sm + 1$ $(0 \leq s < (m-1)!)$. Then $\frac{m}{km!+sm+1} = \frac{1}{k(m-1)!+s+1} + \frac{m-1}{(k(m-1)!+s+1)(km!+sm+1)}$. Also, $(k\,(m-1)! + s + 1)\,(km! + sm + 1) \equiv (s+1)\,(sm+1) \pmod{(m-1)!}$.

This formula is true for $m = 3$. Assume that it is true for some $m - 1 \geq 3$. Then for s such that $ms^2\,(m+1)\,s \equiv t\,(m-1) \pmod{(m-1)!}$ $(t \in S_{m-1})$, the formula is also true for m. Since that equivalence is how S_m is defined, the formula is true for m. □

The smallest n such that $\frac{m}{n}$ has m terms in its greedy algorithm expansion is:

- $m = 3$: $n = 1 \cdot 3! + 0 \cdot 3 + 1 = 7$
- $m = 4$: $n = 0 \cdot 4! + 4 \cdot 4 + 1 = 17$
- $m = 5$: $n = 0 \cdot 5! + 6 \cdot 5 + 1 = 31$
- $m = 6$: $n = 0 \cdot 6! + 18 \cdot 6 + 1 = 109$

1.4.3 Irrational numbers

The greedy algorithm can also be used for irrational numbers (from the MathWorld entry "Egyptian Fraction"), although then the series will be infinite (because an irrational number cannot be represented as the sum of rational numbers). For instance,

- $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{13} + \frac{1}{253} + \frac{1}{218201} + \cdots$
- $e = 2 + \frac{1}{2} + \frac{1}{5} + \frac{1}{55} + \frac{1}{9999} + \frac{1}{3620211523} + \cdots$
- $\pi = 3 + \frac{1}{8} + \frac{1}{61} + \frac{1}{5020} + \frac{1}{128541455} + \cdots$
- $\log 2 = \frac{1}{2} + \frac{1}{6} + \frac{1}{38} + \frac{1}{6071} + \cdots$

1.5 Sylvester's sequence

Some of these ideas are from the Wikipedia entry "Sylvester's sequence."

1.5.1 Introduction and relation to Egyptian fractions

Sylvester's sequence is a sequence related to the greedy algorithm for Egyptian fractions. One way to find it is to apply the greedy algorithm for Egyptian fractions to 1, but at each step use the largest unit fraction that keeps the sum of the unit fractions less than 1 (change $\dfrac{m}{n} = \dfrac{1}{\lceil \frac{n}{m} \rceil} + \dfrac{-n \pmod{m}}{n\lceil \frac{n}{m} \rceil}$ to $\dfrac{m}{n} = \dfrac{1}{\lfloor \frac{n}{m} \rfloor + 1} + \dfrac{-n \pmod{m}}{n\left(\lfloor \frac{n}{m} \rfloor + 1\right)}$). The sequence is comprised of the denominators of the resulting fractions.

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3263443} + \cdots$, so the first 6 terms are $2, 3, 7, 43, 1807, 3263443$. (Sloane.)

1.5.2 Formal (sequential) definition

Definition. $e_0 = 2$, $e_n = 1 + \prod_{i=0}^{n-1} e_i$.

Corollary. $e_0 = 2$, $e_n = e_{n-1}\left(e_{n-1} - 1\right) + 1$.

Proof. An exercise. $\square$

The sequence's values grow doubly exponentially, which means that they grow at the rate of a^{b^x}. Specifically, $e_n = \lfloor E^{2^{n+1}} + \frac{1}{2} \rfloor$, where $E = \frac{1}{2}\sqrt{6}\exp\left[\sum_{i=1}^{\infty} 2^{-i-1}\log\left(1 + (2e_i - 1)^{-2}\right)\right] = 1.2640847\ldots$. (Aho and Sloane 1973, Vardi 1991, Graham et al. 1994.)

Fact. $\sum_{i=0}^{\infty} \frac{1}{e_i} = 1$. *So,* $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots = 1$.

