Worksheet #8 Solutions

I. 1.

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$	$a_n = \frac{f^{(n)}(\pi/3)}{n!}$
0	$\sin x$	$\sqrt{3}/2$	$\frac{\sqrt{3}/2}{0!} = \frac{\sqrt{3}}{2}$
1	$\cos x$	$1/2$	$\frac{1/2}{1!} = \frac{1}{2}$
2	$-\sin x$	$-\sqrt{3}/2$	$-\frac{\sqrt{3}/2}{2!} = -\frac{\sqrt{3}}{4}$
3	$-\cos x$	$-1/2$	$-\frac{1/2}{3!} = -\frac{1}{12}$

So: $\sin x = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$

$$\boxed{\sin x = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x-\frac{\pi}{3}\right)^3 + \ldots}$$

2.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$a_n = \frac{f^{(n)}(2)}{n!}$
0	$\ln(1+3x)$	$\ln 7$	$\frac{\ln 7}{0!} = \ln 7$
1	$\frac{3}{1+3x}$	$\frac{3}{7}$	$\frac{3/7}{1!} = \frac{3}{7}$
2	$-\frac{9}{(1+3x)^2}$	$-\frac{9}{49}$	$\frac{-9/49}{2!} = -\frac{9}{98}$
3	$\frac{54}{(1+3x)^3}$	$\frac{54}{343}$	$\frac{54/343}{3!} = \frac{9}{343}$

So: $\ln(1+3x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$

$$\boxed{\ln(1+3x) = \ln 7 + \frac{3}{7}(x-2) - \frac{9}{98}(x-2)^2 + \frac{9}{343}(x-2)^2 + \ldots}$$

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = \frac{f^{(n)}(0)}{n!}$
0	$(1+x)^{1/2}$	1	$\frac{1}{0!} = 1$
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$	$\frac{1/2}{1!} = \frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$	$\frac{-1/4}{2!} = -\frac{1}{8}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$	$\frac{3/8}{3!} = \frac{1}{16}$.

So: $\sqrt{1+x} = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots}$$

4.

n	$f^{(n)}(x)$	$f^{(n)}$([redacted])	$a_n = \frac{f^{(n)}(\text{[redacted]})}{n!}$
0	e^{3x}	e^3	$e^3/0! = e^3$
1	$3e^{3x}$	$3e^3$	$3e^3/1! = 3e^3$
2	$9e^{3x}$	$9e^3$	$9e^3/2! = \frac{9}{2}e^3$
3	$27e^{3x}$	$27e^3$	$27e^3/3! = \frac{9}{2}e^3$

$$e^{3x} = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

$$\boxed{e^{3x} = e^3 + 3e^3(x-1) + \frac{9}{2}e^3(x-1)^2 + \frac{9}{2}e^3(x-1)^3 + \ldots}$$

II. 5. Use Ratio Test:

Let $L = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}\, x^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{\cancel{(-1)^{n+1}}\, x^{2n}} \right|$ ← absolute value kills $(-1)^n$.

$= \lim_{n\to\infty} \left| \frac{x^{2n+2}}{x^{2n}} \cdot \frac{3^n}{3^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{x^{2n}}\, x^2}{\cancel{x^{2n}}} \cdot \frac{\cancel{3^n}}{\cancel{3^n}\, 3^1} \right|$.

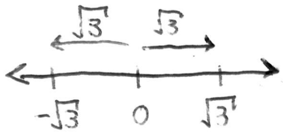
$L = \lim_{n\to\infty} \left| \frac{x^2}{3} \right|$.

The Ratio Test assures the series will converge for any x-value for which $L(x) < 1$, so:

$$L(x) = \frac{1}{3}|x|^2 < 1$$
$$|x|^2 < 3$$
$$|x| < \sqrt{3}$$

$\boxed{\text{The radius of convergence is } \sqrt{3}}$

To find the interval of convergence, note the series is centered at $x=0$.



We know the series converges for any x with $-\sqrt{3} < x < \sqrt{3}$
We must <u>ALWAYS</u> check endpoints separately, as this is where the geometric behavior vanishes $(L=1)$:

$\underline{x=\sqrt{3}}$: $$f(\sqrt{3}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\sqrt{3})^{2n}}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left[(\sqrt{3})^2\right]^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{3^n}}{\cancel{3^n}}$$

This diverges by the Divergence Test!

$\underline{x=-\sqrt{3}}$: $$f(-\sqrt{3}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-\sqrt{3})^{2n}}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left[(-\sqrt{3})^2\right]^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{3^n}}{\cancel{3^n}}$$

This diverges by the Divergence Test!

Thus, $\boxed{\text{the Interval of convergence is } (-\sqrt{3}, \sqrt{3})}$.

6. Use the Ratio test to check for absolute convergence ←

Fact: If a power series converges absolutely by the Ratio Test, it converges for precisely the same set of x-values

$$L(x) = \lim_{n\to\infty} \left| \frac{[(n+1)x]^{n+1}}{(n+1)!} \cdot \frac{n!}{(nx)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n \cdot \cancel{(n+1)}}{n^n} \cdot \frac{\cancel{n!}}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{x^n}x}{\cancel{x^n}} \right|$$

$$= |x| \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^n \right|$$

$$= |x| \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

Note $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ (you should be able to show this!), so:

$$L(x) = e|x|$$

The ratio test assures that the series converges when $L<1 \Rightarrow e|x|<1$
$|x|<\frac{1}{e}$.

$$\boxed{\text{The radius of convergence is } \frac{1}{e}}$$

For the IOC, we do not have the necessary tools to check for the convergence at the endpoints, but we know the series converges at least for all x in $\left(-\frac{1}{e}, \frac{1}{e}\right)$

7. Use the ratio test to check for absolute convergence.

$$L(x) = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{2^n}2}{\cancel{2^n}} \cdot \frac{\cancel{x^n}x}{\cancel{x^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} \right|$$

$$= 2|x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

The ratio test ensures the series will converge for all x-values for which $L(x)<1$. Since $L(x)=0$ for any fixed value of x, we see the series converges for all x

$$\boxed{\begin{array}{l}\text{ROC: } \infty \\ \text{IOC: } (-\infty, \infty)\end{array}}$$

8. Use the Ratio Test:

$$L(x) = \lim_{n\to\infty} \left| \frac{[2(n+1)]!\ x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n)!\ x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)!}{(2n)!} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n^n}{(n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \frac{\cancel{x^n}\,x}{\cancel{x^n}} \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{(n+1)} \right|$$

$$= |x| \lim_{n\to\infty} \left| 2\cancel{(n+1)}\,(2n+1) \cdot \left[\frac{n}{n+1}\right]^n \cdot \frac{1}{\cancel{n+1}} \right|$$

Since $\left(\frac{n+1}{n}\right)^n \to e$, $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$

$$= \frac{|x|}{e} \lim_{n\to\infty} (2n+1)$$

$$= \infty$$

No matter which value of $x \neq 0$ we pick, the limit is infinite! Hence, the series diverges for all $x \neq 0$.

A power series always converges at its center since if $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$, $f(c) = a_0$.

