What is Kuratowski’s 14 set theorem?
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Introduction

In 1920, Kazimierz Kuratowski (1896–1980) published the following theorem as part of his dissertation.

**Theorem 1** (Kuratowski). Let $X$ be a topological space and $E \subset X$. Then, at most 14 distinct subsets of $X$ can be formed from $E$ by taking closures and complements.

This theorem is fairly well known today and shows up as a (difficult) exercise in many general topology books (such as Munkre’s *Topology*), perhaps due to the mystique of the number 14. In this paper we will present a proof of the theorem, and in addition, investigate how the number 14 changes if we include intersections, unions and interior operators.

1 Background knowledge

Let us begin by recalling some basic definitions. Let $X$ be a set, a set $T \subset \mathcal{P}(X)$ is called a topology on $X$ if the following hold:

1. $\emptyset, X \in T$.
2. If $\{E_\alpha\}$ is a collection of sets in $T$, then $\bigcup_{\alpha} E_\alpha \in T$.
3. If $E_1, \ldots, E_n \in T$, then $\bigcap_{i=1}^n E_i \in T$.

Given a pair $(X, T)$, we call an element $E \in T$ an open set of $X$, the complement of an open set is called a closed set. The closure of a set $E \subset X$, denoted $\text{cl}(E)$, is the intersection of all closed sets containing $E$ and the interior of $E$, denoted $\text{int}(E)$, is the union of all open sets contained in $E$.

Moreover, for each $E$ the closure and interior of $E$ are uniquely determined. So, we can view $E \mapsto \text{cl}(E), E \mapsto \text{int}(E)$ as functions from $\mathcal{P}(X)$ to itself. In general, we denote by $\text{End}(\mathcal{P}(X))$ the set of all functions $\varphi: \mathcal{P}(X) \to \mathcal{P}(X)$. A general element $\varphi \in \text{End}(\mathcal{P}(X))$ is called an endomorphism of $\mathcal{P}(X)$. For convenience sake, we will drop the generally used when composing functions, and denote the closure of $E$ by $kE$, the interior by $iE$ and complement by $cE$. Functions will be applied to the left, so that, for example, the closure of the complement of $E$ can be succinctly written $kcE$.

We say a function $k \in \text{End}(\mathcal{P}(X))$ is a Kuratowski closure operator if for all sets $E, F \subset X$ the following hold:

1. $k\emptyset = \emptyset$.
2. $kkE = kE$.
3. $E \subset kE$.
4. $kE \cup kF = k(E \cup F)$.

One can verify that the Kuratowski closure operator is indeed the closure operator from topology if we insist that $X$ be given the topology consisting of sets $\{ckE : E \subset X\}$.

Let $I \in \text{End}(\mathcal{P}(X))$ represent the identity function, then one can verify that:

\[
k^2 = k, \ c^2 = I, \ i = ckc, \ i^2 = i, \ ic = ck, \ kc = ci.
\]

We leave it as an exercise to prove that these relations indeed hold.

Let us recall that a set $P$ is a poset (or partially-ordered set) if there is a relation binary relation $\leq$ on $P$ such that:

1. $a \leq a$ for all $a \in P$ (reflexivity).
2. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).
3. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).
We will create a poset on $\text{End}(\mathcal{P}(X))$ by asserting

$$\varphi \leq \psi \iff \varphi(E) \subseteq \psi(E), \, \forall E \subset X.$$  

We leave it to the reader as an exercise to prove that this is indeed a poset. Note that in addition to being a poset, our $\text{End}(\mathcal{P}(X))$ is also a monoid. That is, a set together with a binary operation (in our case $\circ$) that is associative and has a neutral element.

2 Main theorem

In this section we will present a proof of Theorem 1. To begin, we will make use of the following lemma.

Lemma 1. The following relations hold ($\varphi, \psi \in \text{End}(\mathcal{P}(X))$).

