Worksheet #1: Review of Differentiation and Basic Integration Skills

The following worksheet is designed to help review and/or sharpen your ability to differentiate and integrate functions encountered in a typical Calculus 1 course. Section IV also addresses some good conceptual questions about the relationship between a fiction and its antiderivatives.

I. Differentiation Practice
Differentiate the following functions.

$a)\ y = (2x-7)^4$ $\quad b)\ y = e^{\frac{x}{4}}$ $\quad c)\ y = 7x^4 - 3\sqrt[5]{x} + \dfrac{2}{5x^2}$

$d)\ y = \ln(2x+\cos x)$ $\quad e)\ y = 2xe^{-x}$ $\quad f)\ y = \dfrac{\tan(3x)}{\sqrt{4-x}}$

$g)\ y = \csc\left(e^{4x}\right)$ $\quad h)\ y = [\ln(4x^3-2x)]^3$ $\quad i)\ y = e^{4\sqrt{x}}$

$j)\ y = 4e^{x\sin x}$ $\quad k)\ y = 6x^9 - \dfrac{1}{8x^4} + \dfrac{2}{\sqrt[3]{2x-1}}$ $\quad l)\ y = \dfrac{2}{(3x^2-1)^2}$

II. Integration Practice
Compute the following integrals. If an integral cannot be algebraically reduced to one of the basic functions (powers of x, trig functions, exponentials, etc) that can be easily integrated, state so!

$a)\ \displaystyle\int \left(3x^4 - \sqrt[3]{x^2} + \frac{2}{\sqrt[7]{x}}\right) dx$ $\quad b)\ \displaystyle\int e^{x^2}\,dx$ $\quad c)\ \displaystyle\int_0^{\pi/6} 4\sin(2x)\,dx$

$d)\ \displaystyle\int e^{-\frac{x}{3}}\,dx$ $\quad e)\ \displaystyle\int \ln x\,dx$ $\quad f)\ \displaystyle\int \frac{4x^3-3x}{2x^2}\,dx$

$g)\ \displaystyle\int \left(\sec(4x)\tan(4x) + 3\sec^2\frac{x}{5}\right) dx$ $\quad h)\ \displaystyle\int_1^4 (\sqrt{x}-1)^2\,dx$ $\quad i)\ \displaystyle\int_0^1 \sqrt{e^{3x}}\,dx$

$j)\ \displaystyle\int \cot^2(3x)\sec^2(3x)\,dx$ $\quad k)\ \displaystyle\int \cos\sqrt{x}\,dx$ $\quad l)\ \displaystyle\int \frac{2}{(3x)^2}\,dx$

III. Miscellaneous
The following questions help dispel common integration errors and allow for one to gain some insight as to why these incorrect methods fail.

1. Consider the function $f(x) = e^{2x}$. We know that $\dfrac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$ by the Chain Rule, and this lets us easily conclude that $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$. This could of course be verified by u-substitution (if you know/remember this technique), but can also be understood the following way:

The symbol $\int e^{2x}\,dx$ represents a function whose derivative is e^{2x}. Since taking a derivative of e^{2x} results in multiplying e^{2x} by 2, when we antidifferentiate e^{2x}, we must multiply by $\frac{1}{2}$.

You must be careful with this type of thought! Indeed, it works only when the argument of the function (in this case, the expression in the exponent) is LINEAR[1] in x!

a) Calculate $\dfrac{d}{dx}\left(e^{x^2}\right)$.

b) Suppose a student tries to apply the above logic to compute $\int e^{x^2}\,dx$. The student concludes that since $\dfrac{d}{dx}e^{x^2}\,dx = 2xe^{x^2}$, then:

$$\int e^{x^2}\,dx = \frac{1}{2x}e^{x^2} \tag{1}$$

Since you know that $\int e^{x^2}\,dx$ is a function whose derivative is e^{x^2}, prove this student wrong by differentiating his/her answer (i.e. the RHS of Eqn 1).

c) What insight does this reveal as to why this students' answer is wrong? Why can we think of antidifferentiating e^{2x} differently than antidifferentiating e^{x^2}?

2. Another student sees the following integral on an exam:

$$\int \frac{7x^6 - 3x^2}{4x^3}\,dx$$

The student answers the question the following way:

$$\begin{aligned}\int \frac{7x^6 - 3x^2}{4x^3}\,dx &= \frac{\int \left(7x^6 - 3x^2\right)\,dx}{\int 4x^3\,dx}\\ &= \frac{x^7 - x^3}{x^4} + C \end{aligned} \tag{2}$$

a) By using Eqn. 2 exactly as it is written above (i.e. WITHOUT simplifying it!), show that the derivative of the RHS of Eqn. 2 is NOT equal to the expression in the original integrand.

b) What insight does this yield? Why can one not simply just integrate the numerator and denominator of a fraction separately?

c) Compute the antiderivative of this function correctly.

IV. Conceptual Questions about Antidifferentiation

3. Find a function $f(x)$ such that:

A. $\displaystyle\int f(x)\,dx = 3x^2\cos(x^3) + C.$

[1] "linear in x" means the argument is of the form $ax + b$

B. $\displaystyle\int xf(x)\,dx = xe^{2x} + C.$

4. Determine if the following are true or false. If they are false, provide a clear argument that justifies your claim!

   A. Consider the integral $\displaystyle\int \sin(x^2)\,dx$. Upon performing the substitution $u = x^2$, the integral becomes $\displaystyle\int \sin u\,du$.

   B. For $-\pi/2 \leq \theta \leq \pi/2$, $\displaystyle\int \frac{1}{\sec\theta}\,d\theta = \ln(\sec\theta) + C$

   C. $\displaystyle\int x\sin x\,dx = \frac{1}{2}x^2\cos x + C$

   D. For $-1 < x < 1$, $\displaystyle\int \frac{1}{1-x^2}\,dx = \ln(1-x^2) + C$

5. Find $\displaystyle\int \frac{e^{2x}+2x}{2e^{2x}}\,dx.$

   Hint: For any real number $a \neq 0$, $\displaystyle\frac{d}{dx}\left[\frac{(ax-1)e^{ax}}{a^2}\right] = xe^{ax}.$

6. The following question explores a common misconception that arose on Quiz 1 this semester.

   On an exam, a student is asked to evaluate $\displaystyle\int \frac{2x}{e^{x^2}}\,dx$ and provides the following:

   Student's Response:

   Let $u = x^2$. Then $du = 2x\,dx$ so $dx = \dfrac{du}{2x}$. Substituting this into the integral gives:

$$\int \frac{2x}{e^{x^2}}\,dx = \int \frac{\cancel{2x}}{e^u}\frac{du}{\cancel{2x}} = \ln e^u + C = \boxed{\ln e^{x^2} + C}$$

   A. What error has the student made? Why can't his/her antiderivative be correct?

   B. Compute $\displaystyle\int \frac{2x}{e^{x^2}}\,dx.$

   C. Find all positive, differentiable functions $u(x)$ such that:

$$\int \frac{1}{u(x)}\,dx = \ln[u(x)] + C.$$

   Hint: The absolute value on the ln( ) term is dropped here because of the assumption $u(x)$ is positive. See the solutions for a generalization.