Worksheet #1 Answers

Detailed Solutions follow this sheet!

I.

a) $8(2x-7)^3$

b) $\frac{1}{4}e^{\frac{1}{4}x}$

c) $28x^3 - \frac{3}{5}x^{-\frac{4}{5}} - \frac{4}{5}x^{-3}$

d) $\dfrac{2-\sin x}{2x+\cos x}$.

e) $2e^{-x} - 2xe^{-x}$

f) $\dfrac{3(4-x)^{1/2}\sec^2 3x + \frac{1}{2}(4-x)^{-1/2}\tan 3x}{4-x}$

g) $-4e^{4x}\csc(e^{4x})\cot(e^{4x})$

h) $3[\ln(4x^3-2x)]^2 \cdot \dfrac{12x^2-2}{4x^3-2x}$

i) $2x^{-1/2}e^{4x^{1/2}}$

j) $4e^{x\sin x}(\sin x + x\cos x)$.

k) $54x^8 + \frac{1}{2}x^{-5} - \frac{4}{3}(2x-1)^{-4/3}$

l) $-24x(3x^2-1)^{-3}$

II.

a) $\frac{3}{5}x^5 - \frac{3}{5}x^{5/3} - \frac{14}{6}x^{6/7} + C$

b) Cannot Integrate

c) 1.

d) $-3e^{-\frac{1}{3}x} + C$

e) Cannot Integrate yet

f) $x^2 - \frac{3}{2}\ln|x| + C$

g) $\frac{1}{4}\sec 4x + 15\tan\frac{1}{5}x + C$

h) $\frac{5}{6}$

i) $\frac{2}{3}e^{3/2} - \frac{2}{3}$

j) $-\frac{1}{3}\cot 3x + C.$

k) Cannot integrate!

l) $-\frac{2}{9}x^{-1} + C$

III. <See solutions sheet>

IV. 3 A. $f(x) = 6x\cos x^3 - 9x^4 \sin x^3$

B. $f(x) = \dfrac{e^{2x}}{x} + 2e^{2x}$

4 <see solutions sheet>. ← These were inspired by actual student errors on old midterms and quizzes!!!

5. $\frac{1}{2}x - \frac{(2x+1)}{4}e^{-2x} + C$

Worksheet # 1 Solutions

I. a) $y = (2x-7)^4$

$y' = 4(2x-7)^3(2x-7)'$

$y' = 4(2x-7)^3 \cdot 2$

$\boxed{y' = 8(2x-7)^3}$ ← It's ok just to write this! The previous steps are meant to make sure everyone's ~~to~~ on the same page

b) $y = e^{\frac{x}{4}}$

$y = e^{\frac{1}{4}x}$ ← It's helpful to separate numbers and variables. Once again, you don't have to show this if you're comfortable with the algebra

$y' = e^{\frac{1}{4}x}\left(\frac{1}{4}x\right)'$

$\boxed{y' = \frac{1}{4}e^{\frac{1}{4}x}}$

c) $y = 7x^4 - 3\sqrt[5]{x} + \frac{2}{5x^2}$ ←

$y = 7x^4 - 3x^{\frac{1}{5}} + \frac{2}{5}\frac{1}{x^2}$ ←

In order to apply $\frac{d}{dx}(x^n) = nx^{n-1}$ we have to rewrite all of these terms as fractional / negative powers of x!

