WHAT IS ANDERSON'S LOG-ALGEBRAICITY?

RUDOLPH BRONSON PERKINS

Abstract. Motivated by certain classical conjectures over number fields that logarithms of algebraic numbers should be algebraically independent over $\overline{\mathbb{Q}}$ whenever they are linearly independent over $\mathbb{Q}$, G. Anderson developed machinery called t-motives to attack this problem over function fields. Later, as a way of constructing explicit logarithms of algebraic numbers in the function field setting, Anderson developed a technique that he referred to as *log-algebraicity*. Due to the monumental work of M. Papanikolas we know now that Andersons machinery of t-motives works, and further we have the following remarkable theorem, due to Papanikolas:

If $\lambda_1, \ldots, \lambda_r$ are (Carlitz) logarithms of numbers which are algebraic over the rational function field K with coefficients in a finite field, and if $\lambda_1, \ldots, \lambda_r$ are linearly independent over K, then they are algebraically independent over the algebraic closure of K.

We shall use transcendence as motivation for studying Anderson's log-algebraicity in the simplest case of polynomial rings over finite fields. We will introduce the Carlitz module and related functions, and we will draw analogies with classical constructions as we move along.

This set of notes is intended for a general mathematical audience, in particular, non-number theorists.

Date: July 2, 2013.

1. Analogies between $\mathbb{Z}$ and $A := \mathbb{F}_q[\theta]$ (and Notation!)

Rings of integers, both Euclidean domains	$\mathbb{Z}$	$A := \mathbb{F}_q[\theta]$
Finitely many multiplicative units and zero	$\{-1, 0, 1\}$	$\mathbb{F}_q$
"Positive" elements	$\mathbb{N}$	$A_+ :=$ "monics"
Fields of fractions	$\mathbb{Q}$	$K := \mathbb{F}_q(\theta)$
Completions, w.r.t. which the rings of integers embed discretely and co-compactly	$\mathbb{R}$	$K_\infty := \mathbb{F}_q((\theta^{-1}))$
Smallest complete algebraically closed field containing the integers	$\mathbb{C}$	$\mathbb{C}_\infty$
Polynomials giving another module structure ("exponentiation") to algebraic closures, closed under composition	$\{x^n : n \in \mathbb{Z}\}$	$\{C_a(x) : a \in A\ \}$
Entire exponential functions giving module homomorphisms from regular multiplication action to "exponentiation" action	$e^x : \mathbb{C} \to \mathbb{C}^\times$	$e_C(x) : \mathbb{C}_\infty \to \mathbb{C}_\infty$
Functional equations, good for all integers	$e^{(nx)} = (e^x)^n$	$e_C(ax) = C_a(e_C(x))$
Imaginary "units", i and ι, respectively, defined as roots of the equations	$x^2 + 1 = 0$	$x^{q-1} + \theta = 0$
Fundamental periods of exponentials	$2\pi i$	$\widetilde{\pi} = \iota^q \prod_{j \geq 1} \left(1 - \theta/\theta^{q^j}\right)^{-1}$
Kernel of the exponential functions	$2\pi i\mathbb{Z}$	$\widetilde{\pi}A$
Compact "unit circles" in $\mathbb{C}$, resp. $\mathbb{C}_\infty$	$\{e^{2\pi i x} : x \in \mathbb{R}/\mathbb{Z}\}$	$\{e_C(\widetilde{\pi}x) : x \in K_\infty/A\}$

2. Motivation for us: Transcendence

2.1. **Number fields.** Recall that a complex number $x \in \mathbb{C}$ is called *algebraic* (over $\mathbb{Q}$) if it satisfies (is a root or a zero of, all different words for the same thing) a polynomial with coefficients in $\mathbb{Q}$. If there is no polynomial with coefficients in $\mathbb{Q}$ which x satisfies, we call it *transcendental* (over $\mathbb{Q}$). We shall say $s \geq 1$ complex numbers $x_1, \ldots, x_s$ are *algebraically independent* (over $\mathbb{Q}$) if there is no polynomial f in $\mathbb{Q}[z_1, \ldots, z_s]$ such that $f(x_1, \ldots, x_s) = 0$. Observe that the algebraic independence of a set of complex numbers implies the transcendence of each individual member of the set (this is easy).