(since $f(x) = a_0 + a_1(x-c) + \ldots$ so $f(c) = a_0$).

$$\boxed{\text{ROC: } 0, \quad \text{IOC: } x = 0}$$

9. We could use Ratio or Root Test, but note:

$$3^{2n+1}(x-4)^n = 3^1 \cdot 3^{2n}(x-4)^n = 3\cdot(3^2)^n(x-4)^n = 3\cdot 9^n (x-4)^n$$

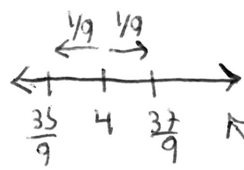
$$= 3[9\cdot(x-4)]^n$$

$$= 3[9x-36]^n.$$

Thus, $\sum_{n=1}^{\infty} 3^{2n+1}(x-4)^n = \sum_{n=1}^{\infty} 3[9x-36]^n$. Using geometric series results, this converges when $|9x-36| < 1 \to 9|x-4| < 1 \to |x-4| < \frac{1}{9}$.

gives both ROC, IOC since we established $\sum ar^k$ diverges when $|r| = 1$!

$$\boxed{\begin{array}{l}\text{The ROC is thus } \frac{1}{9} \\ \text{The IOC is } \left(\frac{35}{9}, \frac{37}{9}\right)\end{array}}$$



series is in powers of $(x-4)$, so center is at $x=4$

10. $L(x) = \lim_{n\to\infty} \left| \dfrac{(n+2)\,x^{3(n+1)}}{8^{n+1}} \cdot \dfrac{8^n}{(n+1)\,x^{3n}} \right|$

$= \lim_{n\to\infty} \left| \dfrac{n+2}{n+1} \, \dfrac{x^{3n+3}}{x^{3n}} \, \dfrac{8^n}{8^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \dfrac{n+2}{n+1} \, \dfrac{\cancel{x^{3n}}\,x^3}{\cancel{x^{3n}}} \cdot \dfrac{\cancel{8^n}}{\cancel{8^n}\,8} \right|$

$= \frac{1}{8} |x^3| \lim_{n\to\infty} \frac{n+2}{n+1}$

$= \frac{1}{8} |x|^3$

The ratio test ensures the series will converges for all x with $L(x) < 1$ so

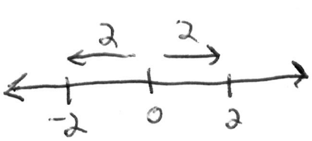
$L(x) = \frac{1}{8}|x|^3 < 1$

$|x|^3 < 8$

$|x| < 2$

$\boxed{\text{The Radius of convergence is } 2}$

To find the IOC, note the series is centered at $x=0$, so it must converge for any x in $(-2, 2)$.



We check the endpoints, which is where $L=1$

$\underline{x=2}$: $f(2) = \sum_{n=1}^{\infty} \dfrac{(n+1)\,2^{3n}}{8^n} = \sum_{n=1}^{\infty} \dfrac{(n+1)\,(2^3)^n}{8^n} = \sum_{n=1}^{\infty} \dfrac{(n+1)\,\cancel{8^n}}{\cancel{8^n}}$

$\underline{x=-2}$: $f(-2) = \sum_{n=1}^{\infty} \dfrac{(n+1)\,(-2)^{3n}}{8^n} = \sum_{n=1}^{\infty} \dfrac{(n+1)[(-2)^3]^n}{8^n} = \sum_{n=1}^{\infty} \dfrac{(n+1)\,(-8)^n}{8^n} = \sum_{n=1}^{\infty} (n+1)\dfrac{(-1)^n\,\cancel{8^n}}{\cancel{8^n}}$

In both cases, $\lim_{n\to\infty} (n+1)$ and $\lim_{n\to\infty} (-1)^n(n+1)$ DNE, so the series diverge by divergence test.

$\boxed{\text{The IOC is } (-2, 2)}$

11. Use the Ratio Test:

$L(x) = \lim_{n\to\infty} \left| \dfrac{4^{n+1}\, x^{2(n+1)+1}}{(n+1)!} \cdot \dfrac{n!}{4^n\, x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \dfrac{4^{n+1}}{4^n} \cdot \dfrac{x^{2n+3}}{x^{2n+1}} \, \dfrac{n!}{(n+1)!} \right|$

$$= \lim_{n\to\infty} \left| \frac{\cancel{4^n}\, 4}{\cancel{4^n}} \cdot \frac{\cancel{x^{2n}}\, x^3}{\cancel{x^{2n}}\, x} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} \right|$$

x^2

$$= 4|x^2| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0.$$

The ratio test ensures the series will converge for all x-values that make $L(x) < 1$. Since $L(x) = 0$ regardless of what x is, we find the series converges for all x: $\boxed{\text{ROC: } \infty \quad \text{IOC: } (-\infty, \infty)}$

12. Use the Ratio Test:

$$L(x) = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\,(x+1)^{n+1}}{4(n+1)+1} \cdot \frac{4n+1}{\cancel{(-1)^n}\,(x+1)^n} \right|$$

(Absolute value kills $(-1)^n$ terms!).

$$= \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \cdot \frac{4n+1}{4n+5} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(x+1)^n}\,(x+1)}{\cancel{(x+1)^n}} \cdot \frac{4n+1}{4n+5} \right|$$

$$= |x+1| \lim_{n\to\infty} \frac{4n+1}{4n+5}$$

$$= |x+1|$$

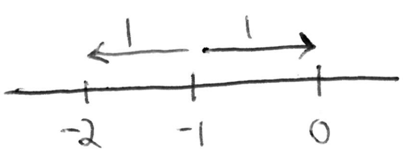
The ratio test assures that the series will converge for all x-values for which $L(x) < 1$, so set: $L(x) = |x+1| < 1$.

$\boxed{\text{The radius of convergence is } 1.}$

This does NOT mean $|x|+1 < 1$!

$|a+b| \neq |a| + |b|$ (set $a=-2$, $b=1$).

To find the IOC, note the series is centered at $x=-1$.



The series will converge for any x in $(-2, 0)$.

We now have to check the endpoints.

$\underline{x=-2}$: $f(-2) = \sum_{n=1}^{\infty} \frac{(-1)^n(-2+1)^n}{4n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{4n+1} = \sum_{n=1}^{\infty} \frac{1}{4n+1}$.

Since $\frac{1}{4n+1} > 0$, use Limit Comparison Test with $\frac{1}{4n}$.

Note $\lim_{n\to\infty} \left(\frac{1}{4n+1}\right) / \left(\frac{1}{4n}\right) = \lim_{n\to\infty} \frac{4n}{4n+1} = 1$ is non-zero and finite

So by LCT, $\sum \frac{1}{4n+1}$ and $\sum \frac{1}{4n} = \frac{1}{4}\sum \frac{1}{n}$ either both converge or diverge. Since $\sum \frac{1}{n}$ is the harmonic series, it diverges. Hence, $\sum \frac{1}{4n+1}$ $\underline{\text{div.}}$

$\underline{x=0}$: $f(0) = \sum_{n=1}^{\infty} \frac{(-1)^n(0+1)^n}{4n+1} = \sum \frac{(-1)^n}{4n+1}$.

This is alternating and $\frac{1}{4n+1}$ is decreasing and $\lim_{n\to\infty} \frac{1}{4n+1} = 0$, so this series $\underline{\text{converges}}$ by Alternating series test

$\boxed{\text{The IOC is thus } (-2, 0]}$

13. $f(x) = xe^{5x}$.

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, all x

$\Downarrow$

$\rightarrow e^{5x} = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$, all x.

$\Downarrow$

So: $\boxed{xe^{5x} = x + 5x^2 + \frac{25}{2}x^3 + \frac{125}{6}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{5^n x^{n+1}}{n!}}$ $\boxed{\text{ROC}: \infty}$

14. $f(x) = \frac{4}{3-2x}$ $\leftarrow$ Need to look like $\frac{1}{1-r}$.

$\frac{4}{3-2x} = \frac{4}{3(1-\frac{2}{3}x)} = \frac{4}{3} \frac{1}{1-\frac{2}{3}x}$ $\leftarrow$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

$= \frac{4}{3} \sum_{n=0}^{\infty} (\frac{2}{3}x)^n$, $|\frac{2}{3}x| < 1$.

