WHAT IS... AN INTRINSICALLY KNOTTED GRAPH?

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1. Introduction

Definition 1 (Graph)**.** *A* graph *is pair $G = (V, E)$ where V is a finite set whose elements we call vertices and E is a finite set whose elements, called edges, are two element subsets of V.*

Example 1.

(1) *Let n be a positive integer. Let $V = \{1, ..., n\}$ and let $E = \{(a, b) : 1 \leq a < b \leq n\}$. The graph $G = (V, E)$ is called the complete graph with n vertices and is denoted by K_n.*

(2) *More generally, for any positive integer n and any positive integers $r_1, ..., r_n$, the complete n-partite graph $K_{r_1,r_2,...,r_n}$ is the following graph: let A_i be a finite set with r_i, for each $i = 1, ..., n$ and let $V = \bigcup A_i$. Then let $E := \{(a, b) : a \in A_i, b \in A_j, 1 \leq i < j \leq n\}$.*

(3) *Given a graph $G = (V, E)$ we can form the graph $G + 1 = (V', E')$ with $V' = V \cup \{1\}$ (of course assuming $1 \notin V$) and $E' = E \cup \{(1, a) : a \in V\}$. For instance $K_n + 1 = K_{n+1}$ and $K_{r_1,...,r_n} + 1 = K_{r_1,...,r_n,1}$*

Definition 2 (Cycle)**.** *A* cycle *in a graph $G = (V, E)$ is a subset $\{v_1, v_2, ..., v_n\} \subset V$ such that $(v_1, v_2) \in E, (v_2, v_3) \in E, ..., (v_n, v_1) \in E$.*

Definition 3 (Embedding of a Graph)**.** *An* embedding of a graph *$G = (V, E)$ into $\mathbb{R}^d$ is an injective map $\phi : V \to \mathbb{R}^d$ together with a piecewise straight line between $\phi(a)$ and $\phi(b)$ for each edge $(a, b) \in E$.*

Definition 4 (Planar Graph)**.** *A graph G is called* planar *if there exists an embedding of G into $\mathbb{R}^2$ for which the image of two edges never intersect.*

Definition 5 (Minor of a graph)**.** *$H = (V', E')$ is a* subgraph *of $G = (V, E)$ if $V' \subset V$, $E' \subset E$ and H is a graph. L is a* minor *of G is L is homeomorphic to some subgraph of G, equivalently if L can be obtained from G by collapsing and deleting edges.*

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Theorem 1 (Kuratowski)**.** *A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

2. Knots and links

Definition 6 (Knots and links)**.** *A* knot *is a closed piecewise straight line in $\mathbb{R}^3$ not crossing itself. A* link *is a finite disjoint union of knots.*

Definition 7 (Ambient isotopy)**.** *Let $\psi : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ be a continuous map such that $\psi(x,0) = x$ for every $x \in \mathbb{R}^3$ and for each $t \in [0,1]$ the map $x \mapsto \psi(x,t)$ is an homeomorphism of $\mathbb{R}^3$. Then a homeomorphism of $\mathbb{R}^3$ is called an* ambient isotopy *if it can be realized as the map $x \mapsto \psi(x,1)$ for some ψ.*

Definition 8 (Trivial knot, nonsplit link)**.** *A knot is called* trivial *if there is an ambient isotopy taking it into a circle. A link is called* nonsplit *if all the components bound disjoint discs.*

3. Intrinsically knotted and linked graphs

Definition 9 (Intrinsically knotted/linked graphs)**.** *A graph G is intrinsically knotted if in every embedding of G, one of the cycles is a non-trivial knot. A graph G is intrinsically linked if in every embedding of G, two of the cycles form a non-split link.*

Theorem 2 (J. H. Conway, C. Gordon [1])**.**

(1) *K_6 is intrinsically linked.*
(2) *K_7 is intrinsically knotted.*

Proposition 1 (Sachs, 1984 [7])**.** *If G is non-planar then $G+1$ is intrinsically linked. Reciprocally if $G+1$ is intrinsically linked then G is non-planar.*

Definition 10 (ΔY transformation)**.** *Let $G = (V, E)$ be a graph, let $a, b, c \in V$ be a triangle, i.e., such that $(a,b),(a,c),(b,c) \in E$. Now consider the graph $G' = (V', E')$ where $V' = V \cup \{1\}$ and $E' = (E \cup \{(1,a),(1,b),(1,c)\}) \setminus \{(a,b),(a,c),(b,c)\}$. We say that G' is obtained from G by a ΔY transformation and that G is obtained from G' by a $Y\Delta$ transformation.*

Theorem 3 (Montwani, Raghunathan, Saran, 1988 [3])**.** *If G is intrinsically knotted (resp. linked) and G' is obtained from G by either a ΔY or a $Y\Delta$ transformation, then G' is also intrinsically knotted (resp. linked).*

Definition 11 (Petersen family)**.** *The* Petersen family *is the family of all graphs that can be obtained from K_6 through successive $Y\Delta$ or ΔY transformations.*

Proposition 2. *The Petersen family has 7 elements, including the Petersen graph and $K_{3,3,1}$.*

4. Classifying intrinsically linked graphs

Observation 1. *If H is a minor of G and H is intrinsically knotted (resp. linked) then G is intrinsically knotted (resp. linked) as well.*

Definition 12 (Minimal minor)**.** *Let P be a property of graphs such that if a minor of G has P then G also has P. Then a graph G is* minimal minor *with respect to P if G has P and no proper minor of G has P.*

Example 2. *If P is the property of being planar then K_5 and $K_{3,3}$ are the only minimal minors.*

Theorem 4 (Robertson-Seymour's graph minor theorem a.k.a. Wagner's conjecture, 2004 [5])**.** *The set of minimal minors is always finite.*

Theorem 5 (Robertson, Seymour, Thomas, 1993 [6])**.** *The Petersen family is the set of minimal minors for the intrinsically linked graphs.*

The full family of minimal minors for the intrinsically knotted property is not known. Here are some known facts

Theorem 6 (2007, [4])**.** *Let G be a graph. Then $(G+1)+1$ is intrinsically knotted if and only G is non-planar.*

Remark 1. *It was conjectured that $G+1$ is an intrinsically knotted graph if and only if G is intrinsically linked, but a counter example was found.*

There are other known intrinsically minimal minor knotted graphs that do not arise from theorems 6 and 3.

5. Some proofs

Definition 13 (Knot diagrams)**.** *A knot diagram is a projection of a knot into a plane such that at most two points of the knot are projected to the same point of the plane. We also use the convention that the lower portion in each crossing is interrupted in the crossing.*

We will use the same conventions to deal with graph diagrams.

Definition 14 (Linking number). *The linking number of a link with two components is the number modulo* 2 *of times one of the components crosses the other from below.*

Proposition 3. *The linking number does not depend on the projection and is invariant under ambient isotopies.*

Observation 2. *The linking number of a splitting link is* 0*, hence if the linking number of a link is* 1*, the link in non-split.*

Proof of Theorem 2, part (1) (Sketch). The sum of the linking number over all pairs of 3-cycles of K_6 is unchanged when we change a cross on a diagram of K_6. It is also easy to exhibit a diagram where this sum is 1. □

Definition 15 (Arf invariant [2]). *The arf invariant of a knot is number in* $\mathbb{Z}/2\mathbb{Z}$ *such that the arf invariant of the trivial knot is* 0*.*

Proof of Theorem 2, part (2) (Sketch). The sum of the arf invariants over all Hamiltonian cycles of K_7 is unchanged when we change a cross on a diagram of K_7. It is also easy to exhibit a diagram where this sum is 1. □

References

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