For this talk, the most important algebraic numbers will be the *roots of unity* $e^{2\pi i q}$, where q ranges through the set $\in \mathbb{Q}/\mathbb{Z}$. Each root of unity x is algebraic as it satisfies $x^n - 1 = 0$ for some $n \geq 1$.

Some well known transcendental numbers are e, π, and $\sqrt{2}^{\sqrt{2}}$. For anyone not working on the theory of transcendence, these are NOT easy things to prove. Further, we don't even know if $\pi + e$ is transcendental! The odd special values of the Riemann zeta function $\zeta(k) := \sum_{n \geq 1} n^{-k}$, for $k \geq 0$ and odd, are expected to be transcendental and algebraically independent, but it appears that a proof of this result may be a long way off. Indeed, while it is known that Apery's constant $\zeta(3)$ is irrational, we do not know of the transcendence of any of the odd special values given above.

We mentioned the following expected generalization of *Baker's Theorem* in the abstract.

Conjecture: Let $x_1, \ldots, x_s$ be distinct complex numbers such that $e^{x_1}, \ldots, e^{x_s}$ are all algebraic over $\mathbb{Q}$ (we say the x_i are *logarithms of algebraic numbers*). If $x_1, \ldots, x_s$ are linearly independent over $\mathbb{Q}$, then they are algebraically independent over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Let us emphasize that linear independence over $\mathbb{Q}$ and *linear independence* over the algebraic closure of $\mathbb{Q} \subseteq \mathbb{C}$ are worlds away! This is Baker's theorem. Moving from linear independence to algebraic independence is a much bigger leap.

For example, we (non-transcendence theorists) may have some hope of proving that π and e are linearly independent, but again we do not know of their algebraic independence. A similar conjecture to the above, Schanuel's conjecture, would imply their algebraic independence, and the transcendence of $\pi + e$ would also follow.

2.2. **Function Fields.** The obvious definitions for *algebraic*, *transcendental*, and *algebraically independent* (over K) are left to be formulated by the reader.

Thanks to the work of G. Anderson on t-motives, we are now able to prove the transcendence and algebraic independence of many familiar looking analytically defined "numbers" in function field arithmetic. Parhaps the game may be summarized by the following principle:

Define in terms of the elements of K (or some finite extension thereof) numbers that are transcendental over K.

As we shall see, ζ values and L-values in positive characteristic give great examples of the game. An amazing thing that has happened is that Anderson's work was inspired in part by certain philosophy coming from the land of number fields. In other words, in the number field setting Grothendiek and others dreamed up a world in which one could prove various transcendence results, etc., and while his vision remains mostly philosophy for number fields it is a complete reality in the land of function fields.

For example, we have the *Carlitz zeta values*, defined for all positive integers k by,

$$\zeta(k) := \sum_{a \in A_+} a^{-k} \in K_\infty.$$

Since all that is required for a sum to converge in non-archimedean analysis is that the terms of the sum go to zero in absolute value, the number $\zeta(1)$ is a well defined element of K_∞. In fact, Carlitz knew in the 1930's that $e_C(\zeta(1)) = 1$, i.e. that $\zeta(1)$ is the Carlitz logarithm of the algebraic number 1. A theorem of Chang and Yu [2] tells us exactly how many of the elements $\widetilde{\pi}, \zeta(1), \ldots, \zeta(n)$ are algebraically independent! (Exercise 3.7 gives some hint as to the connection between $\widetilde{\pi}$ and the Carlitz zeta values.)

Let $\chi : (A/vA)^\times \to \mathbb{F}_{q^d}^\times \subseteq \mathbb{C}_\infty$ be a group homomorphism, where $v \in A$ is a monic irreducible polynomial of degree $d \geq 1$ (for us, a *prime number* or *prime*, for short). Extend χ to all of A by the rule $\chi(a) = 0$ if v divides a. One of Anderson's motivations in proving his log-algebraicity theorem was to express the Dirichlet L-values

$$L(1, \chi) := \sum_{a \in A_+} \frac{\chi(a)}{a} \in \mathbb{C}_\infty,$$

as Carlitz logarithms of algebraic numbers (over K) in $\mathbb{C}_\infty$. As a historical remark, we note that Carlitz initiated his study of the zeta values $\zeta(k)$ in the thirties and fourties, and then this area lay dormant until D. Goss rediscovered these things in the late seventies in the context of Drinfeld

modular forms and put the study of these L-values into a very general framework. In particular, Goss defined the L-values mentioned above.

