Conceptual Problems Involving Partial Sums

The following questions provide practice with concepts involving partial sums. These are important and should be studied and understood in preparation for the second midterm.

I: True or False

Directions: CIRCLE ALL of the statements that *MUST* be TRUE. No explanation is necessary. Note that there may be several statements that are true for each question!

Problem 1: Suppose $\{a_n\}_{n\geq 1}$ is a sequence and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to $L > 0$. Let $s_n = \displaystyle\sum_{k=1}^{n} a_k$.

A. $\displaystyle\lim_{n\to\infty} a_n = L$

B. $\displaystyle\lim_{n\to\infty} a_n = 0$

C. $\displaystyle\lim_{n\to\infty} s_n = 0$

D. $\displaystyle\lim_{n\to\infty} s_n = L$

E. $\displaystyle\sum_{n=1}^{\infty} s_n$ MUST diverge.

F. $\displaystyle\sum_{n=1}^{\infty} (a_n + 1) = L + 1$

G. The divergence test tells us $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to L.

Solution:

A. FALSE - Since $\{a_n\}_{n\geq 1} = L$, $\{a_n\}_{n\geq 1}$ is a *convergent* series, so $\lim_{n\to\infty} a_n = 0$. Since $L > 0$, there is no way that $\lim_{n\to\infty} a_n = L$.

B. TRUE - If $\lim_{n\to\infty} a_n \neq 0$, the divergence test *immediately* implies $\sum_{n=1}^{\infty} a_n$ diverges!

$$\longrightarrow \boxed{\textit{Anytime} \text{ a series } \sum_{n=1}^{\infty} a_n \text{ converges, it MUST be true that } \lim_{n\to\infty} a_n = 0} \longleftarrow$$

C. FALSE

D. TRUE - Some essential facts are:

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ converges iff } \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \text{ exists}}$$

$$\boxed{\text{When } \lim_{n\to\infty} s_n \text{ does exist, } \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.}$$

$$\boxed{\text{The series } \sum_{n=1}^{\infty} a_n \text{ likewise diverges iff the } \lim_{n\to\infty} s_n \text{ does not exist.}}$$

Here, we are given $\sum_{n=1}^{\infty} a_n$ converges to $L > 0$, which tells us immediately that $\lim_{n\to\infty} s_n = L$.

E. TRUE - Since $\lim_{n\to\infty} s_n = L \neq 0$, the divergence test tells us immediately that $\sum_{n=1}^{\infty} s_n$ MUST diverge.

F. FALSE - Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Thus, $\lim_{n\to\infty} (a_n + 1) = 1$, and the divergence test immediately tells us that $\sum_{n=1}^{\infty} (a_n + 1)$ MUST diverge!

G. FALSE - The divergence test *NEVER* can be used to conclude that a series converges!

Problem 2: Suppose that $\{a_n\}_{n\geq 1}$ is a *decreasing* sequence. Let $s_n = \sum_{k=1}^{n} a_k$ and suppose $\lim_{n\to\infty} s_n$ does not exist.

A. $\lim_{n\to\infty} a_n$ does not exist.

B. $\sum_{k=1}^{\infty} a_k$ could converge.

C. $\sum_{n=1}^{\infty} s_n$ MUST diverge.

D. $\{s_n\}$ MUST be monotonic.

E. $\{s_n\}$ MUST be bounded.

F. $\lim_{n\to\infty} s_n = -\infty$

G. The divergence test applied to $\sum_{k=1}^{\infty} a_k$ would guarantee that $\sum_{k=1}^{\infty} a_k$ diverges.

Solution:

A. FALSE - A counterexample is $a_k = \dfrac{1}{k}$. We know this is a decreasing sequence and that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. However, $\displaystyle\lim_{n\to\infty} a_n = 0$.

B. FALSE - The series $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges iff the $\displaystyle\lim_{n\to\infty} s_n$ does not exist.

C. TRUE - Since $\displaystyle\lim_{n\to\infty} s_n \neq 0$, the divergence test *immediately* implies $\displaystyle\sum_{n=1}^{\infty} s_n$ diverges!