1. $i \leq I \leq k$.
2. If $\varphi \leq \psi$ then $c\varphi \geq c\psi$ that is, $c$ switches the order.
3. If $\varphi \leq \psi$ then $k\varphi \leq k\psi$ and $i\varphi \leq i\psi$, that is $k, i$ do not switch order.
4. If $\varphi \leq \psi$ then $\varphi\sigma \leq \psi\sigma$ for any $\sigma \in \text{End}(\mathcal{P}(X))$.

Proof. Left as an exercise to the reader.  

We will make use of one more lemma in our proof of Theorem 1.

Lemma 2. Let $k, i \in \text{End}(\mathcal{P}(X))$ be closure and interior operators respectively. Then, the cardinality of the monoid generated by $k, i$ is at most 7.

For convenience sake let the monoid be represented by $(k, i)$, and in general agree to represent our monoids in such a way.

Proof. From Lemma 1, we know $I \leq k$. So that $i = Ii \leq ki$. Applying $i$ on the left, we find that $ii \leq iki$. Since $ii = i$, we then have $i \leq iki$. Similarly, $i \leq I$. So that $ik \leq Ik = k$. Therefore, $kik \leq kk = k$. So that $kik \leq k$.

Now since $i \leq iki$ we have $ik \leq (iki)k = i(kik)$. But $kik \leq k$, so that $i(kik) \leq k$. Thus, $ik \leq ikik \leq ik$ and so $ik = ikik$. Similarly, $ki \leq k(iki) = (kik)i \leq ki$. So that $ki = kiki$.

Thus, given any word on symbols $k, i$ we can apply $k^2 = k, i^2 = i$ to reduce to a string of alternating $k, i$. But, using $ki = kiki, ik = ikik$, we know the string can be at most 3 terms long. Therefore, we may only produce the following strings (some may be equal, but we know that this is the largest collection that has the ability not be equal)

$$(I, i, ik, iki, k, ki, kik).$$

Since there are 7 terms it follows that $|\langle k, i \rangle| \leq 7$.

Below, we provide the Hasse diagram illustrating the structure of the poset generated by $(k, i)$.

![Hasse diagram](image)

Figure 1: Hasse diagram for $(k, i)$.

We are now prepared to prove Theorem 1, which we restate below.
Theorem 1 (Kuratowski). Let $X$ be a topological space and $E \subset X$. Then, at most 14 distinct subsets of $X$ can be formed from $E$ by taking closures and complements.

Proof. Recall from (1) that $i = c k$. So, the monoids generated by $k, i, c$ and $k, c$ are the same. Now, from (1) as well we know that $i c = c k, k c = c i$. So, given a word on symbols $k, i, c$ we can assume without loss of generality that all the $c$'s appear on the left. Last, we use the relation that $c^2 = I$ and find that any word can be reduced to one with either one or zero $c$'s at the leftmost position. Thus, $|(k, i, c)| \leq 2|(k, i)|$ (since we cannot guarantee that each be unique). To the right, then, are simply the words generated by $k, i$. But $|(k, i)| \leq 7$. So that

$$|(k, c)| = |(k, i, c)| \leq 2|(k, i)| \leq 14.$$  

So, given $E \subset X$ we can produce no more than 14 distinct sets from $E$ by taking closures and complements. \qed

Let’s agree to call a set $E \subset X$ that produces 14 distinct sets from closure and complements a Kuratowski 14 set. Perhaps unsurprisingly, there is a Kuratowski set in $\mathbb{R}$. The following set does the trick (we leave the computation as an exercise to the reader, or refer to [3] for the solution)

$$S = \{0\} \cup \{1, 2\} \cup \{2, 3\} \cup \{(\mathbb{Q} \cap (4, 5))\}.$$  

(3)

However, we have given no intuition as to why such a set is indeed a 14-set. The site in [6] features an interactive applet where the user can pick various subsets of the real line to observe how the number of distinct sets produced by $(k, c)$ changes. There are many other interesting questions to ask about Kuratowski sets. For example: Does every space have a Kuratowski set? How many Kuratowski sets in $\mathbb{R}$ are there? and, are they all measurable? Can a Kuratowski set be countable?

