WHAT IS THE LOGISTIC MODEL?

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1. History
1.1 Exponential model: \( x_{n+1} = \mu x_n \), where \( x_n \) is the population of the \( n \)th year and \( \mu \) is the changing rate of the population.

Exponential model developed by Thomas Robert Malthus (1766-1834) is considered to be the first principle of population dynamics. This model describes any species can potentially increase in numbers according to a geometric series.

Assumptions of Exponential Model: Environment is constant in space and time (e.g., resources are unlimited).

1.2 Logistic model: \( x_{n+1} = \mu x_n (1 - \frac{x_n}{K}) \) where \( K \) is the carrying capacity of the environment.

Logistic model was proposed by Pierre-François Verhulst in 1838, where the rate of reproduction is proportional to both the existing population and the amount of available resources, all else being equal.

\[ x_{n+1} = \mu x_n (1 - \frac{x_n}{K}) \] (by normalization).

2. Dynamics dependent on \( \mu \)

Define \( F(x) = \mu x (1 - x) \).

Def 1. (cf. Definition 3.2 in [2]). The point \( x \) is called a fixed point if \( F(x) = x \). The point \( x \) is a periodic point of period \( n \) if \( F^n(x) = x \).

\( \cdot \) There are two fixed points \( p_1 = 0 \) and \( p_2 = (\mu - 1)/\mu \).

Def 2. (cf. Definitions 4.5&4.7 in [2]). If \( x \) is a fixed point and \( |F'(x)| < 1 \), then \( x \) is called an attracting (stable) fixed point; if \( x \) is a fixed point and \( |F'(x)| > 1 \), then \( x \) is called a repelling (unstable) fixed point.

Proposition 1. (cf. Proposition 4.4 in [2]). If \( p \) is a fixed point with \( |F'(x)| < 1 \), then there is an open interval \( U \) containing \( p \) such that if \( x \in U \), then \( \lim F^n(x) = p \).

* Case 1. \( \mu \in (0,1) \).

Proposition 2. If \( \mu \in (0,1) \), \( p_1 \) is an attracting fixed point.

Physically, it means that the population would die away to zero.

* Case 2. \( \mu \in (1,3) \).

Proposition 3. (cf. Proposition 5.3 in [2]). If \( \mu \in (1,3) \),

a) \( p_1 \) is a repelling fixed point and \( p_2 \) is an attracting fixed point.

b) If \( 0 < x < 1 \), \( \lim F^n(x) = p_2 \).

No matter what the initial population is, the eventual population will be \( p_2 K \).
Case 3. \( \mu > 4 \).

**Def 3.** \( A_0 \) is the set of points which immediately (i.e., at the first iteration) escape from \([0, 1]\); 
\[ A_n = \{ x \in [0, 1] \mid F^n(x) \in [0, 1], F^{n+1}(x) \notin [0, 1] \} \], so that \( A_n \) consists of all points which escape from \([0, 1]\) at the \( n+1 \)st iteration. Let \( \Lambda = [0, 1] - (\cup_{n=0}^{\infty} A_n) \) (i.e., \( \Lambda \) consists of those points which never escape from \([0, 1]\)).

**Def 4.** (cf. Definition 5.4 in [2]). A set \( A \) is a Cantor set if it is a closed, totally disconnected, and perfect subset of \([0, 1]\). A set is totally disconnected if it contains no intervals (no interior points), a set is perfect if every point in it is a limit point of other points in the set.

**Theorem 1.** (cf. Theorem 5.6 in [2]). If \( \mu > 2 + \sqrt{5}, \Lambda \) is a Cantor set.

Remark. The result is also true for \( \mu > 4 \). But the proof is more delicate.

**Def 5.** (cf. Definition 7.4 in [2]). Let \( f : A \to A \) and \( g : B \to B \) be two maps. \( f \) and \( g \) are said to be topologically conjugate if there exists a homeomorphism \( h : A \to B \) such that \( h \circ f = g \circ h \). The homeomorphism \( h \) is called a topological conjugacy.

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics. So in order to study the dynamics of the logistic map, one can study a map which is topologically conjugate to it.

Some concepts and propositions about symbolic dynamics.

