What is Kuratowski's 14 set theorem?

Duncan Clark, 1 July 2014

Introduction

In 1920, Kazimierz Kuratowski (1896–1980) published the following theorem as part of his dissertation.

Theorem 1 (Kuratowski)**.** *Let X be a topological space and $E \subset X$. Then, at most 14 distinct subsets of X can be formed from E by taking closures and complements.*

This theorem is fairly well known today and shows up as a (difficult) exercise in many general topology books (such as Munkre's *Topology*), perhaps due to the mystique of the number 14. In this paper we will present a proof of the theorem, and in addition, investigate how the number 14 changes if we include intersections, unions and interior operators.

1 Background knowledge

Let us begin by recalling some basic definitions. Let X be a set, a set $\mathcal{T} \subset \mathcal{P}(X)$ is called a **topology** on X if the following hold:

1. $\emptyset, X \in \mathcal{T}$.
2. If $\{E_\alpha\}$ is a collection of sets in $\mathcal{T}$, then $\bigcup_\alpha E_\alpha \in \mathcal{T}$.
3. If $E_1, \ldots, E_n \in \mathcal{T}$, then $\bigcap_{i=1}^n E_i \in \mathcal{T}$.

Given a pair $(X, \mathcal{T})$, we call an element $E \in \mathcal{T}$ an **open set** of X, the complement of an open set is called a **closed set**. The **closure** of a set $E \subset X$, denoted $\mathrm{cl}(E)$, is the intersection of all closed sets containing E and the **interior** of E, denoted $\mathrm{int}(E)$, is the union of all open sets contained in E.

Moreover, for each E the closure and interior of E are uniquely determined. So, we can view $E \mapsto \mathrm{cl}(E)$, $E \mapsto \mathrm{int}(E)$ as functions from $\mathcal{P}(X)$ to itself. In general, we denote by $\mathrm{End}(\mathcal{P}(X))$ the set of all functions $\varphi\colon \mathcal{P}(X) \to \mathcal{P}(X)$. A general element $\varphi \in \mathrm{End}(\mathcal{P}(X))$ is called an **endomorphism** of $\mathcal{P}(X)$. For convenience sake, we will drop the $\circ$ generally used when composing functions, and denote the closure of E by kE, the interior by iE and complement by cE. Functions will be applied to the left, so that, for example, the closure of the complement of E can be succinctly written kcE.

We say a function $k \in \mathrm{End}(\mathcal{P}(X))$ is a **Kuratowski closure operator** if for all sets $E, F \subset X$ the following hold:

1. $k\emptyset = \emptyset$.
2. $kkE = kE$.
3. $E \subset kE$.
4. $kE \cup kF = k(E \cup F)$.

One can verify that the Kuratowski closure operator is indeed the closure operator from topology if we insist that X be given the topology consisting of sets $\{ckE : E \subset X\}$.

Let $I \in \mathrm{End}(\mathcal{P}(X))$ represent the identity function, then one can verify that:

$$k^2 = k,\ c^2 = I,\ i = ckc,\ i^2 = i,\ ic = ck,\ kc = ci. \tag{1}$$

We leave it as an exercise to prove that these relations indeed hold.

Let us recall that a set P is a **poset** (or partially-ordered set) if there is a relation binary relation $\leq$ on P such that:

1. $a \leq a$ for all $a \in P$ (reflexivity).
2. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).
3. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

We will create a poset on $\mathrm{End}(\mathcal{P}(X))$ by asserting

$$\varphi \leq \psi \text{ iff } \varphi(E) \subseteq \psi(E),\ \forall E \subset X.$$

We leave it to the reader as an exercise to prove that this is indeed a poset. Note that in addition to being a poset, our $\mathrm{End}(\mathcal{P}(X))$ is also a **monoid**. That is, a set together with a binary operation (in our case $\circ$) that is associative and has a neutral element.

2 Main theorem

In this section we will present a proof of Theorem 1. To begin, we will make use of the following lemma.

Lemma 1. *The following relations hold ($\varphi, \psi \in \mathrm{End}(\mathcal{P}(X))$).*

1. $i \leq I \leq k$.
2. *If* $\varphi \leq \psi$ *then* $c\varphi \geq c\psi$ *that is,* c *switches the order.*
3. *If* $\varphi \leq \psi$ *then* $k\varphi \leq k\psi$ *and* $i\varphi \leq i\psi$*, that is* k, i *do not switch order.*
4. *If* $\varphi \leq \psi$ *then* $\varphi\sigma \leq \psi\sigma$ *for any* $\sigma \in \mathrm{End}(\mathcal{P}(X))$.