$\boxed{= \frac{4}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n, \quad |x| < \frac{3}{2}}$

Writing out the first 4 terms:

$$\frac{4}{3-2x} = \frac{4}{3} + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{32}{81}x^3 + \dots$$

15. $f(x) = 5\sin x^2.$

We know $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, all x

$$\rightarrow \sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$$

So:
$$\boxed{5\sin x^2 = 5x^2 - \frac{5}{6}x^6 + \frac{1}{24}x^{10} - \frac{5}{7!}x^{14} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \text{ for all } x}$$

(RoC $= \infty$)

16. $\frac{2x}{1+x^2} = 2x\left(\frac{1}{1+x^2}\right)$ ← only need this Taylor series!

↑ already in powers of x

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2| < 1.$$

$|x|^2 < 1$

$|x| < 1$

$$= 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

Hence:
$$\boxed{2x\left(\frac{1}{1+x^2}\right) = 2x - 2x^3 + 2x^5 - 2x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}, \quad |x| < 1}$$

17. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, all x.

$\cos 5x = 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \frac{(5x)^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n}}{(2n)!}$, all x.

$$\boxed{\cos 5x = 1 - \frac{25}{2}x^2 + \frac{625}{24}x^4 - \frac{3125}{144}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} x^{2n}, \text{ all } x}$$

18. $\frac{1}{1+4x} = \frac{1}{1-(-4x)}$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$, $|x|<1$

$\frac{1}{1-(-4x)} = 1 + (-4x) + (-4x)^2 + (-4x)^3 + \ldots = \sum_{n=0}^{\infty} (-4x)^n$, $|-4x|<1$

$$\boxed{\frac{1}{1+4x} = 1 - 4x + 16x^2 - 64x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n 4^n x^n, \quad |x| < \tfrac{1}{4}}$$

19. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, all x.

→ $\sin 4x = (4x) - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} - \frac{(4x)^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!}$, all x.

$\sin 4x = 4x - \frac{32}{3}x^3 + \frac{128}{15}x^5 - \frac{1024}{315}x^7 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n+1}}{(2n+1)!} x^{2n+1}$

→ $$\boxed{4x^3 \sin 4x = 16x^4 - \frac{128}{3}x^6 + \frac{512}{15}x^8 - \frac{4096}{315}x^{10} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n+2}}{(2n+1)!} x^{2n+4}, \text{ all } x}$$

20. $\frac{4}{8x-1} = -4 \cdot \frac{1}{1-8x}$

$= -4\left[1 + 8x + (8x)^2 + (8x)^3 + \ldots\right] = -4\sum_{n=0}^{\infty} (8x)^n$, $|8x|<1$

$$\boxed{\frac{4}{8x-1} = -4 - 32x - 256x^2 - 2048x^3 - \ldots = \sum_{n=0}^{\infty} -4 \cdot 8^n x^n, \quad |x| < \tfrac{1}{8}}$$

IV.

21. Note: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ on one hand.

On the other:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1+x+x^2+x^3+x^4\ldots\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right), \ |x|<1$$

$$= 1+2x+3x^2+4x^3+\ldots = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n), \ |x|<1.$$

$$= 1+2x+3x^2+4x^3+\ldots = \sum_{n=0}^{\infty} n x^{n-1}, \ |x|<1$$

Hence, $\boxed{\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots = \sum_{n=0}^{\infty} n x^{n-1}, \ |x|<1}$

22. $\frac{3x}{(1+x)^2} = 3x \cdot \frac{1}{(1+x)^2}$ So we just need the series for $\frac{1}{(1+x)^2}$

We could immediately obtain this from above:

$$\frac{1}{(1+x)^2} = \frac{1}{[1-(-x)]^2} = \sum_{n=0}^{\infty} n(-x)^{n-1} = \sum_{n=0}^{\infty} (-1)^{n-1} n \cdot x^{n-1}$$

or derive it (since you may not have the series for $\frac{1}{(1-x)^2}$ given on the exam):

Note: $\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = -1(1+x)^{-2}(1+x)' = -\frac{1}{(1+x)^2}.$

Also: $-\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{d}{dx}\left(\frac{1}{1-(-x)}\right)$ use Taylor series for $\frac{1}{1-u}$

$$= -\frac{d}{dx}\left(1-x+x^2-x^3+x^4-\ldots\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^n\right), \ |x|<1.$$

Note: $(-1)^{n-1} = (-1)^{n+1}$ since $(-1)^{n+1} = (-1)^{n-1}(-1)^2 = (-1)^{n-1}$

$$= 1-2x+3x^2-4x^3+\ldots = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$$

Hence: $\boxed{3x\frac{1}{(1+x)^2} = 3x-6x^2+9x^3-12x^4+\ldots = \sum_{n=0}^{\infty} (-1)^{n+1} 3n x^n, \ |x|<1}$

23. $\dfrac{2x}{(3+4x^3)^2} = 2x \dfrac{1}{(3+4x^3)^2}$.

We know we must differentiate $\dfrac{1}{3+4x^3}$ to get $(3+4x^3)^2$ in the denom:

- $\dfrac{d}{dx}\left(\dfrac{1}{3+4x^3}\right) = \dfrac{d}{dx}(3+4x^3)^{-1} = -1\,(3+4x^3)^{-2}(3+4x^3)'$

$$= -\frac{12x^2}{(3+4x^3)^2}.$$

We'll worry about the $-12x^2$ later. For now:

- $\dfrac{d}{dx}\left(\dfrac{1}{3+4x^3}\right) = \dfrac{d}{dx}\left(\dfrac{1}{3(1+\frac{4}{3}x^3)}\right) = \dfrac{1}{3}\dfrac{d}{dx}\left(\dfrac{1}{1-(-\frac{4}{3}x^3)}\right).$

$$= \frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\left(-\frac{4}{3}x^3\right)^n = \frac{1}{3}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}x^{3n}, \quad \left|-\frac{4}{3}x^3\right|<1$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}\frac{d}{dx}\left(x^{3n}\right), \quad |x^3|<\frac{3}{4}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}\cdot 3n\,x^{3n-1}, \quad |x|<\sqrt[3]{\frac{3}{4}}$$

Hence: $\dfrac{-12x^2}{(3+4x^3)^2} = \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}\cdot 3n\,x^{3n-1}$

We want $2x$ on the LHS to recover $f(x)$; so divide both sides by $-6x$:

$$-\frac{1}{6x}\left[-\frac{12x^2}{(3+4x^3)^2}\right] = -\frac{1}{6x}\left[\cancel{\frac{1}{3}}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}\cdot\cancel{3}n\,x^{3n-1}\right].$$

$$\boxed{\frac{2x}{(3+4x^3)^2} = -\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{3^n}\cdot n\,x^{3n-2}, \quad |x|<\sqrt[3]{\frac{3}{4}}}$$

24. We know $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$

so $a = \frac{1}{3}$ here and:

$$\int \frac{1}{x^2+\frac{1}{9}}\,dx = 3\arctan 3x + C.$$

$$\int \frac{1}{\frac{1}{9}(1+9x^2)}\,dx = 3\arctan 3x + C$$

Use series for $\frac{1}{1-u}$

$$9\int \frac{1}{1-(-9x^2)}\,dx = 3\arctan 3x + C$$

So:

$$3\int \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}\,dx = \arctan 3x + \tilde{C}, \quad |-9x^2| < 1.$$

$$3\sum_{n=0}^{\infty} (-1)^n 9^n \int x^{2n}\,dx = \arctan 3x + \tilde{C}, \quad |x^2| < \frac{1}{9}$$

$$3\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan 3x + \tilde{C}, \quad |x| < \frac{1}{3}.$$

Letting $x = 0 \Rightarrow C = 0$ so.

$$\boxed{\arctan 3x = 3\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < \frac{1}{3}}$$

25. We know $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$ so $\ln(1+x) = \int \frac{1}{1+x}\,dx.$

Also: $\int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx, \quad |x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C.$$

So: $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C, \quad |x| < 1.$

Letting $x = 0 \Rightarrow C = 0$, so: $\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad |x| < 1}$

26. $f(x) = x^2 \ln(1-3x^2)$

Already a polynomial!

