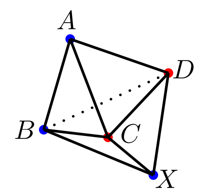
2016 Gordon exam solutions

1. *Is there a continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ which takes on rational values at the irrational points and irrational values at the rational points?*

Solution. No. Such a function would be nonconstant (since it has at least one rational value and at least one irrational value), and take at most countably many values: at most countably many values at irrational points (since the set of rational values is countable), and at most countably many values at rational points (since the set of rational points is countable). However, the image of a nonconstant continuous function contains an interval, so such a function must take on uncountably many values.

2. *Each point of the three dimensional space $\mathbb{R}^3$ is colored either red or blue. Prove that there exists an equilateral triangle with side length 1 whose vertices have the same color.*



Solution. Assume, in the way of contradiction, that the statement is false. Let h be the height of a regular tetrahedron in $\mathbb{R}^3$ with side length 1. ($h = \sqrt{2/3}$.) Take any point A in $\mathbb{R}^3$, and assume, w.l.o.g., that A is blue. Let X be any point of $\mathbb{R}^3$ at the distance of $2h$ from A. Consider a polyhedron $ABCDX$ consisting of two regular tetrahedrons with side length 1, vertices at A and X, and sharing the same base. If, say, the points B and C are blue, then the equilateral triangle ABC has blue vertices; so, at most one of the points B, C, D is blue, and the other two are red. But then the point X must be blue. Hence, all points at the distance of $2h$ from A are blue.

We have proved that any two points in $\mathbb{R}^3$ at the distance of $2h$ from each other share the same color; thus all spheres in $\mathbb{R}^3$ of radius $2h$ are monochromatic, having the color of their centers. But if not all points in $\mathbb{R}^3$ have the same color, there are two points of different colors at the distance $< 4h$ from each other, and the spheres of radius $2h$ centered at these points have a nonempty intersection, contradiction.

3. *Prove that for any $n, d \in \mathbb{N}$ and any vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ one has $\displaystyle\sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq n^2$.*

Solution. For any $x \in \mathbb{R}$ we have $e^x \geq 1 + x$, so

$$\sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq \sum_{i,j=1}^{n} (1 + v_i \cdot v_j) = n^2 + \sum_{i,j=1}^{n} v_i \cdot v_j = n^2 + \Big(\sum_{i=1}^{n} v_i\Big) \cdot \Big(\sum_{i=1}^{n} v_i\Big) = n^2 + \Big\|\sum_{i=1}^{n} v_i\Big\|^2 \geq n^2.$$

4. *Evaluate $\int_{x^2+y^2 \leq R^2} \sin x^2 \cos y^2 \, dxdy$.*

Solution. Let $I = \int_{x^2+y^2 \leq R^2} \sin x^2 \cos y^2 \, dxdy$. Notice that $I = \int_{x^2+y^2 \leq R^2} \sin y^2 \cos x^2 \, dxdy$ as well, so

$$I = \frac{1}{2}\Big(\int_{x^2+y^2 \leq R^2} \sin x^2 \cos y^2 \, dxdy + \int_{x^2+y^2 \leq R^2} \sin y^2 \cos x^2 \, dxdy\Big)$$
$$= \frac{1}{2} \int_{x^2+y^2 \leq R^2} \Big(\sin x^2 \cos y^2 + \sin y^2 \cos x^2\Big) \, dxdy.$$

Now, we have $\sin x^2 \cos y^2 + \sin y^2 \cos x^2 = \sin(x^2 + y^2)$, and passing to the polar coordinates we compute:

$$I = \frac{1}{2} \int_{x^2+y^2 \leq R^2} \sin(x^2 + y^2) \, dxdy = \frac{1}{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \sin(r^2) r \, dr = \frac{1}{2}\pi \int_{0}^{R} \sin(r^2) \, dr^2 = \frac{1}{2}\pi \int_{0}^{R^2} \sin(t) \, dt$$
$$= \frac{1}{2}\pi(1 - \cos(R^2)).$$

