What is inversive geometry?

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Throughout, Greek letters $(\alpha, \beta, \dots)$ denote geometric objects like circles or lines; small Roman letters $(a, b, \dots)$ denote distances, and large Roman letters $(A, B, \dots)$ denote points.. The symbol $\overline{AB}$ denotes the line segment between points A and B and $\ell(\overline{AB})$ denotes its length. The symbol $\overrightarrow{AB}$ denotes the ray with endpoint at A and passing through B.

1 Steiner's Porism

Suppose we have two nonintersecting circles α and β in the plane. When is it possible to draw n circles $\gamma_1, \dots, \gamma_n$ such that each γ_i is tangent to both α and β, as well as to γ_{i+1} and γ_{i-1} (indices taken mod n)? How many different such drawings are there for fixed circles α and β?

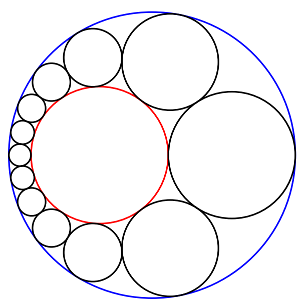

Figure 1: A solution for $n = 12$ [Credit: WillowW / Wikimedia Commons / CC-BY-SA]

Suppose α, β are concentric. Let a and b denote the radii of α and β respectively, and assume $a > b$. Clearly, such an arrangement of circles γ_i exists only when the centers $C_1, \dots, C_n$ of the n circles $\gamma_1, \dots \gamma_n$ are the vertices of a regular n-gon, whose center is the common center O of α and β. Consider then a point of tangency T between two circles γ_i and γ_{i+1}, and the triangle $\triangle OC_iT$. The edge $\overline{OT}$ is a tangent to γ_i, and $\overline{C_iT}$ is a radius, so $\overline{OT} \perp \overline{C_iT}$. Furthermore, $\ell(\overline{C_iT}) = \frac{a-b}{2}$ and $\ell(\overline{OC_i}) = \frac{a+b}{2}$, by straightforward geometry. So, we have

$$\sin\left(\frac{\pi}{n}\right) = \frac{\ell(\overline{C_iT})}{\ell(\overline{OC_i})} = \frac{a-b}{a+b} = \frac{(a/b)-1}{(a/b)+1} = \frac{\rho-1}{\rho+1}$$

where $\rho := \frac{a}{b}$.

Rearranging the equation $\sin(\pi/n) = (\rho-1)/(\rho+1)$ to solve for ρ gives us:

$$\rho\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right) = \rho-1 \Leftrightarrow \rho\left(\sin\left(\frac{\pi}{n}\right)-1\right) = -\sin\left(\frac{\pi}{n}\right)-1 \Leftrightarrow \rho = \frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} = \left(\frac{1+\sin(\pi/n)}{cos(\pi/n)}\right)^2$$

So, it is possible to draw n circles $\gamma_1, \dots, \gamma_n$ according to the rules of Steiner's Porism precisely when the ratio $\rho = \frac{a}{b}$ satisfies $\rho = (\sec(\pi/n) + \tan(\pi/n))^2$ (after simplifying the trigonometric functions from the

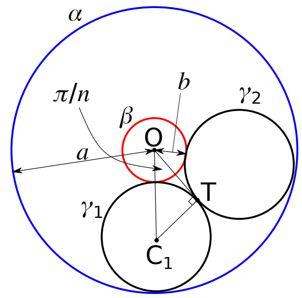

Figure 2: Geometry of Steiner's Porism for concentric circles A and B

previous equation). Furthermore, we can obtain uncountably many different drawings just by rotating about O through any angle.

Direct analysis of this sort is significantly more difficult in the general (non-concentric) case. So, let's look for another method.

2 Inversive Geometry

We will work in the Euclidean plane $\mathbb{R}^2$, with an additional point at infinity (denoted ∞) adjoined. We will consider a line in the plane to be a circle of infinite radius, which included the point at infinity. Any two lines intersect at ∞. If the lines are parallel in the Euclidean sense, so that they meet only at ∞, we consider them to be tangent.

The first example of an inversion map: $f(z) : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^{-1}$ for $z \neq 0$, $f(0) = \infty$, $f(\infty) = 0$. This turns the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ inside-out, exchanging external points and internal points. We will generalize this to any circle in the Eucldean plane, and express inversion in purely geometric terms.

