# Worksheet #2 Answers

## I. Areas

1. $A=\int_{y=-4}^{y=8} \left[\left(6-\frac{1}{2}y\right)-\frac{1}{4}y\right] dy = 54$

2. $A=\int_{x=-2}^{x=2} \left[\left(18-3x^2\right)-\left(x^2+2\right)\right] dx = \frac{128}{3}$

3. a) $A=\int_{x=0}^{x=4} \sqrt{x}\, dx = \frac{16}{3}$

   b) $a=\frac{1}{3}$

4. a) $A=\int_{x=0}^{x=\frac{\sqrt{2}}{2}} (\arccos x - \arcsin x)\, dx.$ ← We can't evaluate this yet!!! (But we'll be able to soon!)

   b) $A=\int_{y=0}^{y=\frac{\pi}{4}} \sin y\, dy + \int_{y=\frac{\pi}{4}}^{y=\pi} \cos y\, dy.$ ← So we're forced to use 2 integrals!

   c) $A=2-\sqrt{2}$

5. a) $A=\int_{x=0}^{x=1} \left[3-\left(\frac{1}{3}x^2+1\right)\right] dx + \int_{x=1}^{x=3} \left[\left(\frac{1}{2}x+\frac{5}{2}\right)-\left(\frac{1}{3}x^2+1\right)\right] dx.$

   b) $A=\int_{x=0}^{x=1} \left[(3-x)-(1-x)\right] dx + \int_{x=1}^{x=3} (3-x)\, dx.$

## II. Washer / Shell Method

1. a) Washer: $V=\pi\int_{y=0}^{y=6} \left(\frac{1}{3}y\right)^2 dy$

   Shell: $V=2\pi\int_{x=0}^{x=2} x(6-3x)\, dx$

   b) Washer: $V=\pi\int_{x=0}^{x=2} \left[6^2-(3x)^2\right] dx$

   Shell: $V=2\pi\int_{y=0}^{y=6} y\left(\frac{1}{3}y\right) dy$

c) Washer: $V = \pi \int_{x=0}^{x=2} (6-3x)^2 \, dx$

Shell: $V = 2\pi \int_{y=0}^{y=6} (6-y)(\frac{1}{3}y) \, dy$.

d) Washer: $V = \pi \int_{y=0}^{y=6} \left[(\frac{1}{3}y+5)^2 - 5^2\right] dy$.

Shell: $V = 2\pi \int_{x=0}^{x=2} (x+5)(6-3x) \, dx$

2. a) i) $V = 2\pi \int_{x=0}^{x=1} x(10-3x^2) \, dx$

ii) $V = \frac{17\pi}{2}$

b) i) $V = \pi \int_{x=0}^{x=1} (10-3x^2)^2 \, dx$

ii) $V = \frac{409\pi}{5}$

c) i) $V = \pi \int_{x=0}^{x=1} \left[(15-3x^2)^2 - 5^2\right] dx$

ii) $V = \frac{859\pi}{5}$

d) i) $V = 2\pi \int_{x=0}^{x=1} (2+x)(10-3x^2) \, dx$

ii) $V = \frac{89\pi}{2}$.

3. a) i) $V = \pi \int_{y=0}^{y=1} \left[(\frac{3}{2} - \frac{1}{2}y)^2 - (y^2)^2\right] dy$

ii) $V = \frac{83\pi}{60}$

b) i) $V = 2\pi \int_{y=0}^{y=1} y(\frac{3}{2} - \frac{1}{2}y - y^2) dy$

ii) $V = \frac{2\pi}{3}$

c) i) $V = 2\pi \int_{y=0}^{y=1} (10-y)(\frac{3}{2} - \frac{1}{2}y - y^2) dy$

ii) $V = \frac{53\pi}{3}$

d) i) $V = \pi \int_{y=0}^{y=1} \left[(\frac{5}{2} - \frac{1}{2}y)^2 - (y^2+1)^2\right] dy$

ii) $V = \frac{193\pi}{60}$

4. a) $V = 2\pi \int_{x=-1}^{x=3} (4-x)(18-6x) \, dx = 352\pi$

b) $a \approx -.095$ (Use a graphing utility... I won't make it this bad on an In-Class Quiz).

