What is... Fractional Calculus?

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Abstract

Differentiation and integration of non-integer order have been of interest since Leibniz. We will approach the fractional calculus through the differintegral operator and derive the differintegrals of familiar functions from the standard calculus. We will also solve Abel's integral equation using fractional methods.

The Grünwald-Letnikov Definition

A plethora of approaches exist for derivatives and integrals of arbitrary order. We will consider only a few. The first, and most intuitive definition given here was first proposed by Grünwald in 1867, and later Letnikov in 1868. We begin with the definition of a derivative as a difference quotient, namely,

$$\frac{d^1 f}{dx^1} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

It is an exercise in induction to demonstrate that, more generally,

$$\frac{d^n f}{dx^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - jh)$$

We will assume that all functions described here are sufficiently differentiable.

Differentiation and integration are often regarded as inverse operations, so we wish now to attach a meaning to the symbol $\frac{d^{-1}}{dx^{-1}}$, what might commonly be referred to as anti-differentiation. However, integration of a function is dependent on the lower limit of integration, which is why the two operations cannot be regarded as truly inverse. We will select a definitive lower limit of 0 for convenience, so that,

$$\frac{d^{-n} f}{dx^{-n}} \equiv \int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \cdots \int_0^{x_2} dx_1 \int_0^{x_1} f(x_0) dx_0$$

By instead evaluating this multiple intgral as the limit of a sum, we find

$$\frac{d^{-n}f}{dx^{-n}} = \lim_{N\to\infty} \left(\frac{x}{N}\right)^{n} \sum_{j=0}^{N-1} \binom{j+n-1}{j} f(x - j\left[\frac{x}{N}\right])$$

in which the interval $[0, x]$ has been partitioned into N equal subintervals. This expression bears a striking similarity to our formula derived for integer-order differentiation, provided we allow $h \to 0$ through discrete values $\frac{x}{N}$. Our suspicion is confirmed by writing the binomial coefficients as their Γ function equivalents and unifying the two formulas as

$$\frac{d^{q}f}{dx^{q}} = \lim_{N\to\infty} \left(\frac{x}{N}\right)^{-q} \frac{1}{\Gamma(q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(x - j\left[\frac{x}{N}\right])$$

which is valid for integers q. We will call this the *differintegral* of f of order q. There is no reason to exclude non-integer values of q in the differintegral, except that the resulting limit may not exist. We will presently establish the existence of differintegrable functions of all orders. The simplest such example is $f(x) \equiv 0$, for which we have

$$\frac{d^{q}f}{dx^{q}} = 0, \quad \forall q$$

Less trivial is the case of a general constant C. We find,

$$\frac{d^{q}[C]}{dx^{q}} = C\frac{x^{-q}}{\Gamma(1-q)}$$

Lastly,

$$\frac{d^{q}[x^{p}]}{dx^{q}} = \frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}, \quad p > -1$$

For $p \leq -1$ our definition breaks down, due to the poor behavior of the function at 0. As for more complex functions, the formulas can become quite cumbersome. The interested reader is directed to [1] for details.

Properties

Many of the basic properties of the derivative carry over without trouble. If f, g are differintegrable, then cf, $f+g$ and $h(x) = f(cx)$ are differintegrable, with

$$\frac{d^{q}[cf]}{dx^{q}} = c\frac{d^{q}f}{dx^{q}}$$

$$\frac{d^{q}[f+g]}{dx^{q}} = \frac{d^{q}f}{dx^{q}} + \frac{d^{q}g}{dx^{q}}$$

$$\frac{d^{q}[h]}{dx^{q}} = c^{q}\frac{d^{q}f}{dx^{q}}(cx)$$

Even Leibniz's rule extends in a natural manner, provided f and g are analytic (otherwise convergence issues are encountered):

$$\frac{d^q[fg]}{dx^q} = \sum_{j=0}^{\infty} \binom{q}{j} \frac{d^{q-j}f}{dx^{q-j}} \frac{d^j g}{dx^j}$$

The chain rule, as one may expect, is much too complex for practical application. Again, [1] has the details.