Proof. From the recurrence equation,

$$\begin{aligned}
e_n &= e_{n-1}\left(e_{n-1} - 1\right) + 1 \\
e_{n+1} &= e_n\left(e_n - 1\right) + 1 \\
e_{n+1} - 1 &= e_n\left(e_n - 1\right) \\
\frac{1}{e_{n+1} - 1} &= \frac{1}{e_n\left(e_n - 1\right)} \\
\frac{1}{e_{n+1} - 1} &= \frac{1}{e_n - 1} - \frac{1}{e_n} \\
\frac{1}{e_n} &= \frac{1}{e_n - 1} - \frac{1}{e_{n+1} - 1}
\end{aligned}$$

So, $\sum_{i=0}^{n-1} \frac{1}{e_i} = \sum_{i=0}^{n-1}\left(\frac{1}{e_i - 1} - \frac{1}{e_{i+1} - 1}\right) = \frac{1}{e_0 - 1} - \frac{1}{e_n - 1} = 1 - \frac{1}{e_n - 1} \rightarrow_{(n\rightarrow\infty)} 1$. $\square$

1.5.3 Relation to prime numbers

Theorem. *(Euclid) There are infinitely many prime numbers.*

Proof. For two terms e_i and e_j $(i < j)$, $e_j \equiv 1 \pmod{e_i}$ since $\frac{e_n(e_n-1)+1}{e_n} = (e_n - 1) + \frac{1}{e_n}$. So, any two numbers in Sylvester's sequence are relatively prime. Any prime p divides no more than one number in Sylvester's sequence, because if it divides more than one, those numbers would not be relatively prime. Since there are infinitely many numbers in Sylvester's sequence, there are also infinitely many prime numbers. $\square$

Note. The second part of the first statement assumes e_n and e_{n+1}. For e_n and e_{n+2}, repeat the operation to get

$$\frac{\left[e_n\left(e_n - 1\right) + 1\right]\left(\left[e_n\left(e_n - 1\right) + 1\right] - 1\right) + 1}{e_n} = \left[e_n\left(e_n - 1\right) + 1\right]\left(e_n - 1\right) + \frac{1}{e_n}$$

For the general case, e_n and e_{n+k+1}, it's

$$\frac{\left[e_{n+k}\left(e_{n+k} - 1\right) + 1\right]\left(\left[e_{n+k}\left(e_{n+k} - 1\right) + 1\right] - 1\right) + 1}{e_n} = \left[e_{n+k}\left(e_{n+k} - 1\right) + 1\right]\left(e_{n+k} - 1\right) + \frac{1}{e_n}$$

1.5.4 Convergence of series

According to Badea (1993), given any sequence that grows such that $e_n \geq e_{n-1}^2 - e_{n-1} + 1$ and $\sum \frac{1}{e_i} = E \in \mathbb{Q}$, there exists an N such that for all $n > N$ the sequence is defined by $e_n = e_{n-1}^2 - e_{n-1} + 1$.

1.6 The Erdős-Straus conjecture

1.6.1 Introduction

Definition. A Diophantine equation is an equation where its solutions are restricted to integer values.

Conjecture. (Erdős 1950) *The Diophantine equation* $\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ *can be solved for any natural number* $n \geq 2$.

The greedy algorithm gives 3 or fewer terms for most cases of n. The greedy expansion gives 4 terms when $n = km! + sm + 1$ ($k \in \mathbb{N}$, $m = 4$, $s \in \{0, 4\}$), $n = 25, 49, 73, \ldots$ or $17, 41, 65, \ldots \equiv 1$ or $17 \pmod{24}$.

$\dfrac{4}{n} = \dfrac{1}{n} + \dfrac{1}{\frac{n-2}{3} + 1} + \dfrac{1}{n\left(\frac{n-2}{3} + 1\right)}$. When $n = 2 \pmod 3$, $\frac{n-2}{3} + 1 \in \mathbb{N}$, so this expansion is an Egyptian fraction representation of $\frac{4}{n}$. $\{n : n \equiv 17 \pmod{24}\} \subset \{n : n \equiv 2 \pmod 3\}$, so that expansion also holds for the case $n = 17 \pmod{24}$. No similar solution exists for the case $n = 1 \pmod{24}$ (Mordell 1967).