D. FALSE - Although $\{a_n\}$ is monotonic, it does NOT imply $\{s_n\}$ is monotonic! This is because the terms in the sequence $\{a_k\}$ could change from positive to negative. For example, let $a_k = 3 - k, k \geq 1$. Then, computing the s_n:

$$\begin{aligned}
a_1 &= 2 & s_1 &= a_1 = \underline{2} \\
a_2 &= 1 & s_2 &= a_1 + a_2 = \underline{3} \\
a_3 &= 0 & s_3 &= a_1 + a_2 + a_3 = \underline{3} \\
a_4 &= -1 & s_4 &= a_1 + a_2 + a_3 + a_4 = \underline{2} \\
a_5 &= -2 & s_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = \underline{0}
\end{aligned}$$

Hence, s_n is NOT monotonic!

E. FALSE - The harmonic series (i.e. the series described in the explanation of A.) provides a counterexample.

F. FALSE - The harmonic series provides a counterexample.

G. FALSE - The harmonic series provides a counterexample.

Problem 3: Suppose that $\{a_n\}_{n\geq 1}$ and $a_n > 0$ for all $n \geq 1$. Let $s_n = \sum_{k=1}^{n} a_k$ and suppose $\lim_{n\to\infty} s_n = L$.

A. $\sum_{k=1}^{\infty} a_k = L$

B. $\lim_{n\to\infty} a_n = 0$

C. $\{s_n\}$ MUST be monotonic.

D. $\{s_n\}$ MUST be bounded.

E. $\sum_{n=1}^{\infty} (a_n - L) = 0$

F. $\sum_{n=1}^{\infty} s_n$ MUST diverge.

G. The divergence test applied to $\sum_{k=1}^{\infty} a_k$ would guarantee that $\sum_{k=1}^{\infty} a_k$ converges.

H. The Ratio Test can be used to determine that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.

Solution:

A. TRUE - $\sum_{n=1}^{\infty} a_n$ converges iff $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists, and in this case, we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k$.

B. TRUE - If $\lim_{n\to\infty} a_n \neq 0$, the divergence test *immediately* implies $\sum_{n=1}^{\infty} a_n$ diverges!

$\longrightarrow$ $\boxed{\textit{Anytime} \text{ a series } \sum_{n=1}^{\infty} a_n \text{ converges, it MUST be true that } \lim_{n\to\infty} a_n = 0}$ $\longleftarrow$

C. TRUE - Note that $s_n = s_{n-1} + a_n$. Since $a_n > 0$ for all $n \geq 1$, then $s_n > s_{n-1}$.

D. TRUE - Since $\{s_n\}$ is increasing, if it were not bounded, it would not have a limit (since a bounded, monotonic sequence *MUST* have a limit).

E. FALSE - Since it has been established that $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$, and thus: $\lim_{n\to\infty} (a_n - L) = -L$.

Also, since s_n has been established to be increasing, $L = \lim_{n\to\infty} s_n > s_1 = a_1 > 0$.

Hence, $\lim_{n\to\infty} (a_n - L) = -L \neq 0$, so $\sum_{n=1}^{\infty} (a_n - L)$ MUST diverge!

F. TRUE - $\lim_{n\to\infty} s_n = L \neq 0$, and the divergence test *immediately* implies $\sum_{n=1}^{\infty} s_n$ diverges.

G. FALSE - The divergence test can NEVER be used to determine that a series converges!

H. FALSE - The ratio test may *NOT* apply to the series in question! There are convergent series for which $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$!

For a specific example, consider $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$. This sum can be shown to be telescoping by using partial fractions to justify the result below:

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

By noting that $s_n = 1 - \frac{1}{n+1}$, it is clear that this series converges to 1. However, it can be shown using the Ratio Test that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$!

II: Short Answer

Directions: Provide a brief response to the following questions.

<u>**Problem 4**</u> (Exploring the Relationship Between $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} s_k$)

For a sequence $\{a_n\}_{n\geq 1}$ let $s_n = \sum_{k=1}^{n} a_k$ denote its sequence of partial sums.

a) Given that $\sum_{k=1}^{\infty} a_k$ converges, what can be said about $\sum_{k=1}^{\infty} s_k$?

b) Given that $\sum_{k=1}^{\infty} a_k$ diverges, what can be said about $\sum_{k=1}^{\infty} s_k$?

c) Given that $\sum_{k=1}^{\infty} s_k$ converges, what can be said about $\sum_{k=1}^{\infty} a_k$?

d) Given that $\sum_{k=1}^{\infty} s_k$ diverges, what can be said about $\sum_{k=1}^{\infty} a_k$?