**Def 6.** (cf. Definition 6.1 in [2]). The sequence space on two symbols 0 and 1 is defined as \( \Sigma_2 = \{ s = (s_0s_1s_2...) \mid s_i = 0 \text{ or } 1 \} \).

For any \( s = (s_0s_1s_2...) \) and \( t = (t_0t_1t_2...) \) in \( \Sigma_2 \), let \( d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \). One can show that \( d \) is a metric on \( \Sigma_2 \).
Def 7. (cf. Definition 6.4 in [2]). The shift map \( \sigma : \Sigma_2 \to \Sigma_2 \) is given by \( \sigma(s_0s_1s_2...) = (s_1s_2s_3...) \).

**Proposition 4.** (cf. Proposition 6.6 in [2]).
1. Card \( Per_n(\sigma) = 2^n \).
2. Periodic points are dense in \( \Sigma_2 \).
3. There is a dense orbit of shift map in \( \Sigma_2 \).

Now we define a map \( S : \Lambda \to \Sigma_2 \): \( S(x) = (s_0s_1s_2...) \), where \( s_i = 0 \) if \( F^i(x) \in I_0 \), \( s_i = 1 \) if \( F^i(x) \in I_1 \).

**Theorem 2.** (cf. Theorem 7.2 & 7.3 in [2]). If \( \mu > 2 + \sqrt{5} \), the map \( S \) is a homeomorphism and \( S \circ F = \sigma \circ S \).

Remark. The result is also true for \( \mu > 4 \).

Therefore, Card \( Per_n(F) = 2^n \); the periodic point of \( F \) is dense in \( \Lambda \); \( F \) has a dense orbit in \( \Lambda \).

Why these properties are so important?

Def 8. (cf Definition 8.5 in [2]) Let \( V \) be a set, \( F : V \to V \) is said to be chaotic on \( V \) if
1. \( F \) has sensitive dependence on initial conditions;
2. \( F \) is topologically transitive;
3. periodic points are dense in \( V \).

Def 9. (cf Definition 8.2 in [2]). \( F : J \to J \) has sensitive dependence on initial condition if there exists \( \delta > 0 \) such that, for any \( x \in J \) and any neighborhood \( N \) of \( x \), there exists \( y \in N \) and \( n \geq 0 \) such that \( |F^n(x) - F^n(y)| > \delta \). It means there exists points arbitrarily close to \( x \) which eventually separate from \( x \) by at least \( \delta \) under iteration of \( F \).

Remark. One can show that the logistic map possesses sensitive dependence on initial conditions on \( \Lambda \) when \( \mu > 2 + \sqrt{5} \).

Def 10. (cf Definition 8.1 in [2]) \( F : J \to J \) is said to be topologically transitive if for any pair of open sets \( U, V \) contained in \( J \), there exists \( k > 0 \) such that \( F^k(U) \cap V \neq \emptyset \). This means the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map.

Remark. It is equivalent to the property that the map has a dense orbit.

**Theorem 3.** When \( \mu > 2 + \sqrt{5} \), the logistic map is chaotic.

Remark. The result is also true for \( \mu > 4 \).

For the case \( \mu \in (3, 4) \). We consider the period-doubling bifurcations to chaos.

**Theorem 4.** (cf. Theorem 10.1 in [2]). Let \( F : R \to R \) be continuous. Suppose \( F \) has a periodic point of period three. Then \( F \) has periodic points of all other periods.

![Fig. 10.2. The map \( F_{\lambda=0}(x) = 3.832\sigma(1-x) \).](image-url)

As pointed out in [4], “the first period doubling bifurcation occurs at $\mu = 3$. For value of $\mu$ between 3 and 3.449490... a periodic point of period 2 exists. For value of $\mu$ between 3.449490... and 3.544090... a period point of period 4 exists, followed in sequence by period points of period 8, 16, 32, and so on. The sequence carries on, doubling in period each time, until an infinite period is reached at $\mu = 3.569945...$, at which point the map is chaotic. The sequence is known as the period doubling route to chaos. Between 3.569945... and 4, there are not only periodic points of period integer powers of 2, but also periodic points of period 3, 5, 7, 9...”

![Graph](image)

**References**


