WHAT IS THE LOGISTIC MODEL?

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1. History

1.1 **Exponential model**: $x_{n+1} = \mu x_n$, where x_n is the polution of the nth year and μ is the changing rate of the population.

Exponential model developed by Thomas Robert Malthus (1766-1834) is considered to be the first principle of population dynamics. This model describes any species can potentially increase in numbers according to a geometric series.

Assumptions of Exponential Model: Environment is constant in space and time (e.g., resources are unlimited).

1.2 **Logistic model**: $x_{n+1} = \mu x_n(1 - \frac{x_n}{K})$ where K is the carrying capacity of the environment.

Logistic model was proposed by Pierre-François Verhulst in 1838, where the rate of reproduction is proportional to both the existing population and the amount of available resources, all else being equal.

$x_{n+1} = \mu x_n(1 - x_n)$ (by normalization).

2. Dynamics dependent on μ

Define $F(x) = \mu x(1 - x)$.

Def 1. (cf. Definition 3.2 in [2]). The point x is called a fixed point if $F(x) = x$. The point x is a periodic point of period n if $F^n(x) = x$.

· There are two fixed points $p_1 = 0$ and $p_2 = (\mu - 1)/\mu$.

Def 2. (cf. Definitions 4.5&4.7 in [2]). If x is a fixed point and $|F'(x)| < 1$, then x is called an attracting (stable) fixed point; if x is a fixed point and $|F'(x)| > 1$, then x is called a repelling (unstable) fixed point.

Proposition 1. (cf. Proposition 4.4 in [2]). If p is a fixed point with $|F'(x)| < 1$, then there is an open interval U containing p such that if $x \in U$, then $\lim F^n(x) = p$.

∗ **Case 1.** $\mu \in (0, 1)$**.**

Proposition 2. If $\mu \in (0, 1)$, p_1 is an attracting fixed point.
Physically, it means that the population would die away to zero.

∗ **Case 2.** $\mu \in (1, 3)$**.**

Proposition 3. (cf. Proposition 5.3 in [2]). If $\mu \in (1, 3)$,

a) p_1 is a repelling fixed point and p_2 is an attracting fixed point.

b) If $0 < x < 1$, $\lim F^n(x) = p_2$.

No matter what the initial population is, the eventual population will be $p_2 K$.

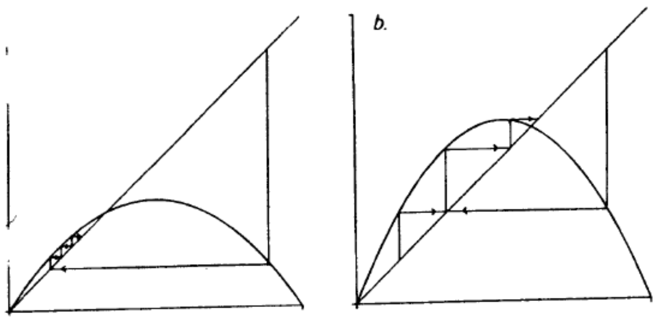


Fig. 5.2. Graphical analysis of $F_\mu(x) = \mu x(1-x)$ when a. $1 < \mu < 2$, and b. $2 < \mu < 3$.

$*$ **Case 3.** $\mu > 4$.

Def 3. A_0 is the set of points which immediately (i.e., at the first iteration) escape from $[0,1]$; $A_n = \{x \in [0,1] \mid F^i(x) \in [0,1],\ F^{n+1}(x) \notin [0,1]\}$, so that A_n consists of all points which escape from$[0,1]$ at the $n+1$st iteration. Let $\Lambda = [0,1] - (\cup_{n=0}^{\infty} A_n)$ (i.e., Λ consists those points which never escape from $[0,1]$)

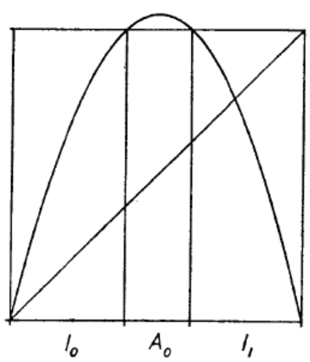


Fig. 5.4.

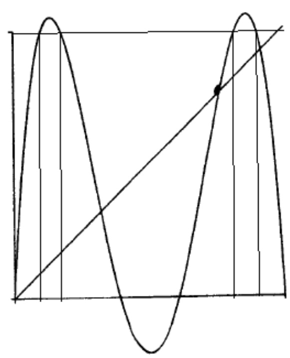


Fig. 5.5. The graph of F^2.

Def 4. (cf. Definition 5.4 in [2]). A set A is a Cantor set if it is a closed, totally disconnected, and perfect subset of $[0,1]$. A set is totally disconnected if it contains no intervals (no interior points), a set is perfect if every point in it is a limit point of other points in the set.

Theorem 1. (cf. Theorem 5.6 in [2]). If $\mu > 2 + \sqrt{5}$,Λ is a Cantor set.

Remark. The result is also true for $\mu > 4$. But the proof is more delicate.

Def 5. (cf. Definition 7.4 in [2]). Let $f: A \to A$ and $g: B \to B$ be two maps. f and g are said to be topologically conjugate if there exists a homeomorphism $h: A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy.