Proof. Left as an exercise to the reader. ☐

We will make use of one more lemma in our proof of Theorem 1.

Lemma 2. *Let* $k, i \in \mathrm{End}(\mathcal{P}(X))$ *be closure and interior operators respectively. Then, the cardinality of the monoid generated by* k, i *is at most* 7.

For convenience sake let the monoid be represented by (k, i), and in general agree to represent our monoids in such a way.

Proof. From Lemma 1, we know $I \leq k$. So that $i = Ii \leq ki$. Applying i on the left, we find that $ii \leq iki$. Since $ii = i$, we then have $i \leq iki$. Similarly, $i \leq I$. So that $ik \leq Ik = k$. Therefore, $kik \leq kk = k$. So that $kik \leq k$.

Now since $i \leq iki$ we have $ik \leq (iki)k = i(kik)$. But $kik \leq k$, so that $i(kik) \leq ik$. Thus, $ik \leq ikik \leq ik$ and so $ik = ikik$. Similarly, $ki \leq k(iki) = (kik)i \leq ki$. So that $ki = kiki$.

Thus, given any word on symbols k, i we can apply $k^2 = k, i^2 = i$ to reduce to a string of alternating k, i. But, using $ki = kiki, ik = ikik$, we know the string can be at most 3 terms long. Therefore, we may only produce the following strings (some may be equal, but we know that this is the largest collection that has the ability not be equal)

$$I, i, ik, iki, k, ki, kik. \tag{2}$$

Since there are 7 terms it follows that $|(k, i)| \leq 7$. ☐

Below, we provide the Hasse diagram illustrating the structure of the poset generated by (k, i).

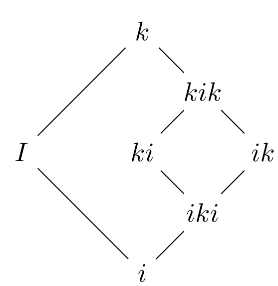

Figure 1: Hasse diagram for (k, i).

We are now prepared to prove Theorem 1, which we restate below.

Theorem 1 (Kuratowski)**.** *Let X be a topological space and $E \subset X$. Then, at most 14 distinct subsets of X can be formed from E by taking closures and complements.*

Proof. Recall from (1) that $i = ckc$. So, the monoids generated by k, i, c and k, c are the same. Now, from (1) as well we know that $ic = ck, kc = ci$. So, given a word on symbols k, i, c we can assume without loss of generality that all the c's appear on the left. Last, we use the relation that $c^2 = I$ and find that any word can be reduced to one with either one or zero c's at the leftmost position. Thus, $|(k, i, c)| \leq 2|(k, i)|$ (since we cannot guarantee that each be unique). To the right, then, are simply the words generated by k, i. But $|(k, i)| \leq 7$. So that

$$|(k, c)| = |(k, i, c)| \leq 2|(k, i)| \leq 14.$$

So, given $E \subset X$ we can produce no more than 14 distinct sets from E by taking closures and complements. $\square$

Let's agree to call a set $E \subset X$ that produces 14 distinct sets from closure and complements a **Kuratowski 14 set**. Perhaps unsurprisingly, there is a Kuratowski set in $\mathbf{R}$. The following set does the trick (we leave the computation as an exercise to the reader, or refer to [3] for the solution)

$$S = \{0\} \cup (1, 2) \cup (2, 3) \cup (\mathbf{Q} \cap (4, 5)). \tag{3}$$

However, we have given no intuition as to why such a set is indeed a 14-set. The site in [6] features an interective applet where the user can pick various subsets of the real line to observe how the number of distinct sets produced by (k, c) changes. There are many other interesting questions to ask about Kuratowski sets. For example: Does every space have a Kuratowski set? How many Kuratowski sets in $\mathbf{R}$ are there? and, are they all measurable? Can a Kuratowski set be countable?

It turns out the lower bound for the cardinality of a Kuratowski set is 3 [7]. A 3 element Kuratowski set can be found in the 7 point space $X = \{1, \ldots, 7\}$. Let $\mathcal{T}$ be a topology on X with basis

$$\mathcal{B} = \{\emptyset, X, \{1\}, \{6\}, \{1, 2\}, \{3, 4\}, \{5, 6\}\}.$$

Then the set $A = \{1, 3, 5\}$ is a Kuratowski set, as the reader may verify.

3 Generalizations

In this section, we will offer a generalized approach to the Kuratowski problem. To do so, we will make use of $\mathrm{End}(\mathcal{P}(X))$ not just as a poset, but as a lattice.