Note: $\frac{d}{dx} \ln(1-3x^2) = \frac{1}{1-3x^2}(1-3x^2)' = -\frac{6x}{1-3x^2}$.

So: $$\ln(1-3x^2) = \int -\frac{6x}{1+3x^2}\,dx \qquad (1)$$

Also: $$-\frac{6x}{1+3x^2} = -6x \frac{1}{1-(-3x^2)}$$

$$= -6x \sum_{n=0}^{\infty} (-3x^2)^n, \quad |-3x^2| < 1$$

$$= -6x \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n}, \quad |x^2| < \frac{1}{3}$$

$$= -6 \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n+1}, \quad |x| < \frac{1}{\sqrt{3}}. \qquad (2)$$

So: $$\ln(1-3x^2) = \int -6 \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n+1}\,dx \leftarrow \text{Plug (2) into (1)}$$

$$= -6 \sum_{n=0}^{\infty} (-1)^n 3^n \int x^{2n+1}\,dx$$

$$\ln(1-3x^2) = -6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+2}}{2n+2} + C$$

Letting $x=0 \Rightarrow C=0$.

So: $$x^2 \ln(1-3x^2) = x^2\left[-6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+2}}{2n+2}\right]$$

$$\boxed{x^2 \ln(1-3x^2) = -6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+4}}{2n+2}}$$

V.

27. $$(x^2+1)\sin x = (x^2+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right).$$

Collect like powers:

$$= x + \left(1 - \frac{1}{3!}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{3!}\right)x^5 + \left(\frac{1}{5!} - \frac{1}{7!}\right)x^7 + \dots$$

$$= \boxed{x + \frac{5}{6}x^3 - \frac{19}{120}x^5 + \frac{41}{5040}x^7} + \dots$$

28. $f(x) = e^x \cos x$

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

Collect like powers:

$$= 1 + x + \underbrace{\left(-\frac{1}{2!} + \frac{1}{2!}\right)}_{0}x^2 + \left(-\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} - \left(\frac{1}{2!}\right)^2 + \frac{1}{4!}\right)x^4 + \dots$$

$$= \boxed{1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots}$$

<u>Note</u>: Here, if we identify $e^x \cos x = \sum_{n=0}^{\infty} a_n x^n$, note $a_2 = 0$.

We also have $a_2 = \frac{f^{(2)}(0)}{2!} \Rightarrow \underline{f''(0) = 0}$.

We could of course use this to compute other derivatives at $x=0$;

$a_4 = -\frac{1}{6} = \frac{f^{(4)}(0)}{4!} \Rightarrow \underline{f^{(4)}(0) = -4}$, etc!

29. $$f(x) = \frac{e^x}{1-x} = e^x \cdot \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 + x + x^2 + x^3 + \dots\right)$$

Collect like powers:

$$= 1 + (1+1)x + \left(1 + 1 + \frac{1}{2!}\right)x^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)x^3$$

$$= \boxed{1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots}$$

30. $f(x) = \sin 2x + 3\cos x$.

Note: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots$

So: $\sin 2x + 3\cos x = \left(2x - \frac{8x^3}{3!} + \frac{2^5x^5}{5!} + \dots\right) + \left(3 - 3\frac{x^2}{2!} + 3\frac{x^4}{4!} - \dots\right)$

$= \boxed{3 + 2x - \frac{3}{2}x^2 - \frac{4}{3}x^3 + \dots}$

31.a) $\lim\limits_{x\to 0} \frac{x\sin x - x^2}{\cos x - 1} = \lim\limits_{x\to 0} \frac{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - x^2}{\left(\cancel{1} + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right) - \cancel{1}}$

$= \lim\limits_{x\to 0} \frac{\cancel{x^2} - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \cancel{x^2}}{\frac{1}{2}x^2 + \dots}$

$= \lim\limits_{x\to 0} \frac{\cancel{x^4}^{\,x^2}\left(-\frac{1}{6} + \frac{1}{120}x^2 + \dots\right)}{\cancel{x^2}\left(\frac{1}{2} - \frac{1}{4}x^2 + \dots\right)}$ 0 in the limit.

$= \boxed{0}$

b) $\lim\limits_{x\to 0} \frac{e^{x^2} - 1 - x^2}{\sin 2x} = \lim\limits_{x\to 0} \frac{\left(\cancel{1} + \cancel{x^2} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots\right) - \cancel{1} - \cancel{x^2}}{2x - \frac{(2x)^3}{3!} + \dots}$

$= \lim\limits_{x\to 0} \frac{x^4\left(\frac{1}{2!} + \frac{1}{3!}x^2 + \dots\right)}{x\left(2 - \frac{8}{6}x^2 + \dots\right)}$

$= \lim\limits_{x\to 0} x^3 \frac{\frac{1}{2!} + \frac{1}{3!}x^2 + \dots}{2 - \frac{8}{6}x^2 + \dots}$ ← Goes to $\frac{\frac{1}{2!}}{2}$ when $x \to 0$!

$= \boxed{0}$

c) $$\lim_{x\to 0} \frac{\cos x^2 - 1}{x^3(e^x-1)} = \lim_{x\to 0} \frac{\left(\cancel{1} - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \cdots\right) \cancel{-1}}{x^3\left(\cancel{1} + x + \frac{x^2}{2!} + \cdots \cancel{-1}\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \cdots}{x^4 + \frac{1}{2}x^5 + \cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(-\frac{1}{2!} + \frac{1}{4!}x^4 - \cdots\right)}{\cancel{x^4}\left(1 + \frac{1}{2}x + \cdots\right)}$$

$$= \boxed{-\frac{1}{2}}$$

d). $$\lim_{x\to 0} \frac{\sin 3x - 3xe^x}{4\cos 4x - 4} = \lim_{x\to 0} \frac{\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots\right) - 3x\left(1 + x + \frac{x^2}{2!} + \cdots\right)}{4\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - 4}$$

$$= \lim_{x\to 0} \frac{\cancel{3x} - \frac{9}{2}x^3 + \frac{81}{40}x^5 - \cdots - \cancel{3x} - 3x^2 - \frac{3}{2}x^3 + \cdots}{\cancel{4} - 2x^2 + \frac{1}{6}x^4 - \cdots \cancel{-4}}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(-3 - 6x + \cdots)}{\cancel{x^2}\left(-2 + \frac{1}{6}x^2 + \cdots\right)}$$

$$= \boxed{\frac{3}{2}}.$$

32. We know $\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Letting $x = 0 \Rightarrow$ $\underline{C = 0}$ so:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

<u>If</u> the series converges at $x = 1$, which we are told it does.

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Note that $x=1$ is an endpt of the IOC!

33. We showed in #25:

write out the sum!