We should hope to be so lucky with composition, that,

$$\frac{d^p}{dx^p} \frac{d^q f}{dx^q} = \frac{d^{p+q} f}{dx^{p+q}} = \frac{d^q}{dx^q} \frac{d^p f}{dx^p}$$

and for $p, q \geq 0$ and $p, q \leq 0$, this is indeed true. However, if either $p > 0$ or $q > 0$ (but not both), then equality may not hold, the reason being that additional factors may appear upon integration that would have been erased by differentiation, if the integration is performed last. A variety of conditions exist for validity of composition: one is,

$$\frac{d^q}{dx^q} \frac{d^{-q} f}{dx^{-q}} = f, \qquad q < 0$$

Alternative Definitions

The assumption that the lower limit of integration is 0 is by no means necessary. The theory does not change if the lower limit is adjusted. There is also no reason why q cannot be complex, except for simplicity. What is remarkable about fractional calculus is that most reasonable definitions for the derivative or integral may be taken as the starting point of the development of the theory. All of these definitions are either equivalent or intimately related.

Infinite series may in fact be differintegrated term-by-term under essentially the same conditions as in standard calculus, which leads to the differintegral of x^p being a natural assumption to take, which allows us to differintegrate any analytic function.

Riemann and Liouville independently arrived at an expression for the differintegral as an integral transform, known today as,

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_a^x (x-y)^{-q-1} f(y) dy, q < 0$$

which may be extended to arbitrary q by imposing the identity,

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n} \frac{d^{q-n} f}{dx^{q-n}}$$

where n is a natural number. This form is equivalent to the Grünwald-Letnikov definition and is often more suited to computations for $q < 0$. Another definition due to Liouville concerns functions expressible as an exponential series,

$$f(x) = \sum_{j=0}^{\infty} c_j e^{b_j x}$$

defining,

$$\frac{d^q f}{dx^q} = \sum_{j=0}^{\infty} c_j b_j^q e^{b_j x}$$

Krug showed in 1890 that if we take the lower limit to be $-\infty$ in the Riemann-Liouville definition, we recover this definition.

Abel's Integral Equation

Abel first studied the equation which bears his name in the context of the tautochrone problem, which asks for "a curve in the (x, y) plane such that the time required for a particle to slide down the curve to its lowest point is independent of initial placement on the curve."(Oldham) He tackled the general form of this problem, which requests instead a curve corresponding to a function f that specifies the time of descent from particular points on the curve. Assuming no loss of energy to friction, the loss of potential energy may be equated to the gain in kinetic energy. This leads to an equation of the form,

$$f(Y) = \frac{1}{\sqrt{2g}} \int_0^Y \frac{\sigma'(y)dy}{\sqrt{Y-y}}, \quad f(0) = 0$$

where we have taken $(0, 0)$ to be the lowest point on the curve, (X, Y) to be the initial point of placement, (x, y) any intermediate point, $\sigma(y)$ the arclength of the curve from 0 to (x, y), and g the gravitational constant. We will make the assumption that σ is differintegrable. If we compare Abel's equation to the Riemann-Liouville definition of the differintegral, we see immediately that the equation defines a fractional differential equation,

$$f(Y) = \frac{\Gamma(\frac{1}{2})}{\sqrt{2g}} \frac{d^{-\frac{1}{2}}\sigma'}{dY^{-\frac{1}{2}}}(Y), \quad f(0) = 0$$

Let us replace y with Y, so that the equation now reads,

$$f(y) = \frac{\Gamma(\frac{1}{2})}{\sqrt{2g}} \frac{d^{\frac{1}{2}}\sigma}{dy^{\frac{1}{2}}}(y), \quad f(0) = 0$$

Since σ is differintegrable and $\sigma(0) = 0$, composition will go through. Upon semi-integrating both sides, we arrive at the deceptively simple solution,

$$\frac{\sqrt{2g}}{\Gamma(\frac{1}{2})} \frac{d^{-\frac{1}{2}}f}{dy^{-\frac{1}{2}}}(y) = \sigma(y)$$

References

[1] Oldham, K., and Spanier, J., *The Fractional Calculus*, Academic Press, Inc., New York, New York, 1974

[2] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, California, 1999

[3] Kilbas, A., Marichev, O., Samko, S., *Fractional Integrals and Derivatives*, George and Breach Science Publishers, Berlin, Germany, 1987