5. a) $V = \pi \int_{y=0}^{y=1} \left[(\pi - \frac{\pi}{2}y)^2 - \arcsin^2 y\right] dy$

b) $V = 2\pi \int_{y=0}^{y=1} y\left[\pi - \frac{\pi}{2}y - \arcsin y\right] dy$

c) $V = \pi \int_{y=0}^{y=1} \left[(6-\arcsin y)^2 - (6-\pi+\frac{\pi}{2}y)^2\right] dy$

d) $V = 2\pi \int_{y=0}^{y=1} \left[(y+4)(\pi - \frac{\pi}{2}y - \arcsin y)\right] dy$

# Worksheet #2 Addendum Answers

6a) $V = \int_{x=0}^{x=4} (8-2x)^2\,dx = \frac{256}{3}$

b) $V = \int_{x=0}^{x=4} \frac{\sqrt{3}}{4}(8-2x)^2\,dx = \frac{64\sqrt{3}}{3}$

c) $V = \int_{x=0}^{x=4} \frac{\pi}{2}(4-x)^2 = \frac{32\pi}{3}$

d) The solid whose cross-sections are squares has the largest volume.

7 a) i) $V = \int_{x=-1}^{x=0} (9+9x)^2\,dx + \int_{x=0}^{x=3} (9-9x^2)^2\,dx = \frac{13743}{5}$

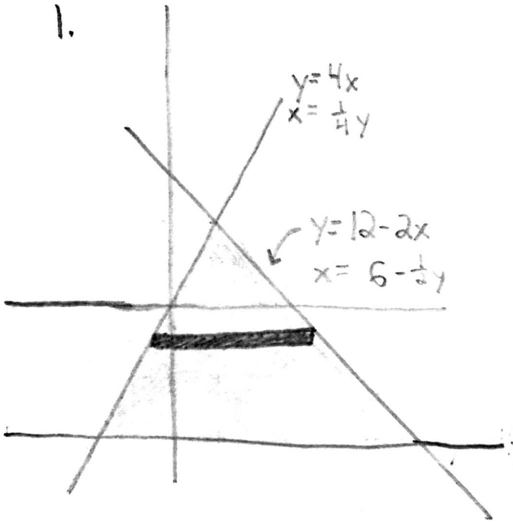
ii) $V = \int_{y=0}^{y=9} \left[ (1-\frac{1}{9}y) - 2(1-\frac{1}{9}y)^{3/2} + (1-\frac{1}{9}y)^2 \right] dy = \frac{3}{10}$

b) $V = \int_{y=0}^{y=9} \frac{\pi}{8}\left[\sqrt{1-\frac{1}{9}y} - (1-\frac{1}{9}y)\right]^2 dy.$

# Worksheet #2 Solutions

## I. Areas

1.




- We want to use horizontal strips since the right and left curves do not change.
- Horizontal Strips ⇒ integrate wrt y.

Intersection Pts:

$\frac{1}{4}y = 6 - \frac{1}{2}y$

$\frac{3}{4}y = 6$

$y = 8$

$A = \int_{y=c}^{y=d} (\text{right} - \text{left})\,dy$

$A = \int_{y=-4}^{y=8} \left[(6 - \frac{1}{2}y) - (\frac{1}{4}y)\right] dy$

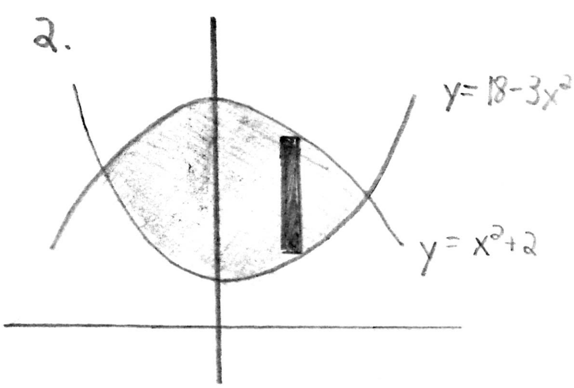
$A = \int_{y=-4}^{y=8} (6 - \frac{3}{4}y)\,dy$

$= 6y - \frac{3}{8}y^2 \Big|_{-4}^{8}$

$= \left[6(8) - \frac{3}{8}(8)^2\right] - \left[6(-4) - \frac{3}{8}(-4)^2\right]$

$\boxed{A = 54}$

2.




