KAKEYA NEEDLE PROBLEM

YISHUN DONG

Introduction

The Kakeya Needle Problem was proposed in 1917, by the Japanese mathematician S. Kakeya. It states:

In the class of figures in which a segment of length 1 can be turned around through 360 degree, remaining always within the figure, which one has the smallest area?

Some figures in the class

1. A circle of diameter 1. It has area of $\pi \times 0.5^2 = \dfrac{\pi}{4} \approx 0.78$.

2. A pentagram with diagonal of length 1. It has area of ≈ 0.59.

3. An equilateral triangle with height 1. It has area of $\dfrac{1}{2} \times (1 \times \dfrac{2}{\sqrt{3}}) = \dfrac{\sqrt{3}}{3} \approx 0.58$

The first assumed "answer" to the problem

A **three-cornered hypocycloid** that is inscribed in a circle of diameter $\dfrac{3}{2}$.

A hypocycloid is the curve that is described by a point on the circumference of a circle, as the circle rolls on the inside of the circumference of a second, fixed circle. It is known that the tangent line at a point M of the hypocycloid meets the hypocycloid at two distinct points K and L, distance 1 from each other. If one end of the segment stays touching the hypocycloid, while the other illustrates it, both ends move on the hypocycloid. Thus, the segment remains within the figure the entire time, and so it fits into the class of figures.

The area of the hypocycloid is given by:

$A = \dfrac{(n-1)(n-2)}{n^2}\pi a^2$, where n is the number of cusps, and a is the radius of the circle.

Thus, here the *three-cornered hypocycloid inscribed in a circle of diameter* $\dfrac{3}{2}$ has the area $\dfrac{\pi}{8} \approx .39$. (Interestingly, this is exactly half that of a circle of diameter 1)

It was assumed to be the minimum area for a period of time, until Besicovitch reported his solution of the Kakeya problem in *Mathematische Zeitschrift* (proved in 1920, published in 1928).

Besicovitch Solution

Problem which led Besicovitch to the success solution of the Kakeya needle problem:

If f is a Riemann integrable function defined on the plane, is it always possible to find orthogonal coordinate axis such that with respect to these coordinates the integral $\int f(x,y)\,dx$ exists as a Riemann integral for all y where the resulting function of y is also Riemann integrable?

Besicovitch noticed that if he constructed a compact set F of plane Lebesgue measure zero containing a unit line segment in every direction he could answer this question by showing a counter example. He indeed succeeded in constructing such a set, which shows not only that the hypocycloid conjecture is false, but there are figures of arbitrarily small areas which allow a unit segment, or needle, to be turned around through 360, remaining always within the figure. His solution has been simplified, with the help of Perron trees, which were introduced by Oskar Perron in 1928, and the idea of joins, which were suggested by Besicovitch's colleague J. Pal, a Hungarian mathematician.

Step 1: Discrete rotations, start with 2 by 2 squares, use parallel translations to reduce the area.

Step 2: Move between elementary triangles using Pal's Joins, allows continuous rotaions

Step 3: Use Perron Trees to construct regions with arbitrarily small areas.

Related definitions, theorems and facts

Definition 1: A Kakeya(Besicovitch) set is a compact set $E \subset \mathbb{R}^n$ which contains a unit line segment in each direction, i.e.

$$\forall \xi \in S^{n-1} \ \ \exists x \in \mathbb{R}^n : x + t\xi \in E \ \ \forall t \in [-\frac{1}{2}, \frac{1}{2}]$$

For example, all those figures mentioned at the beginning are Kakeya sets for $n = 2$. Thus, the problem can be restated as finding Kakeya sets with small Lebesgue measure. As it has shown by Besicovitch, there are Kakeya sets of measure zero.

Theorem 2: For $n \geq 2$, there is a set $F \subset \mathbb{R}^n$ with n-dimensional Lebesgue measure zero which contains a **unit line segment** in every direction.

Theorem 3: For $n > 2$, there is a set $F \subset \mathbb{R}^n$ with n-dimensional Lebesgue measure zero which contains a **line segment** in every direction.

Definition 4: A **Nikodym set** in the unit square $K := [0,1]^2$ in the plane is a subset $N \subset K$ with area 1 such that for every point $x \in N$ there is a straight line L through x such that $L \cap N = \{x\}$.

Definition 5: A (n,k)**-set** is a compact set $E \subset \mathbb{R}^n$ which contains a translate of every k-dimensional subspace, or alternatively an essential portion of that subspace like a disc.

Analogues and Generalizations of the Kakeya problem

1. The solution of Kakeya's problem for **convex sets** is an equilateral triangle of area $\frac{1}{\sqrt{3}}$ (Due to J. Pal in 1921).

2. **Kakeya problem in bounded region:** A kakeya set can have arbitrarily small area and stay inside a circle of radius $2 + \epsilon$, with ϵ being arbitrarily small number (Due to A.H. van Alphen in 1941).

3. **The spherical Kakeya problem:** Instead of a plane, the rotation takes place on the surface of a unit sphere and an arc of great circle plays the role of the needle.

4. **Alternative moving figure:** Instead of a straight line segment consider a broken line segment consisting of three segments, which we call a "bird", with small central segment as the "body" and the end segments as the "wings".