It has been shown that given an interval $[1, N]$, the fraction of n in that interval that could be counterexamples to the conjecture $\to 0$ as $N \to \infty$ (Webb 1970).

This conjecture has been tested valid using computer searches for $n \leq 10^{14}$ (Swett).

1.6.2 Generalizations

Conjecture. *(Sierpiński 1956) There exists some* N *such that the Diophantine equation* $\frac{5}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ *can be solved for any natural number* $n \geq N$.

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Conjecture. *(Schinzel) For any given* $m \in \mathbb{N}$*, there exists some* N *such that the Diophantine equation* $\frac{m}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ *can be solved for any natural number* $n \geq N$ *(Vaughan 1970).*

2 References

- Aho, A. V. and Sloane, N. J. A. "Some Doubly Exponential Sequences." *Fib. Quart.* **11**, 429-437, 1973.
- Badea, Catalin (1993). "A theorem on irrationality of infinite series and applications". *Acta Arithmetica* **63**: 313–323.
- Clagett, Marshall Ancient Egyptian Science, A Source Book. Volume Three: Ancient Egyptian Mathematics (Memoirs of the American Philosophical Society) American Philosophical Society. 1999

- Erdős, Paul (1950), "Az $1/x_1 + 1/x_2 + ... + 1/x_n = a/b$ egyenlet egész számú megoldásairól (On a Diophantine Equation)", *Mat. Lapok.* **1**: 192–210.

- Eves, Howard (1953), *An Introduction to the History of Mathematics*, Holt, Reinhard, and Winston

- Freitag, H. T.; Phillips, G. M. (1999). "Sylvester's algorithm and Fibonacci numbers". *Applications of Fibonacci numbers, Vol. 8* (Rochester, NY, 1998). Dordrecht: Kluwer Acad. Publ.. pp. 155–163.

- Gardner, Milo (2002), "The Egyptian Mathematical Leather Roll, attested short term and long term", in Gratton-Guiness, Ivor, *History of the Mathematical Sciences*, Hindustan Book Co, pp. 119–134

- Graham, R. L.; Knuth, D. E.; and Patashnik, O. *Research problem 4.65 in Concrete Mathematics: A Foundation for Computer Science, 2nd ed.* Reading, MA: Addison-Wesley, 1994.

- Mordell, Louis J. (1967), *Diophantine Equations*, Academic Press, pp. 287–290.

- Sierpiński, Wacław (1956), "Sur les décompositions de nombres rationnels en fractions primaires" (in French), *Mathesis* **65**: 16–32.

- Sigler, Laurence E. (trans.) (2002), *Fibonacci's Liber Abaci*, Springer-Verlag

- Sloane, N. J. A. Sequences A000058/M0865, A014546, and A076393 in "The On-Line Encyclopedia of Integer Sequences."

- Anthony Spalinger , The Rhind Mathematical Papyrus as a Historical Document, Studien zur Altägyptischen Kultur, Bd. 17 (1990), pp. 295-, Helmut Buske Verlag GmbH

- Swett, Allan, *The Erdos-Straus Conjecture*, <http://math.uindy.edu/swett/esc.htm>, retrieved 2006-09-09.

- Sylvester, J. J. (1880). "On a point in the theory of vulgar fractions". *American Journal of Mathematics* **3** (4): 332–335.

- Vardi, I. "Are All Euclid Numbers Squarefree?" and "PowerMod to the Rescue." §5.1 and 5.2 in *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 82-89, 1991.

- Vaughan, R. C. (1970), "On a problem of Erdős, Straus and Schinzel", *Mathematika* **17** (02): 193–198.

- Webb, William A. (1970), "On $4/n = 1/x + 1/y + 1/z$", *Proceedings of the American Mathematical Society* **25** (3): 578–584.