Solution: Thinking about $\displaystyle\sum_{k=1}^{\infty} s_k$ is very difficult conceptually, but we do have several important facts:

- Fact 1: Anytime we are told about convergence or divergence of a series, we know something about the sequence of partial sums:

  i. $\displaystyle\sum_{k=1}^{\infty} a_k$ converges iff $\displaystyle\lim_{n\to\infty} s_n$ exists, and in this case: $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$.

  ii. $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges iff $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ does not exist.

- Fact 2: If $\sum_{k=1}^{\infty} b_k$ converges, then $\lim_{k\to\infty} b_k = 0$.

a) Since $\sum_{k=1}^{\infty} a_k$ converges, say $\sum_{k=1}^{\infty} a_k = L$. Then by Fact 1, $\lim_{n\to\infty} s_n = L$.

   i. If $L \neq 0$, then $\lim_{n\to\infty} s_n = L \neq 0$, so $\sum_{k=1}^{\infty} s_k$ would diverge by the divergence test.

   ii. If $L = 0$, then $\sum_{k=1}^{\infty} s_k$ could converge.

   Further Thinking: Here are examples of an instance where the series $\sum_{k=1}^{\infty} s_k$ converges and an instance in which it diverges.

   Ex 1: An example where $\sum_{k=1}^{\infty} s_k$ converges occurs when $s_k = \dfrac{1}{k^2+k}$. In this case, each a_k can be found via the formula $a_k = s_k - s_{k-1}$.

   Note this formula is not saying much:

$$\begin{aligned} s_k &= a_1 + \cdots + a_{k-1} + a_k \\ s_{k-1} &= a_1 + \cdots + a_{k-1} \end{aligned}$$

   Subtracting gives $s_k - s_{k-1} = a_k$. This may look intimidating, but conceptually, the formula $a_k = s_k - s_{k-1}$ is obvious! So:

$$a_k = s_k - s_{k-1} = \frac{1}{k^2+k} - \frac{1}{(k-1)^2 + k - 1}.$$

   Ex 2: An example where $\sum_{k=1}^{\infty} s_k$ diverges occurs when $s_k = \dfrac{1}{k}$. In this case,

$$a_k = s_k - s_{k-1} = \frac{1}{k} - \frac{1}{k-1} = -\frac{1}{k^2-k}.$$

b) Since $\sum_{k=1}^{\infty} a_k$ diverges, we know that $\lim_{n\to\infty} s_n$ does not exist by Fact 1. Thus, $\lim_{n\to\infty} s_n \neq 0$, so $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by the divergence test.}}$

c) Since $\sum_{k=1}^{\infty} s_k$ converges, we know that $\lim\limits_{n\to\infty} s_n = 0$ by the above. But, note that since $\lim\limits_{n\to\infty} s_n = 0$, this means precisely that $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = 0$.

$\boxed{\text{Hence, } \sum_{k=1}^{\infty} a_k \text{ converges to 0.}}$

d) Since $\sum_{k=1}^{\infty} s_k$ diverges, we know very little about $\lim_{n\to\infty} s_n$.

i. It is possible that the limit exists, in which case, say $\lim_{n\to\infty} s_n = L$. Then, $\sum_{k=1}^{\infty} a_k = L$. Note that any series $\sum_{k=1}^{\infty} a_k$ that converges to any value other than 0 will give rise to a sequence of partial sums for which $\sum_{k=1}^{\infty} s_k$ diverges (since in this case $\lim_{n\to\infty} s_n = L \neq 0$, and thus $\sum_{k=1}^{\infty} s_k$ would diverge by the divergence test).

ii. It is possible that $\lim_{n\to\infty} s_n$ does not exist, in which case $\sum_{k=1}^{\infty} a_k$ diverges by Fact 1.

Further Thinking: An easy example of a series that converges to L can be made from a previous example. Choose L and consider the series $\sum_{k=1}^{\infty} \dfrac{L}{k^2+k}$. This was discussed in Problem 3, H. and can easily be shown to converge to L.