$y = 7x^4 - 3x^{\frac{1}{5}} + \frac{2}{5}x^{-2}$

$\boxed{y' = 28x^3 - \frac{3}{5}x^{-\frac{4}{5}} - \frac{4}{5}x^{-3}}$

d) $y = \ln(2x + \cos x)$

$y' = \frac{1}{2x + \cos x}(2x + \cos x)'$

$\boxed{y' = \frac{1}{2x + \cos x}(2 - \sin x)}$

e) $y = 2xe^{-x}$

$y' = (2x)' e^{-x} + 2x(e^{-x})'$

$y' = 2e^{-x} + 2x \; e^{-x} \; (-x)'$

$\boxed{y' = 2e^{-x} - 2xe^{-x}}$

f) $y = \dfrac{\tan 3x}{\sqrt{4-x}}$

$y = \dfrac{\tan 3x}{(4-x)^{1/2}}$

$y' = \dfrac{(\tan 3x)' \; (4-x)^{1/2} - \tan 3x \left[(4-x)^{1/2}\right]'}{\left[(4-x)^{1/2}\right]^2}$

$y' = \dfrac{\sec^2 3x \cdot (3x)' \; (4-x)^{1/2} - \tan 3x \left[\frac{1}{2}(4-x)^{-1/2} \cdot (4-x)'\right]}{(4-x)}$

$= \boxed{\dfrac{3\sec^2 3x \; (4-x)^{1/2} + \frac{1}{2}\tan 3x \; (4-x)^{-1/2}}{4-x}}$

g) $y = \csc(e^{4x})$

$y' = -\csc e^{4x} \cot e^{4x} \; (e^{4x})'$

$\boxed{y' = -\csc e^{4x} \cot e^{4x} (e^{4x} \cdot 4)}$

h) $y = [\ln(4x^3 - 2x)]^3$

$y' = 3[\ln(4x^3 - 2x)]^2 \cdot [\ln(4x^3 - 2x)]'$

$y' = 3[\ln(4x^3 - 2x)]^2 \frac{1}{4x^3 - 2x} (4x^3 - 2x)'$

$\boxed{y' = 3[\ln(4x^3 - 2x)]^2 \cdot \frac{1}{4x^3 - 2x} (12x^2 - 2).}$

i) $y = e^{4\sqrt{x}}$

$y = e^{4x^{1/2}}$

$y' = e^{4x^{1/2}} (4x^{1/2})'$

$\boxed{y' = e^{4x^{1/2}} \cdot 2x^{-1/2}}$

j) $y = 4e^{x \sin x}$

$y' = 4e^{x \sin x} (\underbrace{x \sin x}_{\text{Product Rule}})'$

$\boxed{y' = 4e^{x \sin x} (\sin x + x \cos x)}$

k) $y = 6x^9 - \frac{1}{8x^4} + \frac{2}{\sqrt[3]{2x-1}}$

$y = 6x^9 - \frac{1}{8} \frac{1}{x^4} + \frac{2}{(2x-1)^{1/3}}$

$y = 6x^9 - \frac{1}{8} x^{-4} + 2(2x-1)^{-1/3}$

$y' = 54x^8 + \frac{1}{2} x^{-5} - \frac{2}{3}(2x-1)^{-4/3} \cdot (2x-1)'$

$\boxed{y' = 54x^8 + \frac{1}{2} x^{-5} - \frac{4}{3}(2x-1)^{-4/3}}$

e) $y = \dfrac{2}{(3x^2-1)^2}$

$y = 2(3x^2-1)^{-2}$

$y' = -4(3x^2-1)^{-3}(3x^2-1)'$

$y' = -4(3x^2-1)^{-3}(6x)$

$\boxed{y' = -24x(3x^2-1)^{-3}}$

II. a) $\int \left(3x^4 - \sqrt[3]{x^2} + \dfrac{2}{\sqrt[7]{x}}\right) dx$

$= \int \left(3x^4 - x^{2/3} + 2 \cdot \dfrac{1}{x^{1/7}}\right) dx$

$= \int \left(3x^4 - x^{2/3} + 2x^{-1/7}\right) dx$

$= \boxed{\dfrac{3}{5}x^5 - \dfrac{3}{5}x^{5/3} - \dfrac{14}{6}x^{6/7} + C}$

b) Cannot be integrated! (In fact, one can prove there's no elementary derivative of e^{x^2}!).

