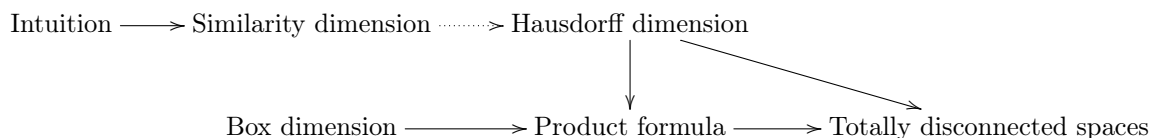


# The Similarity and Hausdorff Dimensions

Ji Hoon Chun

July 26, 2012

## Overview:



**Definition.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a **similarity with ratio  $r$**  ( $r > 0$ ) if  $|f(x) - f(y)| = r|x - y|$ .

**Definition.** A set  $A \subseteq \mathbb{R}^n$  is **self-similar** if there exists a finite  $m$  and similarities  $f_1, \dots, f_m$  with the ratio  $r$  such that  $A = f_1(A) \cup \dots \cup f_m(A)$ .

**Definition.** Similarities  $f_1, \dots, f_m$  are **separated** if there is a bounded open set  $O$  such that  $f_1(O) \cup \dots \cup f_m(O) \subseteq O$  and the  $f_i(O)$ 's are disjoint.

**Definition.** The **similarity dimension** for a self-similar set  $S \subseteq \mathbb{R}^n$  with separated self-similarities is  $\dim_S S = \frac{\log m}{-\log r}$ , where  $m$  = the number of scaled copies of an object (scaled using similarities) needed to occupy the same space as the original object and  $r$  = the similarity ratio.

**Definition.** The **diameter** of a nonempty set  $S \subseteq \mathbb{R}^n$  is  $|S| = \sup \{|x - y| : x, y \in S\}$ .

**Definition.**  $\mathcal{H}_\delta^a(S) = \inf \left\{ \sum_{\alpha \in \Lambda} |S_\alpha|^a : S \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha, |S_\alpha| \leq \delta \right\}$ , where  $\Lambda$  is a countable index set.

**Definition.** The  **$a$ -dimensional Hausdorff measure** of a set  $S \subseteq \mathbb{R}^n$  is  $\mathcal{H}^a(S) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^a(S)$ .

**Definition.** The **Hausdorff dimension** of  $S \subseteq \mathbb{R}^n$  is  $\dim_H S = \inf \{a \geq 0 : \mathcal{H}^a(S) = 0\} = \sup \{a : \mathcal{H}^a(S) = \infty\}$ , with the condition that  $\sup \emptyset = 0$ .

**Theorem.** For self-similar sets  $S$  (with separated self-similarities),  $\dim_S S = \dim_H S$ .

**Proposition.** Let  $S \subseteq \mathbb{R}^n$  and  $\mathbf{x}, \mathbf{y} \in S$ . If a function  $f : S \rightarrow \mathbb{R}^m$  satisfies the Lipschitz condition  $|f(\mathbf{x}) - f(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|$  for some  $C \in [0, \infty)$ , then  $\dim_H f(S) \leq \dim_H S$ .

**Theorem.** If a set  $S \subseteq \mathbb{R}^n$  (with at least 2 points) satisfies  $\dim_H S < 1$ , then  $S$  is totally disconnected.

**Theorem.** Given two sets  $S \subseteq \mathbb{R}^m$  and  $T \subseteq \mathbb{R}^n$ ,  $\dim_H(S \times T) \geq \dim_H S + \dim_H T$  and  $\dim_H(S \times T) \leq \dim_H S + \overline{\dim}_B T$ , where  $\overline{\dim}_B$  is the upper box dimension.

**Corollary.** If  $\dim_H T = \overline{\dim}_B T$ , then  $\dim_H(S \times T) = \dim_H S + \dim_H T$ .

**Theorem.** The product of an arbitrary collection of totally disconnected topological spaces is also totally disconnected.

**Corollary.** There exist totally disconnected sets of arbitrarily high Hausdorff dimension.

## References:

- Falconer, Kenneth, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 2003.
- Patty, C. Wayne, *Foundations of Topology*, Jones and Bartlett, 2009.
- Stein, Elias M. and Shakarchi, Rami, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.