What are Iterated Function Systems?

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Metric Space A pair (X, d) where X is a nonempty set and $d : X \times X \to [0, \infty)$ s.t. $\forall x, y, z \in X$

1) $d(x, y) = d(y, x)$
2) $d(x, y) = 0$ iff $x = y$
3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Cauchy Sequence Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a *Cauchy sequence* if $\forall \varepsilon > 0, \exists N$ s.t. if $n_1, n_2 > N$, then $d(x_{n1}, x_{n2}) < \varepsilon$.

Complete Metric Space A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges in X.

Diameter Let (X, d) be a metric space. Let $A \subseteq X$. Then $\text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$.

<u>Theorem 1</u>

Let (X, d) be a complete metric space. Let $\{A_n\}$ be a sequence of nonempty closed subsets of X s.t. $A_1 \supseteq A_2 \supseteq \ldots$ and $\text{diam}(A_n) \to 0$ as $n \to \infty$. Then there is a single point $x \in \bigcap_{n=1}^{\infty} A_n$.

Lipschitz constant Let (X, d) be a metric space. Let $F : X \to X$. Then the *Lipschitz constant* of F is $\text{Lip}(F) = \sup\{d(F(x), F(y))/d(x, y) \mid x, y \in X, x \neq y\}$

Contraction Let (X, d) be a metric space, $F : X \to X$, and $\text{Lip}(F) = \lambda$. Then we say that F is a *contraction* if $\lambda < 1$.

<u>Theorem 2</u> (Banach Fixed Point Theorem)

Let F be a contraction on a complete metric space (X, d). Then F has a unique fixed point, i.e., $\exists$ a unique $x \in X$ s.t. $F(x) = x$.

Iterated Function System (IFS) An *iterated function system* is a finite set of contractions $S = \{S_1, \ldots, S_N\}$ on a complete metric space (X, d).

<u>Theorem 3</u>

Let S be an IFS on the complete metric space (X, d). Then there is a unique compact set K s.t. $K = \bigcup_{i=1}^{N} S_i(K)$.

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[1] J.E. Hutchinson, *Fractals and Self Similarity*, Indiana University Mathematics Journal **30** (1981), 713-747.