* If you tried to integrate this, take the derivative of your answer. Is it equal to e^{x^2}?

c) $\int_0^{\pi/6} 4 \sin 2x \, dx$

$= -4 \cdot \frac{1}{2} \cos 2x \Big|_0^{\pi/6}$

$= -2\cos\frac{\pi}{3} - (-2\cos 0)$

$= -2 \cdot \frac{1}{2} + 2$

$= \boxed{1}$

d) $\int e^{-\frac{x}{3}}\,dx$

$= \int e^{-\frac{1}{3}x}\,dx$

$= \boxed{-3e^{-\frac{1}{3}x} + C}$

e) $\boxed{\text{Cannot integrate yet!}}$ (We will learn how to do this later in the semester though!)

(For those curious: $\int \ln x\,dx = x\ln x - x + C$; you can check this by differentiating $x\ln x - x + C$!)

f) $\int \frac{4x^3 - 3x}{2x^2}\,dx$

DO NOT WRITE $\int \frac{4x^3-3x}{2x^2}\,dx = \frac{\int (4x^3 - 3x\,dx}{\int 2x^2\,dx}$

We need to write this as a sum of powers of x!

$$\int \frac{4x^3-3x}{2x^2}\,dx = \int \left(\frac{4x^3}{2x^2} - \frac{3x}{2x^2}\right) dx$$

$$= \int \left(\frac{4}{2}\cdot\frac{x^3}{x^2} - \frac{3}{2}\,\frac{x}{x^2}\right) dx$$

$$= \int \left(2x - \frac{3}{2}\,\frac{1}{x}\right) dx$$

$$= \boxed{x^2 - \frac{3}{2}\ln|x| + C}$$

g) $\int \left(\sec 4x \tan 4x + 3\sec^2 \tfrac{1}{5}x\right) dx$

$= \int \sec 4x \tan 4x\,dx + \int 3\sec^2 \tfrac{1}{5}x\,dx$

$$= \boxed{\frac{1}{4}\sec 4x + 3\cdot\left(5\tan\frac{1}{5}x\right) + C}$$

h) $\int_1^4 (\sqrt{x}-1)^2 \, dx$

$\downarrow$ FOIL

$$= \int_1^4 (x - 2\sqrt{x} + 1)\, dx$$

$$= \int_1^4 (x - 2x^{1/2} + 1)\, dx$$

$$= \left[\frac{1}{2}x^2 - \frac{4}{3}x^{3/2} + x \,\right|_1^4$$

$$= \left[\frac{1}{2}(4)^2 - \frac{4}{3}(4)^{3/2} + 4\right] - \left[\frac{1}{2}(1)^2 - \frac{4}{3}(1)^{3/2} + 1\right]$$

$$= \boxed{\frac{5}{6}}$$

i) $\int_0^1 \sqrt{e^{3x}}\, dx$

$$= \int_0^1 e^{\frac{3}{2}x}\, dx$$

$$= \left[\frac{2}{3}\cdot e^{\frac{3}{2}x}\,\right|_0^1$$

$$= \frac{2}{3}e^{\frac{3}{2}} - \frac{2}{3}e^0$$

$$= \boxed{\frac{2}{3}e^{3/2} - \frac{2}{3}}$$

j) $\int \cot^2 3x \sec^2 3x \, dx$

$$= \int \frac{\cos^2 3x}{\sin^2 3x} \sec^2 3x \, dx$$

$$= \int \frac{1}{\sin^2 3x}\, dx$$

j) $= \int \csc^2 3x \, dx$

$= \boxed{-\frac{1}{3}\cot 3x + C}$

k) $\boxed{\text{Cannot integrate!}}$ ~~(This can be shown to possess no elementary antiderivative!)~~.