5. Prove that for $z_1, \dots, z_4 \in \mathbb{C}$, $\max_{|z_1|,\dots,|z_4|\le 1}\big|z_1z_3+z_1z_4+z_2z_3-z_2z_4\big| = 2\sqrt{2}$.
Solution. Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\big|z_1z_3+z_1z_4+z_2z_3-z_2z_4\big|^2 &\le \big(|z_1|\cdot|z_3+z_4|+|z_2|\cdot|z_3-z_4|\big)^2 \le \big(|z_1|^2+|z_2|^2\big)\big(|z_3+z_4|^2+|z_3-z_4|^2\big)\\ &\le 2\big(|z_3|^2+|z_4|^2+z_3\bar z_4+\bar z_3 z_4+|z_3|^2+|z_4|^2-z_3\bar z_4-\bar z_3 z_4\big) = 4(|z_3|^2+|z_4|^2) = 8.\end{aligned}$$

And for $z_1=z_4=1$, $z_2=i$, $z_3=-i$ we indeed have $\big|z_1z_3+z_1z_4+z_2z_3-z_2z_4\big| = |-i+1+1-i| = 2\sqrt{2}$.

Another solution. For any $z_1,z_2,z_3,z_4 \in \mathbb{C}$ with $|z_1|,|z_2| \le 1$ we have

$$\big|z_1z_3+z_1z_4+z_2z_3-z_2z_4\big| \le |z_1|\cdot|z_3+z_4|+|z_2|\cdot|z_3-z_4| \le |z_3+z_4|+|z_3-z_4|.$$

Our goal is now to maximize this expression when $|z_3|,|z_4| \le 1$. After a rotation of $\mathbb{C}$ (the multiplication of z_3 and z_4 by $\overline{z_4}/|z_4|$), we may assume that z_4 is real; let $z_4 = b$ and $z_3 = a\cos t + ia\sin t$ with $0 \le a,b \le 1$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}\big(|z_3+z_4|+|z_3-z_4|\big)^2 &= \Big(\sqrt{(a\cos t+b)^2+a^2\sin^2 t}+\sqrt{(a\cos t-b)^2+a^2\sin^2 t}\Big)^2\\ &= \Big(\sqrt{a^2+b^2+2ab\cos t}+\sqrt{a^2+b^2-2ab\cos t}\Big)^2 = 2a^2+2b^2+2\sqrt{(a^2+b^2)^2-4a^2b^2\cos^2 t}\\ &\le 2a^2+2b^2+2\sqrt{(a^2+b^2)^2} = 4a^2+4b^2 \le 8,\end{aligned}$$

so $|z_3+z_4|+|z_3-z_4| \le 2\sqrt{2}$.

And for $z_1=z_4=1$, $z_2=i$, $z_3=-i$ we indeed have $\big|z_1z_3+z_1z_4+z_2z_3-z_2z_4\big| = |-i+1+1-i| = 2\sqrt{2}$.

6. *Let A be a 2016×2016 matrix such that all diagonal entries of A are zero and the rest of entries are equal to ± 1. Prove that* $\det A \ne 0$.

Solution. Modulo 2, the matrix A is equal to $\overline{A} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1\\ 1 & 0 & 1 & \dots & 1\\ 1 & 1 & 0 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 0\end{pmatrix}$. It is easy to check that $\overline{A}^2 = I$, so $\det\overline{A} \ne 0$, so $\det A \ne 0$ (since $\det\overline{A} = \det A \bmod 2$).

Another solution. The matrix $\overline{B} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 1\end{pmatrix}$ over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ has rank 1, it maps the entire space to the one-dimensional space spanned by the vector $e = (1,1,\dots,1)$, so all its eigenvalues except one are equal to zero; also $\overline{B}e = 2016e = 0$ in $\mathbb{Z}_2$, so the last eigenvalue is equal to zero as well. So, the matrix $\overline{A} = A \bmod 2 = \overline{B} - I$ has n eigenvalues equal to 1, so the determinant of $\overline{A}$ (which is the product of its eigenvalues) is equal to 1 as well.