Returning to Anderson, after proving his log-algebraicity theorem he expresses the L-values mentioned above as logarithms of algebraic numbers. (I presume he had the number field conjecture above (now Papanikolas' theorem [4] over function fields) in mind, among other things, upon looking for such a result.) Thus it follows from Papanikolas' theorem, mentioned in the abstract, that if we can prove the K-linear independence of the numbers $L(1,\chi)$, as χ ranges over various subsets of the group homomorphisms above (there are only finitely many of these for fixed v), we will have proved their algebraic independence! Lutes and Papanikolas do this in [3]. Again, nothing like this exists yet over number fields. Also, we do not know of any general theorem which states that if something has been proven on the function field arithmetic side of things, then it is true or provable on the number field side (e.g. Papanikolas' theorem, Chang and Yu's theorem).

3. The log-algebraicity theorem over A

All of the previous motivation was given to demonstrate that the following theorem is not just trickery or sleight of hand. While not too hard to prove, it is a non-trivial theorem whose many interesting applications and generalizations are still being discovered today. We shall state the theorem with the table of analogies as reference, and then direct the reader in a series of exercises developing the main characters.

For $a \in A$, denote the degree in θ of a by $\deg a$. Let t, z be independent indeterminates over A. The Carlitz exponential is an $\mathbb{F}_q$-linear power series in z with coefficients in K, so it looks like

$$e_C(z) = \sum_{d \geq 0} e_d z^{q^d}, \tag{1}$$

for some $e_d \in K$. These coefficients will be determined explicitly in Exercise 3.5 to follow.

Warning! We view the elements of A as scalars, and never evaluate at θ.

Theorem 3.1 (Anderson)**.** *Let $b \in A[t]$ be a polynomial. Consider the following formal power series in the variable z with coefficients in $K[t]$:*

$$S_b := S_b(t,z) := e_C\left(\sum_{a \in A_+} \frac{b(C_a(t))}{a} z^{q^{\deg a}}\right).$$

Then S_b is actually a polynomial in z with coefficients in $A[t]$.

Because of this result, one says that the power series

$$l(b) := \sum_{a \in A_+} \frac{b(C_a(t))}{a} z^{q^{\deg a}}$$

is *log-algebraic*; it is formally the logarithm of an algebraic object over A, a polynomial. In fact, writing $\log_C$ for the formal power series inverse of e_C (whose power series is given below), one may show that

$$\log_C(tz) = \sum_{a \in A_+} \frac{C_a(t)}{a} z^{q^{\deg a}}.$$

Imagining you set $z = 1$ (and this may be justified if one sums this series correctly), note the formal similarity with the power series expansion of the classical logarithm

$$\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}.$$

Remark 3.2. Fix $b \in A[t]$ so that S_b is a polynomial by Anderson's theorem. In practice, one takes $z = 1$ and $t = e_C(\widetilde{\pi}k)$ for some $k \in K$ (recall, this is the analog of a "root of unity"). It is easy to show that $l(b)$ converges for this choice of t, z, and one obtains via the theorem above an actual Carlitz logarithm of the algebraic number $S_b(e_C(\widetilde{\pi}k), 1)$. Anderson also shows (using the non-vanishing of the Dirichlet L-series mentioned above) exactly which elements from the set $\{l(t), l(t^2), \ldots, l(t^{q^d-1})\}$ are A-linearly independent after specializing with $z = 1$ and $t = e_C(\widetilde{\pi}k)$ as above. (Here "$l(t^m)$ specialized at..." means, first form the formal series

$$l(t^m) := \sum_{a\in A_+} \frac{C_a(t)^m}{a} z^{q^{\deg a}},$$

and then specialize at $z = 1$ and $t = e_C(\widetilde{\pi}\lambda)$.) This A-linear independence result is used by Lutes and Papanikolas in [3] to determine exactly how many of the numbers $L(1, \chi)$ are algebraically independent as χ ranges over the group homomorphisms $(A/vA)^\times \to \mathbb{F}_{q^d}^\times$.