Definition For any circle γ in the plane with center O and radius r, the **inversion map through** γ takes any point $P \neq O$ to the (unique) point P' on the ray $\overrightarrow{OP}$ such that $\ell(OP) \cdot \ell(OP') = r^2$. I will denote the inversion map through γ by T_γ.

Some properties:

1. T_γ exchanges points inside γ with points outside.
2. The closer a point on the inside is to O, the farther away its image is.
3. The fixed points of $T\gamma$ are precisely the points on the circle γ.
4. T_γ is an involution, i.e. $T_\gamma \circ T_\gamma = \mathrm{id}_{\mathbb{R}^2 \cup \{\infty\}}$.
5. If α is a circle concentric with γ, then $T_\gamma(\alpha)$ is also a circle concentric with α and γ.
6. If λ is a line running through the point O, then $T_\gamma(\lambda) = \lambda$.
7. If μ is any line not passing through O, then $T_\gamma(\mu)$ is a circle containing O.
8. If β is any circle which included the point O, then $T_\gamma(\beta)$ is a line not passing through O.

The first 6 properties are straightforward to prove. The last two are only slightly more involved, and they follow from elementary geometry:

Suppose μ is a line not running through O as in Figure 3. We want to show that the image of μ under T_γ is a circle containing O. If we draw the perpendicular $\overline{OA}$ to μ, we can find the image $A' = T\gamma(A)$. Then, we consider the circle with diameter $\overline{OA'}$ and show that any point P on μ maps to this circle. Let P' be the point where $\overrightarrow{OP}$ intersects the circle with diameter $\overline{OA'}$. By similar triangles, $\ell(OP)/\ell(OA) = \ell(OA')/\ell(OP')$, so $\ell(OP) \cdot \ell(OP') = \ell(OA) \cdot \ell(OA') = r^2$, so $P' = T\gamma(P)$. Thus, $T\gamma(\mu)$ is the circle with diameter $\overline{OA'}$.

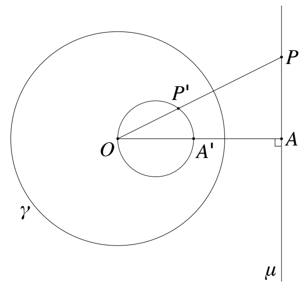

Figure 3: Proof that a line not through O inverts to a circle

Since $T_\gamma = T_\gamma^{-1}$, it follows that the circle with diameter OA' is mapped to a line not containing O, and perpendicular to the ray $\overrightarrow{OA'}$. This holds as we move A' throughout the plane, i.e. any circle through O inverts to a line not passing through O.

3 Concyclic points and separation

Definition If four points A, B, C, D lie on the same circle (or line, since we count lines as circles of infinite radius), we say following [1] that the pair AC *separates* BD – written $AC//BD$ – if either arc from A to C intersects either B or D. In other words, $AC//BD$ if and only if the points lie in the cyclic order A, B, C, D along the circle (up to reversal of the order). We will extend the definition of separation to four points in general position after a theorem.

Theorem (Proof from [1]) When A, B, C, D are not concyclic (or colinear), there are two non-intersecting circles, one through A and C, and another though B and D.

Remark This was a Putnam exam problem in 1965.

Proof. Consider the perpendicular bisectors α and β of $\overline{AC}$ and $\overline{BD}$ respectively. If these bisectors coincide, then A, B, C, D are concyclic. So, these bisectors either intersect once in $\mathbb{R}^2$ (and again at ∞, which intersection we will ignore) or are parallel.

If the bisectors interesect at the (finite) point O, then there are two circles γ and δ with center O, such that A and C lie on γ, and B and D lie on δ. If A, B, C, D are not concycic, these are two distinct concentric circles, which competes this case of the proof.

Otherwise, α and β are parallel. This forces $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ to be parallel as well, so these four lines form a rectangle. Consider the points P and Q, the midpoints of the sides of this rectangle which coincide with α and β respectively. There is a circle through A, P, and C and another through B, Q, and D, and because of the separation between lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, these two circles cannot intersect. This completes the proof.