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics. So in order to study the dynamics of the logistic map, one can study a map which is topologically conjugate to it.

Some concepts and propositions about symbolic dynamics.

Def 6. (cf. Definition 6.1 in [2]). The sequence space on two symbols 0 and 1 is definded as$\sum_2 = \{s = (s_0 s_1 s_2 ...) \mid s_i = 0 \text{ or } 1\}$.

For any $s = (s_0 s_1 s_2 ...)$ and $t = (t_0 t_1 t_2 ...)$ in Σ_2, let $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. One can show that d is a metric on Σ_2 .

Def 7. (cf. Definition 6.4 in [2]). The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is given by $\sigma(s_0 s_1 s_2 ...) = (s_1 s_2 s_3 ...)$.

Proposition 4. (cf. Proposition 6.6 in [2]).

1. Card $Per_n(\sigma) = 2^n$.
2. Periodic points are dense in $\sum_2$.

3 There is a dense orbit of shift map in $\sum_2$.

Now we define a map $S{:}\Lambda \to \sum_2$: $S(x) = (s_0 s_1 s_2 ...)$, where $s_i = 0$ if $F^i(x) \in I_0$, $s_i = 1$ if $F^i(x) \in I_1$.

Theorem 2. (cf. Theorem 7.2 & 7.3 in [2]). If $\mu > 2 + \sqrt{5}$, the map S is a homeomorphism and $S \circ F = \sigma \circ S$.

Remark. The result is also true for $\mu > 4$.

Therefore, Card $Per_n(F) = 2^n$; the perodic point of F is dense in Λ; F has a dense orbit in Λ.

Why these properties are so important?

Def 8. (cf Definition 8.5 in [2]) Let V be a set, $F : V \to V$ is said to be chaotic on V if

1. F has sensitive dependence on initial conditions;
2. F is topologically transitive;
3. periodic points are dense in V.

Def 9. (cf Definition 8.2 in [2]). $F : J \to J$ has sensitive dependence on initial condition if there exists $\delta > 0$ such that, for any $x \in J$ and any neighorhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|F^n(x) - F^n(y)| > \delta$. It means there exists points arbitrarily close to x which eventually separate from x by at least δ under iteration of F.

Remark. One can show that the logistic map possesses sensitive dependence on initial conditions on $\Lambda$when $\mu > 2 + \sqrt{5}$.

Def 10. (cf Definition 8.1 in [2]) $F : J \to J$ is said to be topologically transitive if for any pair of open sets U, V contained in J, there exists $k > 0$ such that $F^k(U) \cap V \neq \emptyset$. This means the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map.

Remark. It is equivalent to the property that the map has a dense orbit.

Theorem 3. When $\mu > 2 + \sqrt{5}$, the logistic map is chaotic.

Remark. The result is also true for $\mu > 4$.

For the case $\mu \in (3, 4)$. We consider the period-doubling bifurcations to chaos.

Theorem 4. (cf. Theorem 10.1 in [2]). Let $F : R \to R$ be continuous. Suppose F has a periodic point of period three. Then F has periodic points of all other periods.

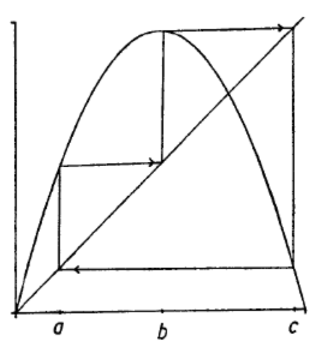
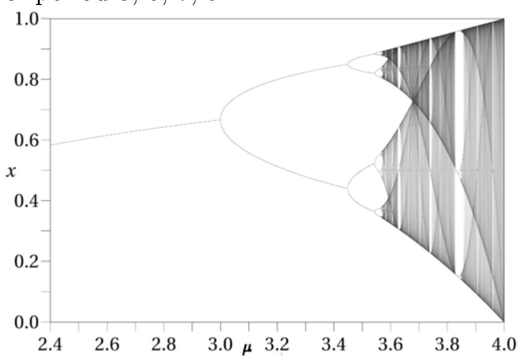


Fig. 10.2. The map $F_{3.839}(x) = 3.839x(1-x)$.

Paper[3]: Period three implies chaos (Tien-Yien Li; James A. Yorke; The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975), pp. 985-992.)

As pointed out in [4], " the first period doubling bifurcation occurs at $\mu = 3$. For value of μ between 3 and 3.449490... a periodic point of period 2 exists. For value of μ between 3.449490... and 3.544090... a period point of period 4 exists, followed in sequence by period points of period 8, 16, 32, and so on. The sequence carries on, doubling in period each time, until an infinite period is reached at $\mu = 3.569945...$, at which point the map is chaotic. The sequence is known as the period doubling route to chaos. Between 3.569945... and 4, there are not only periodic points of peiod integer powers of 2, but also periodic points of period 3, 5, 7, 9...."



References

[1] http://en.wikipedia.org/wiki/Logistic_map

[2] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Second Edition, Addison-Wesley

[3] Tien-Yien Li and James A. Yorke, Period three implies chaos, *The American Mathematical Monthly*, Vol. 82, No. 10. (Dec., 1975), pp. 985-992.)

[4] Paul S. Addison, *Fractals and Chaos: An illustrated course*, CRC Press