* If you tried to antidifferentiate this, differentiate your answer. Is it equal to $\cos\sqrt{x}$?

l) $\int \frac{2}{(3x)^2} \, dx$

$= \int \frac{2}{9x^2} \, dx$

$= \int \frac{2}{9} \frac{1}{x^2} \, dx$

$= \int \frac{2}{9} x^{-2} \, dx$

$= \boxed{-\frac{2}{9} x^{-1} + C}$

III. 1a) $\frac{d}{dx}(e^{x^2}) = e^{x^2}(x^2)'$

$= \boxed{2xe^{x^2}}$

b) First off, the student forgot $+C$!

Also, if the student is correct, then the derivative of his/her answer should be e^{x^2}, but:

$$\frac{d}{dx}\left(\frac{1}{2x}e^{x^2}\right) = \frac{d}{dx}\left(\frac{e^{x^2}}{2x}\right) = \frac{(e^{x^2})' \cdot 2x - e^{x^2}(2x)'}{(2x)^2}$$

$$= \boxed{\frac{4x^2e^{x^2} - 2e^{x^2}}{4x^2}}$$

This is NOT e^{x^2}!!!

c) Note that to differentiate $\frac{1}{2x}e^{x^2}$, we need the quotient rule! When we differentiate $\frac{1}{2}e^{2x}$, we don't need quotient rule!

When the argument is linear in x, the deriv. of the argument will be a constant, so we won't need quotient rule

Indeed, if $F'(x) = f(x)$, then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

since $$\frac{d}{dx}\left[\frac{1}{a}F(ax+b)\right] = \frac{1}{a}\cdot F'(ax+b)\cdot(ax+b)'$$

$$= \frac{1}{\cancel{a}} f(ax+b)\cdot\cancel{a}$$

2 a) $$\frac{d}{dx}\left(\frac{x^7-x^3}{x^4}+C\right) = \frac{(x^7-x^3)'x^4-(x^7-x^3)(x^4)'}{(x^4)^2}$$

$$= \frac{(7x^6-3x^2)x^4-(x^7-x^3)4x^3}{x^8}$$

$$= \frac{7x^{10}-3x^6-4x^{10}+4x^6}{x^8}$$

$$= \frac{3x^{10}+x^6}{x^8}$$

This is certainly NOT the original integrand!

b) When we differentiate our result, we'd need to use quotient rule!

c) $$\int\frac{7x^6-3x^2}{4x^3}\,dx = \int\left(\frac{7}{4}\frac{x^6}{x^3}-\frac{3}{4}\frac{x^2}{x^3}\right)dx = \int \frac{7}{4}x^3-\frac{3}{4}\frac{1}{x}\,dx$$

$$= \boxed{\frac{7}{16}x^4-\frac{3}{4}\ln|x|+C}$$

3 A. By the definition of an antiderivative, if

$$\int f(x)\,dx = 3x^2 \cos(x^3) + C$$

Note: This problem is NOT asking you to find the antideriv of $3x^2\cos(x^3)$; it is asking you to find a function whose antiderivatives are $3x^2\cos(x^3)+C$!

then, $f(x) = \frac{d}{dx}\left[3x^2 \cos(x^3) + C\right]$

↓ product rule and chain rule for $[\cos(x^3)]'$

$$\boxed{f(x) = 6x\cos(x^3) - 9x^4 \sin(x^3)}$$

B. Since $\int x f(x)\,dx = xe^{2x} + C$, we have:

$$x f(x) = \frac{d}{dx}\left[xe^{2x} + C\right]$$

$$x f(x) = e^{2x} + 2xe^{2x}$$

product rule and chain rule for $(e^{2x})'$

$$\boxed{f(x) = \frac{e^{2x}}{x} + 2e^{2x}}$$

4 A. FALSE; if we let $u = x^2$, then $du = 2x\,dx$ so:

$$\frac{du}{2x} = dx$$

$$\int \sin(x^2)\,dx = \int \sin u \cdot \frac{du}{2x}$$ ← can't integrate easily!

IMPORTANT POINT

Note this is NOT enough to prove the student is false though; it may happen that this error still doesn't affect the final answer.