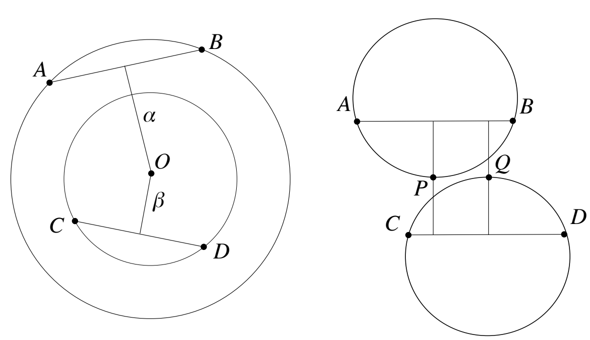

Figure 4: Diagram for proof of theorem [Credit: Reproduction of figure in [1]]

$\square$

With this fact in mind, we can redefine the symbol $AC//BD$ to mean that every circle through A and C intersects or coincides with every circle through B and D. Then by the theorem, $AC//BD$ only if A, B, C, D are concyclic, in the concyclic case, this new definition agrees with the old definition of separation.

Remark In [1], a third characterization of $AC//BD$ is presented. The authors prove that for any four distinct points A, B, C, D,

$$\ell(\overline{AB}) \cdot \ell(\overline{CD}) + \ell(\overline{BC}) \cdot \ell(\overline{AD}) \geq \ell(\overline{AC}) \cdot \ell(\overline{BD})$$

and then prove that equality holds if and only if $AC//BD$. I will use this fact without proof, due to time constraints.

Now, we can give a new method of specifyng circles. For any three points A, B, and C, the circle though A, B, and C is the union of $\{A, B, C\}$ and the set of all points X such that $BC//AX$, $CA//BX$, or $AB//CX$. Clearly, any point X satisfying any of these three separation conditions lies on the same circle as A, B, and C (by the Theorem above), and conversely any point X on the circle is either A, B, or C, or satisfies one of the separation conditions.

An important result is that inversion through a circle preserves separation. The following short Lemmas are due to [1]:

Lemma If γ is any circle with center O and radius r, and A, B are any two points with $T_\gamma(A) = A'$ and $T_\gamma(B) = B'$, then

$$\ell(\overline{A'B'}) = \frac{r^2 \ell(\overline{AB})}{\ell(\overline{OA}) \cdot \ell(\overline{OB})}$$

Proof. The triangles $\triangle OAB$ and $\triangle OB'A'$ are similar, so we have:

$$\frac{\ell(\overline{A'B'})}{\ell(\overline{AB})} = \frac{\ell(\overline{OA'})}{\ell(\overline{OB})} = \frac{\ell(\overline{OA}) \cdot \ell(\overline{OA'})}{\ell(\overline{OA}) \cdot \ell(\overline{OB})} = \frac{r^2}{\ell(\overline{OA}) \cdot \ell(\overline{OB})}$$

$\square$

Definition The *cross ratio* between two pairs of points AC and BD is $\{AB, CD\} = (\ell(\overline{AC}) \cdot \ell(\overline{BD}))/(\ell(\overline{AD}) \cdot \ell(\overline{BC}))$. In particular, $AC//BD$ if and only if $\{AB, CD\} = 1$.

Lemma Inversion through circle γ with center O and radius r preserves cross ratios.

Proof.

$$\{A'B', C'D'\} = \frac{\ell(\overline{A'C'}) \cdot \ell(\overline{B'D'})}{\ell(\overline{A'D'}) \cdot \ell(\overline{B'C'})} = \frac{\frac{r^2 \ell(\overline{AC})}{\ell(\overline{OA}) \cdot \ell(\overline{OC})} \frac{r^2 \ell(\overline{BD})}{\ell(\overline{OB}) \cdot \ell(\overline{OD})}}{\frac{r^2 \ell(\overline{AD})}{\ell(\overline{OA}) \cdot \ell(\overline{OD})} \frac{r^2 \ell(\overline{BC})}{\ell(\overline{OB}) \cdot \ell(\overline{OC})}} = \frac{\ell(\overline{AC}) \cdot \ell(\overline{BD})}{\ell(\overline{AD}) \cdot \ell(\overline{BC})} = \{AB, CD\}$$

$\square$

Combining this with the previous results, we see that inversion also preserves separation. This leads us to the following theorem:

Theorem Any inversion map takes circles to circles (recall that lines are considered circles of infinite radius).