- Use vertical strips since the top and bottom curves do not change
- Vertical Strips ⇒ Integrate wrt x.

Intersection Pts:

$18 - 3x^2 = x^2 + 2$

$4x^2 = 16$

$x = \pm 2$

$A = \int_{x=a}^{x=b} (\text{top} - \text{bot})\,dx$

$= \int_{-2}^{2} \left[(18 - 3x^2) - (x^2 + 2)\right] dx$

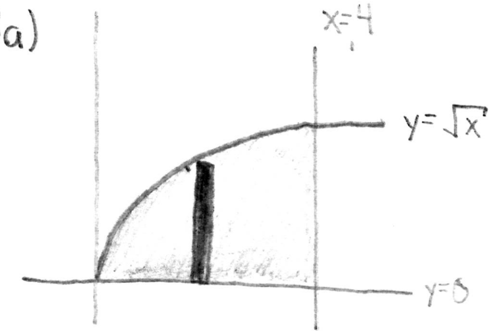
$= \int_{-2}^{2} (16 - 4x^2)\,dx$

$= \left[16x - \frac{4}{3}x^3\right]_{-2}^{2}$

$= \left[16(2) - \frac{4}{3}(2)^3\right] - \left[16(-2) - \frac{4}{3}(2)^3\right]$

$\boxed{A = \frac{128}{3}}$

3a)




$$A = \int_{x=a}^{x=b} (\text{top} - \text{bottom})\, dx$$

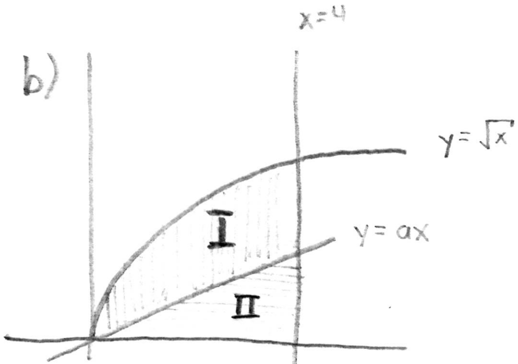
$$A = \int_{x=0}^{x=4} (x^{1/2} - 0)\, dx$$

$$= \left[\tfrac{2}{3} x^{3/2} \right|_0^4$$

$$= \tfrac{2}{3}(4)^{3/2} - \tfrac{2}{3}(0)^{3/2}$$

$$\boxed{A = \tfrac{16}{3}}$$

b)




__Way 1:__ $A_I = \tfrac{1}{2} A = \tfrac{8}{3}$

$$\int_0^4 (x^{1/2} - ax)\, dx = \tfrac{8}{3}$$

$$\tfrac{3}{2} x^{2/3} - \tfrac{1}{2} a x^2 \Big|_0^4 = \tfrac{8}{3}$$

$$\tfrac{16}{3} - \tfrac{1}{2} a (4)^2 = \tfrac{8}{3}$$

$$-8a = -\tfrac{8}{3}$$

$$\boxed{a = \tfrac{1}{3}}$$

__Way 2:__ $A_{II}$ = Area triang

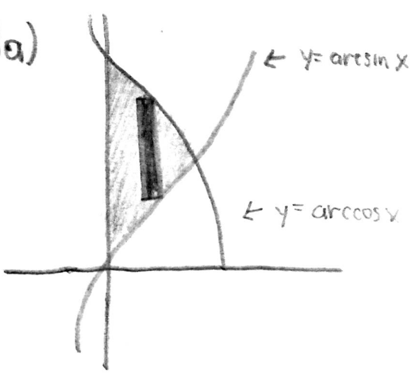
$$\tfrac{8}{3} = \tfrac{1}{2} bh$$

$$\tfrac{8}{3} = \tfrac{1}{2}(4)(4a)$$

$$\tfrac{8}{3} = 8a$$

$$\boxed{\tfrac{1}{3} = a}$$

4a)




·Integrate wrt x ⇒ use vertical strips.