To prove the student is false, note if the substitution were true

$$\int \sin u\,du = -\cos u + C = -\cos(x^2) + C$$

So, the student's claim would be that $\int \sin(x^2)dx = -\cos(x^2)+C$

If this is true, then $\frac{d}{dx}\left[-\cos(x^2)+C\right]$ must be $\sin(x^2)$.

But, $\frac{d}{dx}\left[-\cos(x^2)+C\right] = \sin(x^2)\cdot\frac{d}{dx}(x^2)$ by chain rule

$$= 2x\sin(x^2).$$

which is NOT $\sin(x^2)$! The student is thus incorrect!

B. <u>FALSE</u>; if $\int \frac{1}{\sec\theta}\, d\theta = \ln(\sec\theta) + C$, we must have

$$\frac{1}{\sec\theta} = \frac{d}{d\theta}\left[\ln(\sec\theta) + C\right]$$

But, $\frac{d}{d\theta}\left[\ln(\sec\theta) + C\right] = \frac{1}{\sec\theta}\frac{d}{d\theta}\left[\sec\theta\right] = \frac{1}{\cancel{\sec\theta}}\cancel{\sec\theta}\tan\theta$

Clearly, $\frac{1}{\sec\theta} \neq \tan\theta$ so the statement is false!

<u>Note:</u> $\frac{1}{\sec\theta} = \cos\theta$ by definition, so

$$\int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta = \sin\theta + C.$$

C. <u>FALSE</u>; in general the integral is not the product of the integrals!

IMPORTANT POINT

Note that to <u>prove</u> $\int x\sin x\, dx \neq \frac{1}{2}x^2\cos x + C$ that it is not sufficient just to say this though. We need to check explicitly that $\frac{1}{2}x^2\cos x + C$ is NOT somehow the antiderivative of $x\sin x$.

If $\int x\sin x\, dx = \frac{1}{2}x^2\cos x + C$, then it must be true that $\frac{d}{dx}\left[\frac{1}{2}x^2\cos x + C\right] = x\sin x$.

But, $\frac{d}{dx}\left[\frac{1}{2}x^2\cos x + C\right] = x\cos x - \frac{1}{2}x^2\sin x$

(by product rule)

This is clearly not $x\sin x$; so the student's claim is false.

D. <u>FALSE</u>; if $\int \frac{1}{1-x^2}\, dx = \ln(1-x^2) + C$, then it must be true that

$$\frac{1}{1-x^2} = \frac{d}{dx}\left[\ln(1-x^2) + C\right].$$

But, $\frac{d}{dx}\left[\ln(1-x^2) + C\right] = \frac{1}{1-x^2}\frac{d}{dx}(1-x^2)$ by chain rule

$$= -\frac{2x}{1-x^2}$$

This is clearly not $\frac{1}{1-x^2}$, so the statement is false.

MAJOR POINT: It is NOT enough simply to point out what the student did wrong. The error may somehow lead to the correct antiderivative! A fully correct justification MUST include showing that the derivative of the student's answer does not match the function in the integrand!!!

5. $\displaystyle\int \frac{e^{2x} + 2x}{2e^{2x}}\, dx.$

DO NOT CANCEL THE e^{2x}; $\quad \dfrac{e^{2x}+2x}{2e^{2x}} \neq \dfrac{\cancel{e^{2x}}+2x}{2\cancel{e^{2x}}}$!!!