Proof. Suppose that we have a circle through points A, B, and C. Recall that this circle is $\alpha = \{X : BC//AX, AC//BX,\ \text{or}\ AB//CX\} \cup \{A, B, C\}$. If we invert through some circle γ, we get $A' = T_\gamma(A)$, $B' = T_\gamma(B)$, and $C' = T_\gamma(C)$. Furthermore, for any other point X on the circle, if (say) $BC//AX$, then $B'C'//A'X'$, where $X' = T_\gamma(X)$.

Thus,

$$\begin{aligned} T_\gamma(\alpha) &= \{X' : X' = T_\gamma(X) \text{ and } BC//AX, AC//BX,\ \text{or}\ AB//CX\} \cup \{A', B', C'\} \\ &= \{X' : B'C'//A'X', A'C'//B'X',\ \text{or}\ A'B'//C'X'\} \cup \{A', B', C'\} \end{aligned}$$

is also a circle.

□

Equally important, inversion preserves the number of intersections of any two circles, so tangent circles invert to other tangent circles, etc.

Finally, we shall consider some results about orthogonal circles.

Definition Two circles are called *orthogonal* if they intersect twice at right angles.

Lemma Inversion through a circle γ with center O preserves the angle of intersection (namely, the angle between the two tangents at the point of intersection) of any two intersectings circles.

Proof. First, Let α, β be two intersecting circles. Let λ be tangent to α and μ tangent to β. Under the inversion map, λ maps to a circle λ' which touches the center O of the inversion, and whose tangent at O is parallel to α. Likewise, the μ' has a tangent at O parallel to β. By well-known facts about parallel lines, the angle between the tangents to λ' and μ' is congruent to the angle between λ and μ.

Now, if α and β were tangent to λ and μ respectively at the point of intersection of the two lines (call it P) then α' and β' are tangent to λ' and μ' at P', which forces their angle of intersection to equal that of λ' and μ', which is equal to the angle between λ and μ, which is the angle of intersection of α and β.

□

Corollary Inversion maps orthogonal circles to orthogonal circles.

Theorem Given any two non-intersecting circles, it is possible to invert them into two concentric circles.

Proof. Let α, β be two non-intersecting circles. By geometry (see [1], Section 5.7), there is a line λ, called the *radical axis* of α and β, which is perpendicular to the line of the centers of α and β. For many points P on this line (precisely, those points for which the quantity $\ell(\overline{O_\alpha P})^2 - r_\alpha^2 = \ell(\overline{O_\beta P})^2 - r_\beta^2 > 0$, where O_α is the center of α and r_α its radius), the length of the tangents from P to either A or B are equal. A circle drawn with center P and radius equal to this common length will be orthogonal to both α and β.

We can draw two such circles γ and δ, which will intersect at two points, say O inside of α, and P inside of β. We then invert through any circle with center O.

Let P' be the image of P under this inversion, α' the image of α, and so on. Then, γ' and δ' are two lines which meet at O, since γ and δ were circles touching O. Since α was orthogonal to both γ and δ, we have α' orthogonal to γ' and δ'. Since γ' is a line, it must be a diameter of α'; likewise δ' is another diameter. these diameters meet at P', so P' must be the center of α'. Likewise, P' is the center of β', and we are done.

□

4 Finishing the proof of Steiner's Porism

. . . is now easy. I leave it as an exercise to the reader in applying the prior theorem.

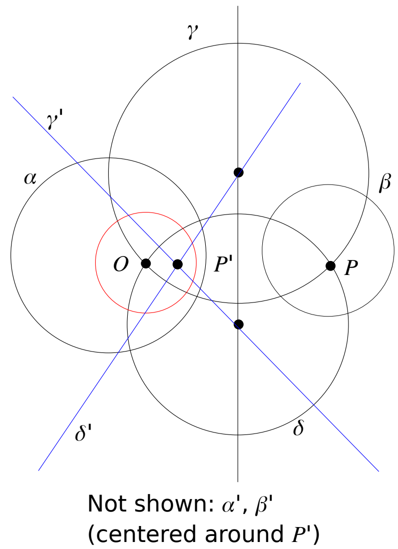

Figure 5: Proof that you can invert two non-intersecting circles into two concentric circles [Credit: This author, based on a proof and figure from [1]]

5 Bibliography

- Coxeter, H.S.M. and Greitzer, S.L. *Geometry Revisited*. New York: Random House, 1967. Print.
- https://en.wikipedia.org/wiki/Steiner_chain

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