It turns out the lower bound for the cardinality of a Kuratowski set is 3 [7]. A 3 element Kuratowski set can be found in the 7 point space $X = \{1, \ldots, 7\}$. Let $T$ be a topology on $X$ with basis

$$B = \{\emptyset, X, \{1\}, \{6\}, \{1, 2\}, \{3, 4\}, \{5, 6\}\}.$$  

Then the set $A = \{1, 3, 5\}$ is a Kuratowski set, as the reader may verify.

3 Generalizations

In this section, we will offer a generalized approach to the Kuratowski problem. To do so, we will make use of $\text{End}(\mathcal{P}(X))$ not just as a poset, but as a lattice.

Recall that a poset $P$ is called a lattice if any two elements $x, y \in P$ have a least upper bound and a greatest lower bound. The least upper bound is called the join of $x$ and $y$, and is written $x \vee y$. Similarly, the greatest lower bound is called the meet and is written $x \wedge y$. A lattice is distributive if for any $x, y, z$ we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. A lattice is complete if it contains a least element, 0, and a greatest element, 1.

On our space $\text{End}(\mathcal{P}(X))$ there is a very natural Boolean lattice structure. We simply assert that for $\varphi, \psi \in \text{End}(\mathcal{P}(X))$ and $E \subset X$

$$(\varphi \lor \psi)(E) = \varphi(E) \cup \psi(E), \quad (\varphi \land \psi)(E) = \varphi(E) \cap \psi(E).$$  

Given $E \subset X$, the complement of $E$ (in terms of Boolean lattice structure) is naturally the complement of $E$ in the topological sense (we leave it to the reader to verify that $\text{End}(\mathcal{P}(X))$ is indeed a distributive, Boolean lattice). Thus, each of $k, i, c, \wedge, \vee$ are unary operations in $\text{End}(\mathcal{P}(X))$. We now propose the following question:

**Question.** Given $E \subset X$, and $\mathcal{O} \subset \{k, c, i, \wedge, \vee\}$, what is the maximal number of distinct subsets of $X$ that can be formed by repeatedly applying operations from $\mathcal{O}$ on the set $E$?
It is clear that when \( \mathcal{O} = \{k, c\} \) this reduces to Kuratowski’s problem. We will now answer this question for the case \( \mathcal{O} = \{k, i, \land\} \). We will first need the following lemma (the proof of which we leave to the reader).

**Lemma 3.** For any \( \varphi, \psi \in \text{End}(\mathcal{P}(X)) \) the following hold:

1. \( i(\varphi \land \psi) = i\varphi \land i\psi, \ k(\varphi \lor \psi) = k\varphi \lor k\psi. \)
2. \( i\varphi \lor i\psi \leq i(\varphi \lor \psi), \ k(\varphi \land \psi) \leq k\varphi \land k\psi. \)

**Theorem 2.** Given a topological space \( X \) and \( E \subset X \). Then, the maximal number of distinct subsets of \( X \) than can be formed by repeating operations from \( \{k, i, \land\} \) is 13.

**Proof.** We begin with the diagram found in figure 1. Add \( ik \land ki \) and notice that \( iki \leq ik \land ki \) since \( iki = iki \land iki \leq ik \land ki \). Now, add the meet of \( I \) to any \( \sigma \in \{I, i, ik, k, ki, kik, ik \land ki\} \). Three of these are obviously redundant since \( I \land I = I, I \land i = i, I \land k = k \). We claim the 13 element set

\[
S = \{I, i, ik, ik, k, ki, kik, I \land ik, I \land ki, I \land ik \land ki, I \land iki, I \land kik\}
\]

is maximal, that is, closed under the operations \( k, i, \land \).

First, \( i \) distributes across \( \land \), so applying \( i \) to any \( \sigma \in S \) and reducing gives another element of \( S \). Second, \( S \) is closed under \( \land \) by construction.