Split the integral:

$$\int \frac{e^{2x}+2x}{2e^{2x}}\,dx = \int \left[\frac{e^{2x}}{2e^{2x}} + \frac{2x}{2e^{2x}}\right] dx$$

$$= \int \frac{\cancel{e^{2x}}}{2\cancel{e^{2x}}}\,dx + \int \frac{\cancel{2}x}{\cancel{2}e^{2x}}\,dx$$

$$= \int \frac{1}{2}\,dx + \underbrace{\int xe^{-2x}\,dx}_{\text{We can't evaluate this yet!}}.$$

Note we cannot evaluate $\int xe^{-2x}\,dx$ (we will once we discuss integration by parts), but we're given

$$\frac{d}{dx}\left[\frac{(ax-1)e^{ax}}{a^2}\right] = xe^{ax} \quad \text{for } a \neq 0.$$

When $a=-2$, this reads:

$$\frac{d}{dx}\left[\frac{(-2x-1)e^{-2x}}{4}\right] = xe^{-2x}$$

Thus,

$$\frac{(-2x-1)e^{-2x}}{4} + C = \int xe^{-2x}\,dx$$

by the definition of an antiderivative!

So,

$$\int \frac{e^{2x}+2x}{2e^{2x}}\,dx = \int \frac{1}{2}\,dx + \int xe^{-2x}\,dx \quad \text{(from before)}$$

$$= \boxed{\frac{1}{2}x + \frac{(-2x-1)e^{-2x}}{4} + C}$$

6. A. The error in the student's argument is that

$$\int \frac{1}{e^u}\,du = \ln(e^u) + C.$$

Indeed, if the student is correct, then the answer $\ln(e^{x^2}) + C$ can be simplied since

$$\ln(e^{x^2}) = x^2.$$

So, the student really claims that $\int \frac{2x}{e^{x^2}}\,dx = x^2 + C$.

If this were true, then $\frac{d}{dx}[x^2+C]$ would be $\frac{2x}{e^{x^2}}$, which is preposterous.

B. To evaluate $\int \frac{2x}{e^{x^2}}\,dx$, make the same substitution the student made:

$$u = x^2$$

$$du = 2x\,dx$$

$$dx = \frac{du}{2x}$$

So $\displaystyle\int \frac{2x}{e^{x^2}}\,dx = \int \frac{\cancel{2x}}{e^u}\,\frac{du}{\cancel{2x}}$ ← Student was good up to here!

We can integrate $\int e^{au}\,du$, so bring the integral into this form:

$$\int \frac{1}{e^u}\,du = \int e^{-u}\,du$$

$$= -e^{-u} + C$$

Make sure you understand why this is here!

$$= \boxed{-e^{-x^2} + C}$$

Indeed, we may check $\displaystyle\frac{d}{dx}\left[-e^{-x^2} + C\right] = -e^{-x^2} \cdot (-2x)$

$$= 2x\,e^{-x^2}$$

$$= \frac{2x}{e^{x^2}}$$

C. If $\displaystyle\int \frac{1}{u(x)}\,dx = \ln[u(x)] + C$, then

$$\frac{1}{u(x)} = \frac{d}{dx}\left(\ln[u(x)] + C\right)$$

$$\frac{1}{u(x)} = \frac{1}{u(x)} \cdot \frac{du}{dx}$$

Multiply by $u(x)$:

$$1 = \frac{du}{dx}. \quad \rightarrow \quad u = \int 1\,dx.$$

Hence $u(x) = x + b$, where b is any constant s.t. $u(x) > 0$ on its domain.

Thus, the <u>ONLY</u> time we get $\ln(\ )$ when integrate a <u>linear</u> function in x!

<u>Note:</u> It can be shown $\frac{d}{dx}|x| = \frac{|x|}{x}$, and

using this result, we can drop the positivity assumption on b and $u(x)$; and write $\int \frac{1}{u(x)}\,dx = \ln\big[|u(x)|\big] + C$

only when $u(x) = x + b$

Note also if $u(x) = ax+b$, we can let $v = ax+b$
$dv = a\,dx$

and find
$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{v}\,\frac{1}{a}\,dv$$
$$= \frac{1}{a}\ln|v| + C$$
$$= \frac{1}{a}\ln|ax+b| + C.$$

→ The only time we get $\ln(\ )$ while integrating is:

1. When the integrand is a fraction whose numerator is a constant and whose denominator is linear in the variable of integration.

2. When we can make a substitution to bring the integrand into the above form