2015 Gordon exam solutions

1. *Prove that there are infinitely many integers not representable as a sum of cubes of three positive integers.*

Solution. For any $n \in \mathbb{Z}$, $n^3 \equiv 0$ or 1 or $-1 \bmod 9$. (Indeed, if $n = 3k + d$ where $d = 0$, 1, or -1, then $n^3 = 27k^3 + 27k^2d + 9kd^2 + d^3$.) So, for any $n, m, k \in \mathbb{N}$, $n^3 + m^3 + k^3 \in \{0, 1, 2, 3, -1, -2, -3\} \bmod 9$, and so, all integers that are congruent to 4, 5, or 6 modulo 9 are not representable as the sum of three cubes of integers.

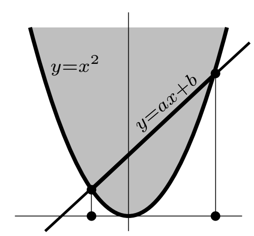
Another solution. For any $N \in \mathbb{N}$, let A_N be the number of elements of $\{1, \ldots, N\}$ representable as a sum of three cubes of positive integers, and let C_N be the number of cubes in $\{1, \ldots, N\}$. Then $C_N \le N^{1/3}$, and $A_N \le \binom{C_N}{3} + 2\binom{C_N}{2} + C_N$ (where the summands correspond to the number of integers representable as sums of three distinct cubes, of two equal and one distinct cubes, and of three equal cubes respectively). So,

$$A_N \le \tfrac{1}{6}C_N^3 + b_2C_N^2 + b_1C_N + b_0 = \tfrac{1}{6}N + b_2N^{2/3} + b_1N^{1/3} + b_0$$

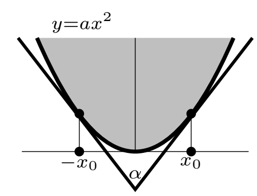
for some $b_i \in \mathbb{R}$. For large enough N, $A_N < \frac{1}{5}N$. So, for large N, at least $4/5$ of integers from $\{1, \ldots, N\}$ are not representable as sums of the three cubes of positive integers.

2. *Can the plane be covered by the interiors of a finite collection of parabolas?*

Solution. No. Any parabola P has the property that for any straight line L not parallel to the axis of P, the intersection of L with the interior of the parabola is a bounded interval in L. (Indeed, after changing coordinates, we may assume that the parabola is given by $y = x^2$ and the line is $y = ax + b$. Then a point (x, y) of the line is in the interior of the parabola iff $ax + b > x^2$, which defines a bounded (or empty) interval of xs between the roots of the polynomial $x^2 - ax - b$.) Given several parabolas, choose a line L which is not parallel to the axis of all these parabolas; then the intersection of L with the union of the interiors of the parabolas is a finite union of bounded intervals, and so, there are points of L that don't belong to this union.

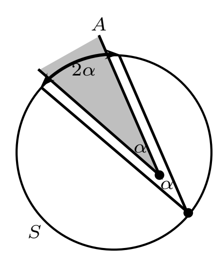


Another solution. The answer is still no. First notice that for any positive number $\alpha < \pi$, any parabola P can be covered by an angle of size α. Indeed, after an appropriate motion (a rotation and a shift) of the plane, we may assume that, in Cartesian coordinates, P is defined by the equation $y = ax^2$ for some $a > 0$. Take $x_0 = \cot(\alpha/2)/(2a)$; then the angle between the tangent lines to P at the points $(-x_0, ax_0^2)$ and (x_0, ax_0^2) is $2\left(\frac{\pi}{2} - \arctan(2ax_0)\right) = 2\,\mathrm{arccot}(2ax_0) = \alpha$.



Next, notice that if S is a circle and A is an angle of size α with vertex inside S, then the arc length of the arc cut off by A on S does not exceed 2α: indeed, A is contained in an angle of size α with vertex on S, and, by a geometry theorem, such an angle cuts off on S an arc of arc length 2α.