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

and that the ROC is 1. We do <u>not</u> know if the series converges at $x=1$ (It actually does for the same reason as in #32, but we don't discuss this!). To get around this, let $x = -\frac{1}{2}$:

$$\ln\left(1-\tfrac{1}{2}\right) = -\tfrac{1}{2} - \frac{(-\frac{1}{2})^2}{2} + \frac{(-\frac{1}{2})^3}{3} - \frac{(-\frac{1}{2})^4}{4} + \dots$$

$$\ln\tfrac{1}{2} = -\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{24} + \tfrac{1}{64} + \dots$$

$$-\ln 2 = -\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{24} + \tfrac{1}{64} + \dots$$

$$\boxed{\ln 2 = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \dots}$$

34. $$\int_0^{.1} \cos x^2 \, dx \approx \int_0^{.1} \left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!}\right) dx$$

$$\approx \int_0^{.1} \left(1 - \tfrac{1}{2}x^4 + \tfrac{1}{24}x^8\right) dx$$

$$= \left[x - \tfrac{1}{10}x^5 + \tfrac{1}{216}x^9 \right]_0^{.1}$$

$$= \left[.1 - \tfrac{1}{10}(.1)^5 + \tfrac{1}{216}(.1)^6\right] - 0$$

$$= \boxed{.09999900463}$$

A computer shows that to 11 decimal places: $\int_0^{.1} \cos x^2 \, dx = .09999900000$

b) $\int_0^{.1} e^{x^2}\,dx \approx \int_0^{.1} \left[1+(x^2)+\frac{(x^2)^2}{2}\right] dx$

$= \int_0^{.1} \left(1+x^2+\frac{1}{2}x^4+\ldots\right) dx$

$= x+\frac{1}{3}x^3+\frac{1}{10}x^5 \Big|_0^{.1}$

$= \left[(.1)+\frac{1}{3}(.1)^3+\frac{1}{10}(.1)^5\right] - 0$

$= \boxed{.1003334333}$

A computer gives to 11 decimal places:

$\int_0^{.1} e^{x^2}\,dx \approx .100334335 7$

Additional Questions Solutions

35. $$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(x-3)^{2k}$$

a) To find the radius of convergence, use Ratio Test:

$$L(x) = \lim_{k\to\infty}\left|\frac{(x-3)^{2(k+1)}}{4^{k+1}} \cdot \frac{4^k}{(x-3)^{2k}}\right| \leftarrow$$

Note: $\left|\frac{(-1)^k (x-3)^k}{4^k}\right| = \frac{|(-1)^k|\,|x-3|^k}{|4^k|} = \frac{|x-3|^k}{|4^k|}$

($|ab| = |a||b|$, so we can ignore the $(-1)^k$ terms!)

$$= \lim_{k\to\infty}\left|\frac{4^k}{4^{k+1}} \cdot \frac{(x-3)^{2k+2}}{(x-3)^{2k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{\cancel{4^k}}{\cancel{4^k}\,4} \cdot \frac{\cancel{(x-3)^{2k}}(x-3)^2}{\cancel{(x-3)^{2k}}}\right|$$

$$L(x) = \frac{|x-3|^2}{4}$$

Ratio Test guarantees the series will converge for all value of x for which $L(x) < 1$. $\rightarrow$ $L(x) = \frac{|x-3|^2}{4} < 1$

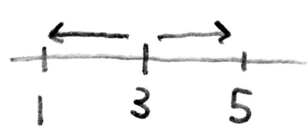
$$|x-3|^2 < 4$$

$$|x-3| < 2. \leftarrow$$

DO NOT say $|x|-3<2$! $|a+b| \neq |a|+|b|$!!! (ex: let $a=-2$, $b=2$)

$\boxed{\text{The radius of convergence is } 2.}$

b). To find the open interval, note the series is centered at $x=3$:



The radius is the maximum distance we can go from the center, so the $\boxed{\text{IOC is: } (1,5)}$.

c) The point $x=6$ is not in the interval of convergence, so the series for $\boxed{f(6) \text{ diverges}}$

d) Write out the first several terms:

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}(x-3)^{2k}$$

$$f(x) = 1 - \frac{1}{16}(x-3)^2 + \frac{1}{64}(x-3)^4 - \dots \quad \rightarrow \boxed{f(3)=1}$$

Compute deriv:

$$f'(x) = -\frac{1}{8}(x-3) + \frac{1}{16}(x-3)^3 - \dots \quad \rightarrow \boxed{f'(3)=0}$$

$$f''(x) = -\frac{1}{8} + \frac{3}{16}(x-3)^2 - \dots \quad \rightarrow \boxed{f''(3) = -\frac{1}{8}}$$

$$f'''(x) = \frac{3}{8}(x-3) - \dots \quad \rightarrow \boxed{f'''(3) = 0}$$

---

* Note: We can also use $a_k = \frac{f^{(k)}(c)}{k!}$ to write:

$$f^{(k)}(3) = k!\, a_k$$

and extract the a_k from the Taylor series $f(x) = \sum_{k=0}^{\infty} a_k (x-3)^k$.

Note that the index k in a_k must match the power k on x^k

Here, $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{16}$, $a_3 = 0$, $a_4 = \frac{1}{64}$, etc

---

36. $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$

a) To find the ROC, use ratio test.

$$L(x) = \lim_{k\to\infty} \left| \frac{x^{(k+1)+1}}{2(k+1)} \cdot \frac{2k}{x^{k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^{k+2}}{x^{k+1}} \cdot \frac{2k}{2k+2} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{x^{k+1}} \cdot x}{\cancel{x^{k+1}}} \cdot \frac{2k}{2k+2} \right|$$

$$= |x| \lim_{k\to\infty} \left(\frac{2k}{2k+2}\right)$$

$L(x) = |x|$

The Ratio Test ensures that the series will converge for all x such that $L(x) < 1$ and diverge for all x where $L(x) > 1$.

$\rightarrow L(x) = |x| < 1$

$\boxed{\text{The ROC is } 1}$

b) MAKE SURE YOU KNOW THAT "the first 4 nonzero terms" means that you need to write the sum of the first 4 powers of $(x-c)$ whose coeff is not 0!

ex: "First 4 nonzero terms" in the series at $x=0$ for $\sin x$ is "$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$"

Here, $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k} = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{6}x^4 + \frac{1}{8}x^5 + \ldots$

$$f(3x) = \frac{1}{2}(3x)^2 + \frac{1}{4}(3x)^3 + \frac{1}{6}(3x)^4 + \frac{1}{8}(3x)^5 + \ldots$$

$$= \frac{9}{2}x + \frac{27}{4}x^2 + \frac{27}{2}x^3 + \frac{243}{8}x^4 + \ldots$$

$$g(x) = x^2 f(3x) = x^2\left[\frac{9}{2}x + \frac{27}{4}x^2 + \frac{27}{2}x^3 + \frac{243}{8}x^4 + \ldots\right]$$

$$\boxed{g(x) = \frac{9}{2}x^3 + \frac{27}{4}x^4 + \frac{27}{2}x^5 + \frac{243}{8}x^6 + \ldots}$$

c) Differentiating does NOT change the ROC. Since the ROC for $f(x)$ is 1, the ROC for $f'(x)$ is 1. The series $f(x)$ is centered at $x=0$, so the open IOC is $(-1,1)$. Hence, the series for $f(x)$ (and $f'(x)$) will converge for any $-1 < x < 1$ and diverge if $|x| > 1$.

$\rightarrow$ $\boxed{\text{The series for } f'(3) \text{ diverges}}$

d) $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$ ← centered at $x=0$

i) <u>Recall:</u> The Taylor series centered at $x=c$ for $f(x)$ is given by $\sum_{k=0}^{\infty} a_k (x-c)^k$

where $a_k = \frac{f^{(k)}(c)}{k!}$.

Implicit to this is that a_k is the coeff of x^k

(the index and powers should match!)

Since we want to find $f''(0)$, we use:

$a_2 = \frac{f''(0)}{2!}$

$f''(0) = 2!\, a_2$.

We find a_2 by inspecting the Taylor series:

- <u>Way 1:</u> Write out $f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \ldots$

  a_2 is the coeff of x^2 → $\underline{a_2 = \frac{1}{2}}$

- <u>Way 2:</u> We need to find the index k so $x^{k+1} = x^2$ → $k=1$.