Third, we show that \( kS = S \). For any \( \sigma \) not of the form \( I \land \tau \), \( k\sigma \) is clearly in \( S \).

So, let \( \sigma \) be any of \( ki \land ik, I \land iki, I \land ik \land ki, I \land ki \). We have \( k\sigma \leq ki \) since each has either a \( iki \) or \( ki \) term and \( k(iki) = k(ki) = ki \). Also, \( i \leq \sigma \) so that \( ki \leq k\sigma \). Thus, \( k\sigma = ki \).

Consider last \( k(I \land ik), (k(I \land kik)). \) We see,

\[
k(I \land ik) \leq kI \land kik \leq k \land kik = kik,
\]
\[
k(I \land kik) \leq kI \land kki \leq k \land kik = kik.
\]

We claim \( k(I \land ik) = k(I \land kik) = kik \). Indeed,

\[
iki = ik \land k = ik \land k((I \land ik) \lor (I \land cik))
\]
\[
= ik \land (k(I \land ik) \lor k(I \land cik)) = (ik \land k(I \land ik)) \lor (ik \land k(I \land cik)).
\]

But \( ik \land k(I \land cik) \leq ik \land k(cik) = ik \land cik = ik \land cik = 0 \). Where 0 is the endomorphism such that \( 0(E) = \emptyset \) for all \( E \subset X \). Thus, \( ik \leq ik \land k(I \land ik) \) and so \( ik \leq k(I \land ik) \). Therefore, since \( I \land ik \leq I \land kik \),

\[
kik \leq k^2(I \land ik) \leq k(I \land kik).
\]

Thus, \( k(I \land ik) = k(I \land kik) = kik \). So \( S \) is closed under \( k \) and the proof is complete.

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\[
\begin{array}{c}
\text{Figure 2: Hasse diagram for } (k, i, \land).
\end{array}
\]
Remark. The following set $T \subset \mathbb{R}$ generates 13 distinct subsets of $\mathbb{R}$ by repeated application of $k,i,\wedge$ (we leave it to the reader to verify at his or her own risk).

$$T = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left( [2,4] \setminus \left\{ 3 + \frac{1}{n} : n \in \mathbb{N} \right\} \right) \cup \left( (5,7] \cap \left( \mathbb{Q} \cup \bigcup_{n=1}^{\infty} \left( 6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi} \right) \right) \right).$$  \hspace{1cm} (4)

A similar question with $\lor$ instead of $\wedge$ was posed as Problem 5996 in the Nov. 1974 edition of The American Mathematical Monthly [4]. C. Y. Yu affirmed that at most 13 distinct sets can be produced with operations $k,i,\lor$ and published a solution four years later, in 1978 [5]. The reader may verify that the following set does the trick:

$$U = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ x \in (2,3) : x \notin \mathbb{Q} \right\} \cup (3,4) \cup (4,5).$$  \hspace{1cm} (5)

In figure 3, we present a table with the answers to our question posited earlier. A more thorough overview, including a proof of the $k,i,\wedge,\lor$ case can be found in [2]. Indeed the number for the $k,i,\wedge,\lor$ case is 35, and the severely dedicated reader may verify that the set $T$ from (4) produces thirty five distinct sets.

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Figure 3: Table providing Kuratowski numbers for $\mathcal{O} \subset \{k,i,c,\wedge,\lor\}$.

We last provide as an example of a set which generates an infinite (countably) number of distinct sets from the operations $k,i,c,\wedge,\lor$. Let $\mathcal{T}$ be the topology on $\mathbb{N}$ generated by the open sets $\{[1,a) : a \in \mathbb{N} \}$. Given a nonempty set $E \subset \mathbb{N}$, one can verify that the $kE = \min E, \infty)$. Now, let $\varphi \in \text{End}(\mathcal{P}(\mathbb{N}))$, $\varphi = I \wedge (k(k \wedge c))$. Take $E = 2\mathbb{N}$. Then, $\varphi^j(E) = E \cap [2j + 2, \infty)$, as the reader may verify.

Acknowledgments

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References