Recall that a poset P is called a **lattice** if any two elements $x, y \in P$ have a least upper bound and a greatest lower bound. The least upper bound is called the **join** of x and y and is written $x \vee y$, similarly the greatest lower bound is called the **meet** and is written $x \wedge y$. A lattice is **distributive** if for any x, y, z we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Given a lattice, P, we say that P is **Boolean** if:

1. It contains a least element, 0, and a greatest element, 1.
2. For any $a \in P$ there is a $b \in P$, called the **complement** of a, such that $a \wedge b = 1$ and $a \vee b = 0$.

On our space $\mathrm{End}(\mathcal{P}(X))$ there is a very natural Boolean lattice structure. We simply assert that for $\varphi, \psi \in \mathrm{End}(\mathcal{P}(X))$ and $E \subset X$

$$(\varphi \vee \psi)(E) = \varphi(E) \cup \psi(E),\ (\varphi \wedge \psi)(E) = \varphi(E) \cap \psi(E).$$

Given $E \subset X$, the complement of E (in terms of Boolean lattice structure) is naturally the complement of E in the topological sense (we leave it to the reader to verify that $\mathrm{End}(\mathcal{P}(X))$ is indeed a distributive, Boolean lattice). Thus, each of $k, i, c, \wedge, \vee$ are unary operations in $\mathrm{End}(\mathcal{P}(X))$. We now propose the following question:

Question. *Given $E \subset X$, and $\mathcal{O} \subset \{k, c, i, \wedge, \vee\}$, what is the maximal number of distinct subsets of X that can be formed by repeatedly applying operations from $\mathcal{O}$ on the set E?*

It is clear that when $\mathcal{O} = \{k, c\}$ this reduces to Kuratowski's problem. We will now answer this question for the case $\mathcal{O} = \{k, i, \wedge\}$. We will first need the following lemma (the proof of which we leave to the reader).

Lemma 3. *For any $\varphi, \psi \in \mathrm{End}(\mathcal{P}(X))$ the following hold:*

1. $i(\varphi \wedge \psi) = i\varphi \wedge i\psi,\ k(\varphi \vee \psi) = k\varphi \vee k\psi.$
2. $i\varphi \vee i\psi \leq i(\varphi \vee \psi),\ k(\varphi \wedge \psi) \leq k\varphi \wedge k\psi.$

Theorem 2. *Given a topological space X and $E \subset X$. Then, the maximal number of distinct subsets of X than can be formed by repeating operations from $\{k, i, \wedge\}$ is 13.*

Proof. We begin with the diagram found in figure 1. Add $ik \wedge ki$ and notice that $iki \leq ik \wedge ki$ since $iki = iki \wedge iki \leq ik \wedge ki$. Now, add the meet of I to any $\sigma \in \{I, i, ik, iki, k, ki, kik, ik \wedge ki\}$. Three of these are obviously redundant since $I \wedge I = I$, $I \wedge i = i$, $I \wedge k = k$. We claim the 13 element set

$$\mathcal{S} = \{I, i, ik, iki, k, ki, kik, ki \wedge ik, I \wedge ik, I \wedge ki, I \wedge ik \wedge ki, I \wedge iki, I \wedge kik\}$$

is maximal, that is, closed under the operations $k, i, \wedge$.

First, i distributes across $\wedge$, so applying i to any $\sigma \in \mathcal{S}$ and reducing gives another element of $\mathcal{S}$. Second, $\mathcal{S}$ is closed under $\wedge$ by construction.

Third, we show that $k\mathcal{S} = \mathcal{S}$. For any σ not of the form $I \wedge \tau$, $k\sigma$ is clearly in $\mathcal{S}$.

So, let σ be any of $ki \wedge ik$, $I \wedge iki$, $I \wedge ik \wedge ki$, $I \wedge ki$. We have $k\sigma \leq ki$ since each has either a iki or ki term and $k(iki) = k(ki) = ki$. Also, $i \leq \sigma$ so that $ki \leq k\sigma$. Thus, $k\sigma = ki$.