  Thus $a_2 = \frac{1}{2(1)} = \underline{\frac{1}{2}}$.

Using $f''(0) = 2!\, a_2$ gives: $f''(0) = 2! \cdot \frac{1}{2}$ → $\boxed{f''(0) = 1}$

ii). Writing out:

$f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{6}x^4 + \frac{1}{8}x^5 + \ldots$

Differentiate:

$f'(x) = x + \frac{3}{4}x^2 + \frac{2}{3}x^3 + \frac{5}{8}x^4 + \ldots$

$f''(x) = 1 + \frac{3}{2}x + 2x^2 + \frac{5}{2}x^3 + \ldots$

Plug in $x=0$:

$f''(0) = 1 + 0$ → $\boxed{f''(0) = 1}$

c) $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$

$$f'(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}\right]$$

Differentiate term-by-term (interchange $\frac{d}{dx}$ and Σ).

$$= \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{x^{k+1}}{2k}\right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k} \frac{d}{dx}(x^{k+1})$$

$$f'(x) = \sum_{k=1}^{\infty} \frac{k+1}{2k} x^k$$

$$f''(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{k+1}{2k} x^k\right]$$

$$= \sum_{k=1}^{\infty} \frac{k+1}{2k} \frac{d}{dx}\left[x^k\right]$$

$$f''(x) = \sum_{k=1}^{\infty} \frac{k+1}{2\cancel{k}} \cancel{k} x^{k-1}$$

Note when $k=1$, $x^{k-1} = x^0$ so when you plug in $x=0$, you do <u>NOT</u> get 0 on the RHS!

$$f''(x) = 1 + \frac{3}{2}x + 2x^2 + \frac{5}{2}x^3 + \ldots$$

→ $\boxed{f''(0) = 1}$

This matches what we found on the previous page!

37. $f(x) = \int_0^x te^{-t^2}\, dt$.

a) To use the definition, we note we are looking for the first 3 nonzero coefficients in the expansion:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

where $a_k = \frac{f^{(k)}(0)}{k}$ (since $c=0$ here).

<u>Note</u>: $f(x) = \int_0^x te^{-t^2}\,dt$ so $f(0) = \int_0^0 te^{-t^2}\,dt = 0.$

$f'(x) = \frac{d}{dx}\left[\int_0^x te^{-t^2}\,dt\right] = xe^{-x^2}$ by the Fund Thm of Calculus

$\rightarrow f'(0) = 0 \rightarrow \underline{a_1 = 0}$

• $f''(x) = e^{-x^2} - 2x^2e^{-x^2} \rightarrow \underline{f''(0) = 1}$. Hence $a_2 = \frac{f''(0)}{2!}$

$\underline{\underline{a_2 = \frac{1}{2}}}$

• $f'''(x) = -2xe^{-x^2} - 4xe^{-x^2} + 4x^3e^{-x^2} \rightarrow \underline{f'''(0)=0} \rightarrow \underline{a_3 = 0}$

$f'''(x) = -6xe^{-x^2} + 4x^3e^{-x^2}$

• $f^{(4)}(x) = -6e^{-x^2} + 12x^2e^{-x^2} + 12x^2e^{-x^2} - 8x^4e^{-x^2}$

$= -6e^{-x^2} + 24x^2e^{-x^2} - 8x^4e^{-x^2}.$

$\rightarrow f^{(4)}(0) = -6$ so $a_4 = \frac{f^{(4)}(0)}{4!} = -\frac{6}{24}$

$\underline{\underline{a_4 = -\frac{1}{4}}}$

• $f^{(5)}(x) = 12xe^{-x^2} + 48xe^{-x^2} - 48x^3e^{-x^2} - 32x^3e^{-x^2} + 16x^5e^{-x^2}$

$= 60xe^{-x^2} - 80x^3e^{-x^2} + 16x^5e^{-x^2}$

$f^{(5)}(0) = 0 \rightarrow a_5 = \frac{f^{(5)}(0)}{5!}$ so $\underline{a_5 = 0}$

• $f^{(6)}(x) = 60e^{-x^2} - 120x^2e^{-x^2} - 240x^2e^{-x^2} + 160x^4e^{-x^2}$

$+ 80x^4e^{-x^2} - 32x^6e^{-x^2}$

$f^{(6)}(0) = 60$ so $a_6 = \frac{f^{(6)}(0)}{6!} = \frac{60}{720}$

$\underline{\underline{a_6 = \frac{1}{12}}}$

Thus, the first 3 nonzero terms in the Taylor series centered at $x=0$ for $f(x)=\int_0^x te^{-t^2}\,dt$ are: $f(x)= a_2x^2 + a_4x^4 + a_6x^6$

$$\boxed{f(x)= \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 + \ldots}$$

Remark: This is PAINFUL! We really never want to use the definition unless we have to!

b) Evaluate the definite integral:

$$\int_0^x te^{-t^2}\,dt = \int_{t=0}^{t=x} \cancel{t}\cdot e^u \frac{du}{-2\cancel{t}}$$

$u=-t^2$

$du=-2t\,dt$

$$= -\frac{1}{2}e^u \Big|_{t=0}^{t=x}$$

$$= -\frac{1}{2}e^{-t^2}\Big|_0^x$$

$$f(x) = -\frac{1}{2}e^{-x^2} + \frac{1}{2}$$

Find the T.S. using rules:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{6}(-x^2)^3$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$$

$$-\frac{1}{2}e^{-x^2} = -\frac{1}{2}\left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \ldots\right]$$

$$= -\frac{1}{2} + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 - \ldots$$

$$f(x) = -\frac{1}{2}e^{-x^2} + \frac{1}{2} = \boxed{\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 - \ldots}$$

c) · Write out the Taylor series first:

$$e^{t} = 1 + t + \frac{1}{2}t^2$$

$$e^{-t^2} = 1 + (-t^2) + \frac{1}{2}(-t^2)^2 + \dots$$

$$= 1 - t^2 + \frac{1}{2}t^4 - \dots$$

$$te^{-t^2} = t - t^3 + \frac{1}{2}t^5 - \dots$$

· Integrate:

$$f(x) = \int_0^x te^{-t^2}\,dt = \int_0^x \left(t - t^3 + \frac{1}{2}t^5 - \dots\right) dt$$

$$= \frac{1}{2}t^2 - \frac{1}{4}t^4 + \frac{1}{12}t^6 - \dots \Big|_0^x$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 - \dots\right) - 0$$

$$\boxed{f(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6}$$

* You get the same result in all 3 cases! This is not an accident!
<u>LESSON</u>: Choose the most efficient way to compute the T.S!

38. $f(x) = \int_0^x \frac{2}{1-t^2}\,dt$, $x \geq 0$.

a) The integrand is unbounded when $t = \pm 1$, so if $x \geq 1$, $\frac{2}{1-t^2}$ will be unbounded on $[0, x]$.

$\boxed{\text{The integral is improper if } x \geq 1}$

b) i) Compute $\int_0^x \frac{2}{1-t^2}\,dt$ ← WARNING: $\int \frac{1}{1-t^2}dt \neq \arctan t + C$!!!