Now, given k parabolas on the plane, cover each of them by an angle of size $< \pi/k$, and let S be a circle big enough so that the vertices of all these k angles are inside S; then the angles, and so the interiors of the parabolas, cut off on S a collection of arcs of total arc length $< 2k(\pi/k) = 2\pi$, and so, don't cover S.



3. *Let A be a 2×2 matrix with integer entries and determinant 1. Prove that A is a product of several matrices of the form $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$.*

Solution. Call the matrices listed in the formulation "elementary". Multiplication of a matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ by the elementary matrices from the left is equivalent to adding/subtracting one of the rows to/from the other one. (For example, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) = \left(\begin{smallmatrix}a+c&b+d\\b&d\end{smallmatrix}\right)$.) Similarly, multiplication by the elementary matrices from the right is equivalent to adding/subtracting one of the columns to/from the other one. We therefore may reformulate the problem this way: show that any integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be reduced to the identity matrix using the operations of adding/subtracting one of the rows/columns to/from the other one.

Ignoring the second row and focusing on the first one, we see that the column operations act this way: $(a, b) \mapsto (a \pm b, b)$ or $(a, b) \mapsto (a, b \pm a)$. Thus, following the Euclidean algorithm, we may reduce the first row to the form $(u, 0)$ or $(0, u)$ where $u = \pm\gcd(a, b)$. Moreover, the transformations $(0, u) \mapsto (u, u) \mapsto (u, 0)$ and $(0, -u) \mapsto (u, -u) \mapsto (u, 0)$ show that we can get $(u, 0)$ with positive u. Hence, the column operations allow us to reduce the matrix to the form $\left(\begin{smallmatrix} u & 0 \\ v & w \end{smallmatrix}\right)$, with $u > 0$. The row/column operations we use don't change the determinant of the matrix, which remains equal to 1; so, it must be that $u = w = 1$. Now, using a row operation v times, we may reduce the matrix to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

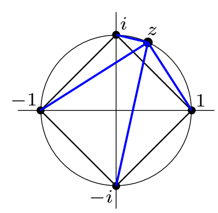
4. *Let $p \in \mathbb{Z}[x]$ and let $a, b, c \in \mathbb{Z}$ be such that $p(a) = b$, $p(b) = c$, and $p(c) = a$. Prove that $a = b = c$.*

Solution. If, say, $a = b$, then also $c = p(b) = p(a) = b$. So, assume that a, b, c are all distinct. Since p has integer coefficients, for any distinct integer u, v the difference $u - v$ divides the difference $p(u) - p(v)$. (Indeed, if $p(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ with $\alpha_i \in \mathbb{Z}$, then $p(u) - p(v) = \alpha_n(u^n - v^n) + \cdots + \alpha_1(u - v)$, where each summand is divisible by $u - v$.) Hence,

$$(a-b) \,\Big|\, (p(a) - p(b)) = b - c, \quad (b-c) \,\Big|\, (c-a), \quad \text{and } (c-a) \,\Big|\, (a-b).$$

It follows that $|a - b| = |b - c| = |c - a|$. But then the equality $(a - b) + (b - c) + (c - a) = 0$ implies that $a - b = b - c = c - a = 0$.

5. *The square $ABCD$ is inscribed in a circle of radius R, and P is a point on the circle. Prove that $|PA|^4 + |PB|^4 + |PC|^4 + |PD|^4 = 24R^4$.*

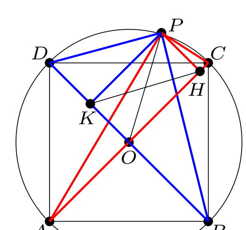


Solution. We may assume that we deal with the unit circle $C = \{z : |z| = 1\}$ in the complex plane and that A, B, C, D are the points $1, i, -1, -i$ on it. For any $z, w \in C$,

$$|z-w|^4 = (z-w)^2\overline{(z-w)}^2 = (z-w)^2(\bar{z}-\bar{w})^2 = (z-w)^2(z^{-1}-w^{-1})^2 = (z-w)^4/z^2w^2.$$