Consider last $k(I \wedge ik)$, $k(I \wedge kik)$. We see,

$$k(I \wedge ik) \leq kI \wedge kik \leq k \wedge kik = kik,$$
$$k(I \wedge kik) \leq kI \wedge kkik \leq k \wedge kik = kik.$$

We claim $k(I \wedge ik) = k(I \wedge kik) = kik$. Indeed,

$$\begin{aligned} ik = ik \wedge k &= ik \wedge k((I \wedge ik) \vee (I \wedge cik)) \\ &= ik \wedge (k(I \wedge ik) \vee k(I \wedge cik)) = (ik \wedge k(I \wedge ik)) \vee (ik \wedge k(I \wedge cik)). \end{aligned}$$

But $ik \wedge k(I \wedge cik) \leq ik \wedge k(cik) = ik \wedge ciik = ik \wedge cik = 0$. Where 0 is the endomorphism such that $0(E) = \emptyset$ for all $E \subset X$. Thus, $ik \leq ik \wedge k(I \wedge ik)$ and so $ik \leq k(I \wedge ik)$. Therefore, since $I \wedge ik \leq I \wedge kik$,

$$kik \leq k^2(I \wedge ik) \leq k(I \wedge kik).$$

Thus, $k(I \wedge ik) = k(I \wedge kik) = kik$. So $\mathcal{S}$ is closed under k and the proof is complete.

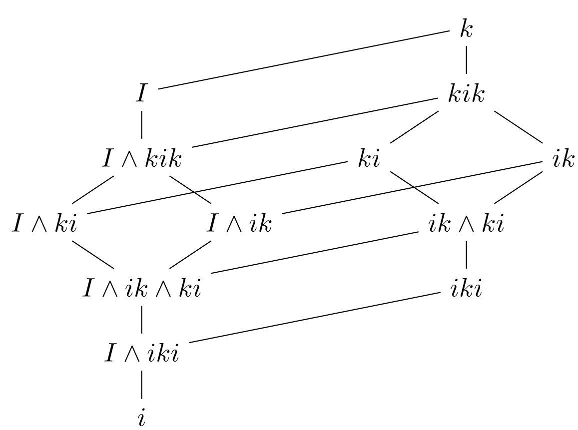

Figure 2: Hasse diagram for $(k, i, \wedge)$.

$\square$

Remark. The following set $T \subset \mathbf{R}$ generates 13 distinct subsets of $\mathbf{R}$ by repeated application of $k, i, \wedge$ (we leave it to the the reader to verify at his or her own risk).

$$T = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\} \bigcup \left([2,4] \setminus \left\{ 3 + \frac{1}{n} : n \in \mathbf{N} \right\} \right) \bigcup \left((5,7] \cap \left(\mathbf{Q} \cup \bigcup_{n=1}^{\infty} \left(6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi} \right) \right) \right). \quad (4)$$

A similar question with $\vee$ instead of $\wedge$ was posed as Problem 5996 in the Nov. 1974 edition of *The American Mathematical Monthly* [4]. C. Y. Yu affirmed that at most 13 distinct sets can be produced with operations $k, i, \vee$ and published a solution four years later, in 1978 [5]. The reader may verify that the following set does the trick:

$$U = \left\{ 1/n : n \in \mathbf{N} \right\} \cup \left\{ x \in (2,3) : x \notin \mathbf{Q} \right\} \cup (3,4) \cup (4,5). \quad (5)$$

In figure 3, we present a table with the answers to our question posited earlier. A more thorough overview, including a proof of the $k, i, \wedge, \vee$ case can be found in [2]. Indeed the number for the $k, i, \wedge, \vee$ case is 35, and the severely dedicated reader may verify that the set T from (4) produces thirty five distinct sets.

	$\{I\}$	$\{\wedge\}$	$\{\vee\}$	$\{\wedge, \vee\}$
$\{I\}$	1	1	1	1
$\{i\}$	2	2	2	2
$\{k\}$	2	2	2	2
$\{c\}$	2	4	4	4
$\{i, k\}$	7	13	13	35
$\{i, c\} = \{i, k, c\} = \{k, c\}$	14	∞	∞	∞

Figure 3: Table providing Kuratowski numbers for $\mathcal{O} \subset \{k, i, c, \wedge, \vee\}$.

We last provide as an example of a set which generates an infinite (countably) number of distinct sets from the operations $k, i, c, \wedge, \vee$. Let $\mathcal{T}$ be the topology on $\mathbf{N}$ generated by the open sets $\{[1, a) : a \in \mathbf{N}\}$. Given a nonempty set $E \subset \mathbf{N}$, one can verify that the $kE = [\min E, \infty)$. Now, let $\varphi \in \mathrm{End}(\mathcal{P}(\mathbf{N}))$, $\varphi = I \wedge (k(k \wedge c))$. Take $E = 2\mathbf{N}$. Then, $\varphi^j(E) = E \cap [2j+2, \infty)$, as the reader may verify.

Acknowledgments

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References

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