<u>Note</u>: $\frac{2}{1-t^2} = \frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

Use partial fractions!

$$2 = (1-t)A + (1+t)B.$$

$t=1$: $2 = 0A + 2B \rightarrow \underline{B=1}$

$t=-1$: $2 = 2A + 0B$

$\underline{A = 1}$

So: $\int_0^x \frac{2}{1-t^2}\,dt = \int_0^x \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt$

$= \Big[\ln|1+t| - \ln|1-t| \Big|_0^x$

$f(x) = \ln(1+x) - \ln(1-x)$

No "| |" necessary since $0 < x < 1$

We're given $\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ so $\ln(1+x) = -\sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1}$

$f(x) = \ln(1+x) - \ln(1-x) = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ lower indices match so add

$f(x) = \sum_{k=0}^{\infty} \left[-(-1)^{k+1} + 1 \right] \frac{1}{k+1} x^{k+1}$

Note: If k is odd, $-(-1)^{k+1} - 1 = 0$ so we only need sum over even indices; letting $k = 2m$ and noting $-(-1)^{2m+1} + 1 = 2$ gives:

$$\boxed{f(x) = \sum_{m=0}^{\infty} \frac{2}{2m+1} x^{2m+1}}$$

ii) Find the Taylor series for $\frac{2}{1-t^2}$ first:

$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$

$\frac{1}{1-t^2} = \sum_{k=0}^{\infty} (t^2)^k$

$\frac{2}{1-t^2} = 2\sum_{k=0}^{\infty} t^{2k} = \sum_{k=0}^{\infty} 2t^{2k}$

Integrate: $f(x) = \int_0^x \sum_{k=0}^{\infty} 2t^{2k}\,dt \overset{\text{int term-by-term (switch } \int \text{ and } \Sigma)}{=} \sum_{k=0}^{\infty} \int_0^x 2t^{2k}\,dt$

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} t^{2k+1} \Big|_0^x$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1}}$$ ← Note: This is exactly $\sum_{m=0}^{\infty} \frac{2}{2m+1} x^{2m+1}$.

* An easier variant of this question would be to ask for the first 3 or 4 nonzero terms in the Taylor series using each method rather than working in summation notation.

39. $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$

a) We have two options for $f'''(0)$.

Way 1: Write out several terms in the series for $f(x)$:

$$f(x) = 1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \frac{1}{9}x^4 - \dots$$

Differentiate:

$$f'(x) = -\frac{1}{3} + \frac{2}{5}x - \frac{3}{7}x^2 + \frac{4}{9}x^3 - \dots$$

$$f''(x) = \frac{2}{5} - \frac{6}{7}x + \frac{4}{3}x^2 - \dots$$

$$f'''(x) = -\frac{6}{7} + \frac{8}{3}x - \dots$$

Plug in $x = 0$:

$$\boxed{f'''(0) = -\frac{6}{7}}$$

This formula is derived using a generalization of the above procedure!

Way 2: Use $a_3 = \frac{f'''(0)}{3!}$ to find $f'''(0) = 3!\, a_3$.

Note: a_3 is the coefficient of the x^3 power. From above, we see $a_3 = -\frac{1}{7}$. Alternatively, the x^k in $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$ is x^3 when $k=3$. The coeff is $\frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$.

So, $f'''(0) = 3!\, a_3 = 3!\left(-\frac{1}{7}\right)$

$$\boxed{f'''(0) = -\frac{6}{7}}$$

b) To find $f^{(10)}(0)$, it would be awful to try to compute the series for $f^{(10)}(x)$. Instead, we'll use

$$a_{10} = \frac{f^{(10)}(0)}{10!} \to f^{(10)}(0) = 10!\, a_{10}$$

The power of x in $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$ is x^{10} when $k=10$.

The coefficient is $a_{10} = \frac{(-1)^{10}}{2(10)+1} = \frac{1}{21}$. Thus, $\boxed{f^{(10)}(0) = \frac{10!}{21}}$

40 a) $f(x) = \sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$

a) We have 2 reasonable options to compute $f''(0)$.

<u>Way 1:</u>
- Write out several terms in the series for $f(x)$.

$$f(x) = x + \frac{1}{2}x^3 + \frac{5}{7}x^5 + \ldots$$

- Take 2 derivatives

$$f'(x) = 1 + \frac{3}{2}x^2 + \frac{25}{7}x^4 + \ldots$$
$$f''(x) = 3x + \frac{100}{7}x^3 + \ldots$$

- Plug in $x=0$:

$$\boxed{f''(0) = 0}$$

<u>Way 2:</u> Use $a_2 = \frac{f''(0)}{2!} \to f''(0) = 2!\, a_2$.

There is no "x^2" term in the series, which means $a_2 = 0$ (if a power is "missing", this tells us the corresponding coefficient is 0) Alternatively, no integer index k makes the x^{2k+1} term be x^2!

Hence, $a_2 = 0$ and $f''(0) = 2!\, a_2 \rightarrow \boxed{f''(0) = 0}$

b) To find $f^{(17)}(0)$, we use the fact $a_{17} = \dfrac{f^{(17)}(0)}{17!}$ to find

$f^{(17)}(0) = 17!\, a_{17}$.

To find a_{17}, we need to find which integer value of k makes the power of x in $\displaystyle\sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$ be x^{17}.

$$\rightarrow \quad 2k+1 = 17$$
$$\underline{k = 8}.$$

Thus, the coefficient in front of x^{17} is $\dfrac{(8)^2+1}{3(8)+1} = \dfrac{65}{25} = \dfrac{13}{5}$

$$\rightarrow \quad \underline{a_{17} = \frac{13}{5}}$$

Hence, $f^{(17)}(0) = 17!\, a_{17}$

$$\boxed{f^{(17)}(0) = 17! \cdot \frac{13}{5}}$$

c) To find $f^{(18)}(0)$, we use the fact $a_{18} = \dfrac{f^{(18)}(0)}{18!}$ to find

$f^{(18)}(0) = 18!\, a_{18}$

To find a_{18}, we need to find which integer value of k makes the power of x in $\displaystyle\sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$ be x^{18}.

$$\rightarrow \quad 2k+1 = 18$$
$$k = \frac{17}{2}$$

For a better understanding, actually write out to x^{19} in the series

There is no integer value of k to make there be an x^{18} term, so $a_{18} = 0$. Hence, $f^{(18)}(0) = 18!\, a_{18} \rightarrow \boxed{f^{(18)}(0) = 0}$

41. Computing $\frac{d^{40}}{dx^{40}}(e^{x^2})$ would be a pain!

To find this derivative at $x=0$, we write the Taylor series for e^{x^2} at $x=0$:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k$$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$

Note when $k=20$, $x^{2k} = x^{40}$, so a_{40} (which is the coefficient in front of x^{40}) is $\frac{1}{20!}$

Since $a_{40} = \frac{f^{(40)}(0)}{40!}$, $\quad f^{(40)}(0) = 40!\, a_{40}$

$$\boxed{f^{(40)}(0) = 40! \cdot \frac{1}{20!}}$$

42. $f(x) = \sum_{k=1}^{\infty} a_k (x-1)^k$.

We're given $\sum_{k=1}^{\infty} 4^k a_k$ converges. To relate this to the function we're given, note that if $(x-1)^k = 4^k$, $x=5$. Plugging this into $f(x)$:

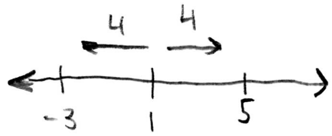
$$f(5) = \sum_{k=1}^{\infty} a_k (5-1)^k = \sum_{k=1}^{\infty} 4^k a_k.$$

Since $f(5)$ converges, and $f(x)$ is centered at $x=1$, we know the radius of convergence for $f(x)$ is at least 4 (since $x=5$ is a distance of 4 away from the center $x=1$)

Note: The series of the form $\sum_{k=1}^{\infty} a_k(x-c)^k$ is centered at $x=c$.

Hence, $\sum_{k=1}^{\infty} a_k(x-1)^k$ is centered at $\boxed{x=1}$

a) As mentioned earlier, since the series is centered at $x=1$, and converges for $x=5$, the minimum ROC is 4, so the minimal open interval of convergence is $(-3, 5)$.