The remainder of this note will consist of a worksheet to be filled in by the reader sketching the proof of this beautiful result.

3.1. **Set-up for the proof: Basic objects and notation.** We begin by introducing more notation.

Let $|\cdot|$ denote the canonical extension to $\mathbb{C}_\infty$ of the unique absolute value on K such that $|\theta| = q$. Note that $|\cdot|$ satisfies the stronger ultrametric inequality

$$|x + y| \leq \max\{|x|, |y|\}$$

for all $x, y \in \mathbb{C}_\infty$, and we call $|\cdot|$ a *non-archimedean absolute value* on K. For a more familiar example, see the p-adic numbers $\mathbb{Q}_p$. Observe that K_∞ is complete with respect to this absolute value.

Exercise 3.3. *Compute the absolute value of a few elements in K_∞ to get a feel for $|\cdot|$.*

Show that if $|y| < |x|$, then $|x - y| = |x|$. (Hint: Write $|x| = |x - y + y|$.)

3.1.1. *The Carlitz module.* The Carlitz module is the $\mathbb{F}_q$-algebra morphism from A into the polynomial ring $A[t]$ (equipped with usual addition of polynomials and composition) determined by the map

$$\theta \mapsto C_\theta := t^q + \theta t.$$

Exercise 3.4. *Calculate $C_{\theta^2} = t^{q^2} + (\theta^q + \theta)t^q + \theta^2 t$, and $C_{\theta^3} = t^{q^3} + \ldots$.*

The Carlitz module map gives a new A-module structure to any A-algebra R via evaluation as follows. We will denote by $C(R)$ the underlying additive group of R equipped with multiplication by $a \in A$ given by $C_a(r)$ for all $r \in R$. Warning! Do not confuse $C(R)$ with continuous functions on R; we make no mention of such things here.

3.1.2. *The Carlitz Exponential.* The Carlitz exponential $e_C(z)$ may be characterized as the unique $\mathbb{F}_q$-linear formal power series, as in (1) such that $e_0 = 1$ and

$$e_C(\theta z) = C_\theta(e_C(z)) = e_C(z)^q + \theta e_C(z). \tag{2}$$

The equation (2) should be thought of as telling you what happens to the action of multiplication by θ under the Carlitz exponential, and this equation determines what happens to the action of multiplication by any element of $a \in A$ by using the $\mathbb{F}_q$-linearity of e_C. This formula expresses that e_C is an A-module homomorphism taking the action of regular multiplication by elements of A to the action of A under the Carlitz module homomorphism.

Let $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$ and let $D_0 := 1$. It's nice to observe that D_i is the product of all monic polynomials in A of degree i. So the D_i are "factorial like," and in fact, they form the basis for the Carlitz factorials via the *digit principle.*

Exercise 3.5. *Use* (2) *to show that* $e_j = D_j^{-1}$ *for all* $j \geq 0$. *Thus,*

$$e_C(z) = \sum_{j \geq 0} \frac{z^{q^j}}{D_j}.$$

Calculate $|D_j|$ *for* $j \geq 0$, *and observe that* e_C *defines an entire* $\mathbb{F}_q$*-linear function on* $\mathbb{C}_\infty$. *Finally, show that* e_C *is an isometry on the open ball of radius* $|\widetilde{\pi}| = q^{q/(q-1)}$.

For us, the following theorem will be a black box.

Theorem 3.6. *The kernel of* $e_C(z)$ *is* $\widetilde{\pi}A$, *and for all* $z \in \mathbb{C}_\infty$ *we have*

$$e_C(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{\widetilde{\pi} a}\right).$$

Exercise 3.7. *See what you can deduce about the Carlitz zeta values defined earlier from the product expansion for the Carlitz exponential.*

Exercise 3.8. *Use the functional equation satisfied by the Carlitz exponential and Theorem 3.6 to show that for each* $a \in A$, *all roots of the polynomial* $C_a \in A[t]$ *are of the form* $e_C(\widetilde{\pi} b/a)$, *where* b *ranges over all polynomials whose degree (in* θ*) is strictly less than the degree of* a.