Take any point $P = z \in C$. Then

$$\begin{aligned} |PA|^4 + |PB|^4 + |PC|^4 + |PD|^4 &= |z-1|^4 + |z-i|^4 + |z+1|^4 + |z+i|^4 \\ &= (z-1)^4/z^2 - (z-i)^4/z^2 + (z+1)^4/z^2 - (z+i)^4/z^2 \end{aligned}$$
$$= \tfrac{1}{z^2}\Big((z^4-4z^3+6z^2-4z+1)-(z^4-4iz^3-6z^2+4iz+1)+(z^4+4z^3+6z^2+4z+1)-(z^4+4iz^3-6z^2-4iz+1)\Big)$$
$$= \tfrac{1}{z^2}24z^2 = 24.$$



Another solution. From the right triangles $\triangle APC$ and $\triangle BPD$ we see that $|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2 = 4R^2$. It follows that

$$|PA|^4 + |PC|^4 + |PB|^4 + |PD|^4 = 32R^4 - 2|PA|^2|PC|^2 - 2|PB|^2|PD|^2.$$

Now, $|PA| \cdot |PC| = 2\text{Area}(\triangle APC) = 2R|PH|$, where PH is the height of $\triangle APC$, and similarly, $|PB| \cdot |PD| = 2R|PK|$ where PK is the height of $\triangle BPD$. So, $|PA|^2|PC|^2 + |PB|^2|PD|^2 = 4R^2(|PH|^2 + |PK|^2)$. The lines AC and BD are orthogonal, so PH and PK are orthogonal, so $OHPK$ is a rectangle, and $|PH|^2 + |PK|^2 = |HK|^2 = |OP|^2 = R^2$. Hence, $|PA|^4 + |PC|^4 + |PB|^4 + |PD|^4 = 32R^4 - 2 \cdot 4R^4 = 24R^4$.

6. *Prove that for any $x, y, z \geq 0$, $\sqrt[2]{x + \sqrt[3]{y + \sqrt[4]{z}}} \geq \sqrt[32]{xyz}$.*

Solution. Assume that, for some $x, y, z \geq 0$, $\sqrt[2]{x + \sqrt[3]{y + \sqrt[4]{z}}} < \sqrt[32]{xyz}$. Then

$$\sqrt[2]{x} < \sqrt[32]{xyz}, \quad \sqrt[6]{y} < \sqrt[32]{xyz}, \quad \text{and } \sqrt[24]{z} < \sqrt[32]{xyz},$$

so

$$x < (xyz)^{1/16}, \quad y < (xyz)^{3/16}, \quad \text{and } z < (xyz)^{12/16},$$

and so, $xyz < (xyz)^{1/16+3/16+12/16} = xyz$, contradiction.

Another solution. We will make use of the inequality $a + b \geq a^{\lambda} b^{1-\lambda}$, which holds for any $a, b \geq 0$ and any $\lambda \in [0, 1]$. (Here is the proof: We may assume that $a, b > 0$. Clearly, $a + b \geq \lambda a + (1-\lambda)b$. By the concavity of log, $\log(\lambda a + (1-\lambda)b) \geq \lambda \log a + (1-\lambda) \log b = \log(a^{\lambda} b^{1-\lambda})$, which implies that $\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}$ since log is a strictly increasing function.) Applying the inequality with $\lambda = 1/5$, we obtain $y + \sqrt[4]{z} \geq y^{1/5} z^{1/5}$, and so $\sqrt[3]{y + \sqrt[4]{z}} \geq (yz)^{1/15}$. Taking $\lambda = 1/16$, we obtain $x + (yz)^{1/15} \geq x^{1/16} (yz)^{1/16}$, so, $\sqrt[2]{x + \sqrt[3]{y + \sqrt[4]{z}}} \geq (xyz)^{1/32}$.