Since $x=3$ is within this interval, $\boxed{\text{the series for } f(3) \text{ converges}}$.

b) To determine if $\sum_{k=1}^{\infty} a_k$ converges, we must relate this to $f(x)=\sum_{k=1}^{\infty} a_k(x-1)^k$. Note when $x=2$,

$$f(2)=\sum_{k=1}^{\infty} a_k(2-1)^k = \sum_{k=1}^{\infty} a_k(1)^k = \sum_{k=1}^{\infty} a_k$$

Since the series converges for all x in $(-3,5)$, the series for $f(2)$ converges so $\boxed{f(2)=\sum_{k=1}^{\infty} a_k \text{ converges.}}$

43. $f(x)=\sum_{k=1}^{\infty} a_k(2x+3)^k = \sum_{k=1}^{\infty} a_k\left[2\left(x+\frac{3}{2}\right)\right]^k = \sum_{k=1}^{\infty} 2^k a_k \left(x+\frac{3}{2}\right)^k$

a) The series is centered at $x=-\frac{3}{2}$

b) It's more useful now to think of $f(x)$ as $\sum_{k=1}^{\infty} a_k(2x+3)^k$.

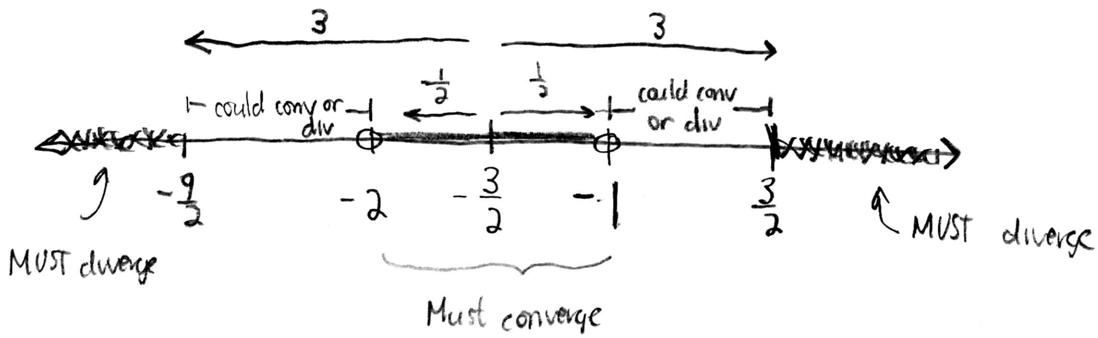
- We know $\sum_{k=1}^{\infty} a_k$ converges. We need to relate this to $f(x)$, which we can do by noting $1^k=1$, so set $2x+3=1$, then $x=-1$.

Hence $f(-1)=\sum_{k=1}^{\infty} a_k(2(-1)+3)^k = \sum_{k=1}^{\infty} a_k 1^k = \sum_{k=1}^{\infty} a_k$

The series is centered at $x = -\frac{3}{2}$ and converges when $x = -1$ which is a distance of $\frac{1}{2}$ from the center, so the ROC of $f(x)$ is at least $\frac{1}{2}$.

Similarly, $f(\frac{3}{2}) = \sum_{k=1}^{\infty} a_k \left[2(\frac{3}{2}) + 3 \right]^k = \sum_{k=1}^{\infty} 6^k a_k$

and since we know this diverges (and $x = \frac{3}{2}$ is 3 units from the center $x = -\frac{3}{2}$), the ROC of $f(x)$ is at most 3.



Note, we have no information to determine what happens if $-\frac{9}{2} < x < -2$ or $-1 < x < \frac{3}{2}$!

Hence, the series for $f(0)$ and $f(1)$ could converge or diverge however, the series for $f(2)$ must diverge!

c) The series for $f(-\frac{5}{4})$ must converge since $-\frac{5}{4}$ is $\frac{1}{4}$ from the center of the series and the ROC is at least $\frac{1}{2}$.
(Alternatively, the minimal IOC is $(-1, -2)$ and $x = -\frac{5}{4}$ is in this interval)

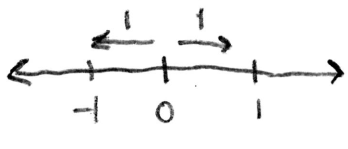
Since differentiating does not change the ROC, the minimal IOC for $f'(x)$ must be $(-1, -2)$, so $f'(-\frac{5}{4})$ must converge.

44. We know the (numeric) series $\sum_{k=0}^{\infty} a_k$ converges.

a) Note that when $x=1$, the power series $\sum_{k=0}^{\infty} a_k x^k$ becomes $\sum_{k=0}^{\infty} a_k$. So, the power series is centered at $x=0$ and converges at $x=1$. Thus the

$\boxed{\text{minimal ROC is } 1}$

b) We found above the minimal ROC is 1, so the minimal interval of convergence is $(-1,1)$, meaning that if $-1<x<1$, the series $\sum_{k=0}^{\infty} a_k x^k$ must converge. In particular, if $x=\frac{1}{2}$, the series converges and.



When $x=\frac{1}{2}$, $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k \left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} a_k \frac{1^k}{2^k} = \sum_{k=0}^{\infty} \frac{a_k}{2^k}$

so $\sum_{k=0}^{\infty} \frac{a_k}{2^k}$ converges.

(Note: If you take a convergent series and "speed up" its rate of convergence by multiplying each a_k by r^k for $|r|<1$, the resulting series not surprisingly converges!)

c) There is no maximal ROC if we aren't given more information about a_k:

• If $a_k = \frac{1}{k^2+k}$, $\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. This is a telescoping

use partial fractions!

series and, converges (you should be able to give details; if this is difficult, ask your instructors!)

Using the ratio test, $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k^2+k}$ can be shown to have ROC=1

In this case the minimal and maximal ROC are 1

• If $a_k = \frac{1}{k!}$, ratio test guarantees $\sum \frac{1}{k!}$ converges, and $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ which is the series for e^x. We know this series has an infinite ROC so in this case, the minimal ROC is 1 and the maximal ROC is infinite.

45. $\sum_{k=0}^{\infty} a_k$ diverges.

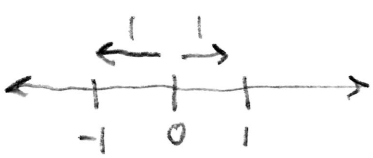
a) There is no minimal ROC for $\sum_{k=0}^{\infty} a_k x^k$; if $a_k = k!$, divergence test ensures $\sum k!$ diverges. Using ratio test on $\sum a_k x^k = \sum k! x^k$ shows the Taylor series has ROC 0.

However, if $a_k = \frac{1}{k}$, $\sum a_k = \sum \frac{1}{k}$ is the harmonic series and thus diverges, but $\sum a_k x^k = \sum \frac{x^k}{k}$ has ROC 1 (use Ratio Test)

$\Rightarrow$ $\boxed{\text{There is no minimal non-zero ROC}}$; we need more info about a_k

$\boxed{\text{There is a maximal ROC of 1}}$ however; the series $\sum a_k x^k$ is centered at $x=0$ and when $x=1$, $\sum a_k x^k = \sum a_k$, which diverges. Hence, the largest possibility for the ROC is 1

b) Since the maximal ROC is 1, the maximal interval of convergence is $(-1, 1)$, meaning if $x > 1$ or $x < 1$, the series $\sum a_k x^k$ MUST diverge!



Note when $x = 2$, the resulting series thus must diverge. When $x = 2$

$\sum a_k x^k = \sum a_k \cdot 2^k$, so this series must diverge!

c) Nothing can be said about $\sum \frac{a_k}{2^k}$;

- If $a_k = k$, Ratio test will ensure $\sum \frac{k}{2^k}$ converges
- If $a_k = 2^k$, divergence test ensures $\sum \frac{a_k}{2^k} = \sum 1$ diverges!

In both cases $\sum a_k$ diverges!