We shall use the following remark to define a metric on the polynomial ring $K[t]$ making $A[t]$ discreet.

Remark 3.9. Using the previous exercise and the continuity of e_C we see that the closure (in $\mathbb{C}_\infty$) of the Carlitz torsion is precisely the image of K_∞ under the map $z \mapsto e_C(\widetilde{\pi} z)$. This map factors through the compact space K_∞/A, and hence is compact. This is the function field analog of the usual unit circle in the complex plane.

Exercise 3.10 (Bonus for those with some Galois theory. Not needed for the proof.)**.** *Show that for any* $a \in A$ *adjoining a root of the polynomial* C_a *to* K *generates an abelian extension of* K. *(Hint: Use the analogous result for cyclotomic fields over* $\mathbb{Q}$ *as inspiration.)*

3.2. **Proof of Anderson's theorem 3.1.** A couple final pieces of notation will be handy before going forward.

For all $b \in K[t]$ we have defined

$$l(b) := \sum_{a \in A_+} \frac{b(C_a(t))}{a} z^{q^{\deg a}} \in K[t][[z]].$$

We define the coefficients $l_d(b) \in K[t]$ of the powers of z^{q^d} by

$$l(b) = \sum_{d \geq 0} l_d(b) z^{q^d}.$$

Observe that there are only finitely many monic polynomials of a given degree d (in fact, precisely q^d), and $l_d(b)$ is given explicitly by "summing $l(b)$ by degree,"

$$l_d(b) = \sum_{\substack{a \in A_+ \\ \deg a = d}} \frac{b(C_a(t))}{a}.$$

Recall, for all $b \in K[t]$ we have defined

$$S_b := e_C \left(\sum_{a \in A_+} \frac{b(C_a(t))}{a} z^{q^{\deg a}} \right) = e_C(l(b)(z)).$$

We now write down a formula for the coefficients of z^{q^d} in S_b in terms of $l_d(b)$ and the coefficients e_i of the Carlitz exponential.

For all $d \geq 0$, let

$$Z_d(b) := \sum_{i=0}^{d} e_i l_{d-i}(b)^{q^i}.$$

Then as formal power series we have

$$S_b = \sum_{d \geq 0} Z_d(b) z^{q^d}.$$

3.2.1. *Map of Theorem 3.1's proof.* We will observe that for all $d \geq 0$ and $b \in A[t]$, the polynomial $Z_d(b)$ is in $A[t]$. Then we will show that there is an ultrametric $||\cdot||$ on $K[t]$ such that if $b \in A[t]$ and $||b|| < 1$, then b is the zero polynomial. Finally, as an exercise, the reader will use the rapid decay of the coefficients of the exponential and the coefficients $l_i(b)$ to show that eventually, $||Z_d(b)|| < 1$.

3.2.2. *Integrality.* Anderson's proof that $Z_i(b) \in A[t]$ for all $i \geq 0$ and $b \in A[t]$ involves valuations on K arising from prime ideals of A and a tricky calculation requiring some versatility with the basic objects of function field arithmetic. Clearly, this is a crucial ingredient in the proof of Theorem 3.1, but because this talk is directed at a general audience, including beginning mathematicians, we state the next proposition for the record and without proof. The interested reader is directed to Anderson's original paper [1].

Proposition 3.11. *For all $i \geq 0$ and $b \in A[t]$ we have*

$$Z_i(b) \in A[t].$$

□

3.2.3. *Norm on $K[t]$.* The proof of the main result of this section should be accessible to anyone with a year long introductory abstract algebra course under their belt.

For each $b \in K[t]$, define

$$||b|| := \sup_{x \in K_\infty/A} |b(e_C(\widetilde{\pi}x))|.$$

This is a finite number by the compactness of K_∞/A.

