The Similarity and Hausdorff Dimensions

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Overview:

$$\begin{tikzcd} \text{Intuition} \arrow[r] & \text{Similarity dimension} \arrow[r, dotted] & \text{Hausdorff dimension} \arrow[d] \arrow[dr] & \\ & \text{Box dimension} \arrow[r] & \text{Product formula} \arrow[r] & \text{Totally disconnected spaces} \end{tikzcd}$$

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **similarity with ratio $\boldsymbol{r}$** $(r > 0)$ if $|f(x) - f(y)| = r|x - y|$.

Definition. A set $A \in \mathbb{R}^n$ is **self-similar** if there exists a finite m and similarities $f_1, \ldots, f_m$ with the ratio r such that $A = f_1(A) \cup \cdots \cup f_m(A)$.

Definition. Similarities $f_1, \ldots, f_m$ are **separated** if there is a bounded open set O such that $f_1(O) \cup \cdots \cup f_m(O) \subseteq O$ and the $f_i(O)$'s are disjoint.

Definition. The **similarity dimension** for a self-similar set $S \subseteq \mathbb{R}^n$ with separated self-similarities is $\dim_S S = \frac{\log m}{-\log r}$, where $m =$ the number of scaled copies of an object (scaled using similarities) needed to occupy the same space as the original object and $r =$ the similarity ratio.

Definition. The **diameter** of a nonempty set $S \subseteq \mathbb{R}^n$ is $|S| = \sup\{|x - y| : x, y \in S\}$.

Definition. $\boldsymbol{\mathcal{H}^a_\delta(S)} = \inf\left\{\sum_{\alpha\in\Lambda} |S_\alpha|^a : S \subseteq \bigcup_{\alpha\in\Lambda} S_\alpha, |S_\alpha| \le \delta\right\}$, where Λ is a countable index set.

Definition. The **$\boldsymbol{a}$-dimensional Hausdorff measure** of a set $S \subseteq \mathbb{R}^n$ is $\mathcal{H}^a(S) = \lim_{\delta\to 0} \mathcal{H}^a_\delta(S)$.

Definition. The **Hausdorff dimension** of $S \subseteq \mathbb{R}^n$ is $\dim_H S = \inf\{a \ge 0 : \mathcal{H}^a(S) = 0\} = \sup\{a : \mathcal{H}^a(S) = \infty\}$, with the condition that $\sup \emptyset = 0$.

Theorem. *For self-similar sets S (with separated self-similarities), $\dim_S S = \dim_H S$.*

Proposition. *Let $S \subseteq \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in S$. If a function $f : S \to \mathbb{R}^m$ satisfies the Lipschitz condition $|f(\mathbf{x}) - f(\mathbf{y})| \le C|\mathbf{x} - \mathbf{y}|$ for some $C \in [0, \infty)$, then $\dim_H f(S) \le \dim_H S$.*

Theorem. *If a set $S \subseteq \mathbb{R}^n$ (with at least 2 points) satisfies $\dim_H S < 1$, then A is totally disconnected.*

Theorem. *Given two sets $S \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$, $\dim_H(S \times T) \ge \dim_H S + \dim_H T$ and $\dim_H(S \times T) \le \dim_H S + \overline{\dim}_B T$, where $\overline{\dim}_B$ is the upper box dimension.*

Corollary. *If $\dim_H T = \overline{\dim}_B T$, then $\dim_H(S \times T) = \dim_H S + \dim_H T$.*

Theorem. *The product of an arbitrary collection of totally disconnected topological spaces is also totally disconnected.*

Corollary. *There exist totally disconnected sets of arbitrarily high Hausdorff dimension.*

References:

- Falconer, Kenneth, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 2003.
- Patty, C. Wayne, *Foundations of Topology*, Jones and Bartlett, 2009.
- Stein, Elias M. and Shakarchi, Rami, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.