Problem 5 For a sequence $\{a_n\}_{n\geq 1}$ let $s_n = \sum_{k=1}^{n} a_k$ denote its sequence of partial sums. Now, suppose that $\{a_n\}_{n\geq 1}$ is a sequence such that $s_n = \dfrac{2n-1}{3n+1}$.

a) Find $a_1 + a_2 + a_3 + a_4$.

b) Find $a_5 + a_6$.

c) Determine whether $\lim_{n\to\infty} a_n$ exists. If it does, find its value.

d) Determine whether $\lim_{n\to\infty} s_n$ exists. If it does, find its value.

e) Determine whether $\displaystyle\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

f) Determine whether $\displaystyle\sum_{k=1}^{\infty} s_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

Solution:

a) Note by definition that $a_1 + a_2 + a_3 + a_4 = s_4$. Using the formula given for s_n with $n = 4$ gives:

$$a_1 + a_2 + a_3 + a_4 = \frac{2(4) - 1}{3(4) + 1} = \boxed{\frac{7}{13}}$$

b) Note that by definition:

$$\begin{aligned} s_6 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

so $a_5 + a_6 = s_6 - s_4$. Using the formula for s_n, we have:

$$s_6 = \frac{2(6) - 1}{3(6) + 1} = \frac{11}{19}, \qquad s_4 = \frac{2(4) - 1}{3(4) + 1} = \frac{7}{13}$$

Thus, $\boxed{a_5 + a_6 = \frac{11}{19} - \frac{7}{13}}$.

c) To determine this, we actually note that:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{2n - 1}{3n + 1} = \frac{2}{3}.$$

Since $\lim_{n \to \infty} s_n$ exists, $\sum_{k=1}^{\infty} a_k$ converges, and thus $\boxed{\lim_{n \to \infty} a_n = 0}$.

d) From the above work, $\boxed{\lim_{n \to \infty} s_n = \frac{2}{3}}$.

e) In c), we showed that $\sum_{k=1}^{\infty} a_k$ converges. Since $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n$ and we determined that $\displaystyle\lim_{n \to \infty} s_n = \frac{2}{3}$, we have $\boxed{\displaystyle\sum_{k=1}^{\infty} a_k \text{ converges to } \frac{2}{3}}$.

f) We showed that $\displaystyle\lim_{n \to \infty} s_n = \frac{2}{3}$, so $\boxed{\displaystyle\sum_{k=1}^{\infty} s_k \text{ diverges by the Divergence Test}}$.

Problem 6 For a sequence $\{a_n\}_{n\geq 1}$ let $s_n = \sum_{k=1}^{n} a_k$ denote its sequence of partial sums. Now, suppose that $\{a_n\}_{n\geq 1}$ is a sequence such that $s_n = \dfrac{4n^2+9}{1-2n}$.

a) Find $a_1 + a_2 + a_3$.

b) Find $a_8 + a_9 + a_{10}$.

c) Determine whether $\displaystyle\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

d) Determine whether $\displaystyle\sum_{k=1}^{\infty} s_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

Solution:

a) Note by definition that $a_1 + a_2 + a_3 = s_3$. Using the formula given for s_n with $n = 3$ gives:

$$a_1 + a_2 + a_3 = \frac{4(3)^2 + 9}{1 - 2(3)} = \boxed{-9}.$$

b) Note that by definition:

$$\begin{aligned} s_{10} &= a_1 + \cdots + a_7 + a_8 + a_9 + a_{10} \\ s_7 &= a_1 + \cdots + a_7 \end{aligned}$$

so $a_8 + a_9 + a_{10} = s_{10} - s_7$. Using the formula for s_n, we have:

$$s_10 = \frac{4(10)^2 + 9}{1 - 2(10)} = -\frac{409}{19}, \qquad s_7 = \frac{4(7)^2 + 9}{1 - 2(7)} = -\frac{205}{13}$$

Thus, $\boxed{a_8 + a_9 + a_{10} = -\frac{409}{19} + \frac{205}{13}}$.

c) To determine this, we note that:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{4n^2 + 9}{1 - 2n} = \infty.$$

Since $\lim_{n \to \infty} s_n$ does not exist, $\boxed{\sum_{k=1}^{\infty} a_k \text{ diverges by the Divergence Test}}$.

d) We showed that $\lim\limits_{n \to \infty} s_n = \infty$, so $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by the Divergence Test}}$.