Exercise 3.12. *Show that $||\cdot||$ is an ultrametric norm on $K[t]$, i.e.*

$$||b+c|| \leq \max\{||b||, ||c||\}, \quad ||bc|| \leq ||b||\ ||c||, \quad \text{if } ||b|| = 0, \text{then } b = 0$$

for all $b, c \in K[t]$. Show for all $a \in A$ that $||b \circ C_a|| = ||b||$, where $\circ$ denotes composition of polynomials.

Proposition 3.13. *Let $b \in A[t]$. If $||b|| \leq 1$, then $b \in \mathbb{F}_q$.*

Proof. Let $x \in K$ so that $\lambda := e_C(\widetilde{\pi}x)$ is an element of Carlitz torsion. Then $b(\lambda)$ is an element of $K(\lambda)$ and satisfies a monic polynomial μ_b with coefficients in A. By definition, $|b(\lambda)| \leq ||b||$. Now the Galois group of $K(\lambda)/K$ permutes the Carlitz torsion, and hence all other roots of μ_b are of the form $b(C_a(\lambda))$ for some $a \in A$. Hence, all roots of μ_b have absolute value less than or equal to 1. As the coefficients of μ_b are polynomials in the roots, we see that they all have absolute value less than or equal to 1. The only such elements in A are in $\mathbb{F}_q$. Hence $b(\lambda)$ is a root of unity (i.e. sits in a finite extension of $\mathbb{F}_q$). The only roots of unity in $K(\lambda)$ are in $\mathbb{F}_q$ (Black-box: The proof may be found in Rosen's corollary to Theorem 12.14 in [5]). Thus $b(\lambda) \in \mathbb{F}_q$. So $b(\lambda)^q - b(\lambda) = 0$. As λ was an arbitrary element of Carlitz torsion, and there are infinitely many distinct such elements, we conclude that the polynomial identity $b^q = b$ holds. Thus $b \in \mathbb{F}_q$. □

3.2.4. *Rapid decay of various coefficients.* We collect here the various estimates needed to show for $b \in A[t]$ that $||Z_i(b)|| \to 0$ as $i \to \infty$.

It follows from Exercise 3.5 that for

$$e_C(z) = \sum_{d \geq 0} e_d z^{q^d}$$

we have $|e_d| = q^{-dq^d}$, for all $d \geq 0$.

Now we need an estimate on the coefficients

$$l_d(b) := \sum_{\substack{a \in A_+ \\ \deg a = d}} \frac{b(C_a(t))}{a}.$$

Exercise 3.14. *Use the non-Archimedean properties of the absolute value $||\cdot||$ to show that for all $d \geq 0$*

$$||l_d(b)|| \leq ||b|| q^{-d}.$$

Recall that we have defined

$$Z_d(b) = \sum_{j=0}^{d} e_j l_{d-j}(b)^{q^j}.$$

Exercise 3.15. *Essentially, finish the proof of Theorem 3.1 by showing that for all $b \in A[t]$*

$$||Z_d(b)|| \leq \max_{j=0}^{d} \{||b||^{q^j} q^{-dq^j}\}.$$

3.2.5. *Final words.* Just to put everything together, we see that choosing I such that $||b||q^{-I} < 1$, for all $i \geq I$ we have $||Z_d(b)|| < 1$. Hence since $Z_d(b) \in A[t]$ whenever $b \in A[t]$, and by Proposition 3.13 $||Z_d(b)|| = 0$ for all $d \geq I$. This completes the proof. □

References

[1] G. Anderson, Log-algebraicity of Twisted A-harmonic Series and Special Values of L-series in Characteristic p, *J. Number Theory* **60**, 1 (1996), 165–209.

[2] C.Y. Chang and J. Yu, Determination of algebraic relations among special zeta values in positive characteristic, *Adv. in Math.* **216** (2007), 321–345.

[3] B. Lutes and M. Papanikolas, Algebraic independence of values of Goss L-functions at $s = 1$, *J. Number Theory* **133**, 3, (2013), 1000–1011.

[4] M. Papanikolas, Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms, *Inv. Math.* **171** (2008), 123–174.

[5] M. Rosen, Number Theory in Function Fields, Springer-Verlag, New York, 2002.

E-mail address: `perkins@math.osu.edu`