2016 Gordon examination problems

1. Is there a continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ which takes on rational values at the irrational points and irrational values at the rational points?

2. Each point of the three dimensional space $\mathbb{R}^3$ is colored either red or blue. Prove that there exists an equilateral triangle with side length 1 whose vertices have the same color.

3. Prove that for any $n, d \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ one has $\displaystyle\sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq n^2$.

(Here $u \cdot v$ represents the dot product of vectors u and v.)

4. Evaluate $\int_{x^2+y^2 \leq R^2} \sin x^2 \cos y^2 \, dxdy$.

5. Prove that for $z_1, \ldots, z_4 \in \mathbb{C}$, $\max_{|z_1|,\ldots,|z_4| \leq 1} \left| z_1 z_3 + z_1 z_4 + z_2 z_3 - z_2 z_4 \right| = 2\sqrt{2}$.

6. Let A be a 2016×2016 matrix such that all diagonal entries of A are zero and the rest of entries are equal to ± 1:

$$A = \begin{pmatrix} 0 & \pm 1 & \pm 1 & \ldots & \pm 1 \\ \pm 1 & 0 & \pm 1 & \ldots & \pm 1 \\ \pm 1 & \pm 1 & 0 & \ldots & \pm 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm 1 & \pm 1 & \pm 1 & \ldots & 0 \end{pmatrix}$$

Prove that $\det A \neq 0$.