__Intersection Points__ :

$y = \arcsin x \rightarrow x = \sin y$

$y = \arccos x \rightarrow x = \cos y$

$\sin y = \cos y$

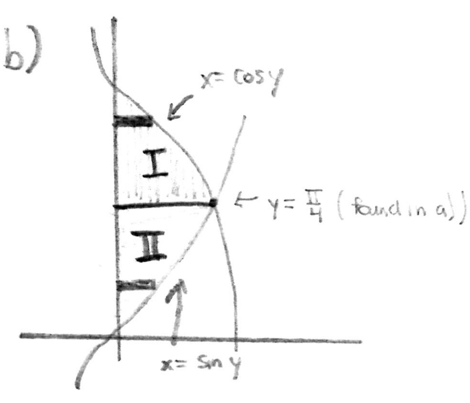
$\tan y = 1$

$\underline{y = \frac{\pi}{4}} \Rightarrow \underline{x = \frac{\sqrt{2}}{2}}$

So: $\boxed{A = \int_{x=0}^{x=\sqrt{2}/2} (\arccos x - \arcsin x)\, dx}$

__Note that we can't integrate this yet!__

b)




$$A_I = \int_{y=\pi/4}^{y=\pi/2} \cos y\, dy$$

$$= \sin y \Big|_{\pi/4}^{\pi/2}$$

$$= \sin \tfrac{\pi}{2} - \sin \tfrac{\pi}{4}$$

$$A_I = 1 - \tfrac{\sqrt{2}}{2}$$

$$A_{II} = \int_{y=0}^{y=\pi/4} \sin y\, dy$$

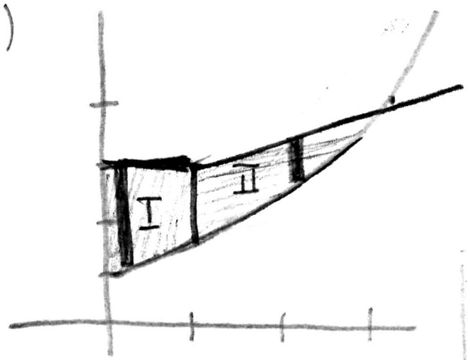
$$= -\cos y \Big|_0^{\pi/4}$$

$$= -\cos \tfrac{\pi}{4} - (-\cos 0)$$

$$A_{II} = -\tfrac{\sqrt{2}}{2} + 1$$

$$A = A_I + A_{II} \rightarrow \boxed{A = 2 - \sqrt{2}}$$

5 a)



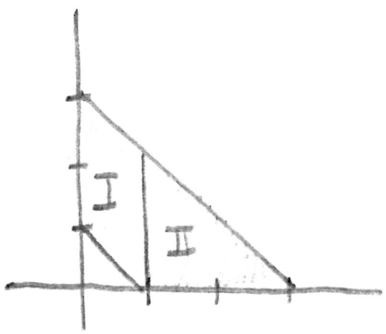

- We need 2 integrals regardless of whether we use vertical or horizontal strips.
- Since y is solved in terms of x, use vertical.

$$A_{I} = \int_{x=0}^{x=1} \left[3-\left(\tfrac{1}{3}x^2+1\right)\right] dx$$

$$A_{II} = \int_{x=1}^{x=3} \left[\left(\tfrac{1}{2}x+\tfrac{5}{2}\right)-\left(\tfrac{1}{3}x^2+1\right)\right] dx$$

$$A = A_{I} + A_{II}$$

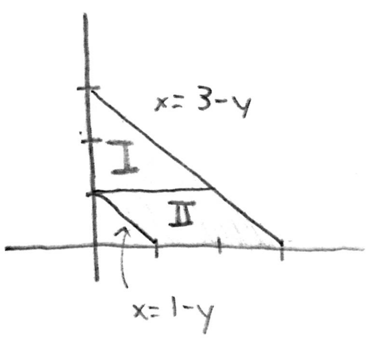
b)




- We need 2 integrals regardless of whether we use vertical or horizontal strips.

Set-Up wrt x (Preferable since y is in terms of x)

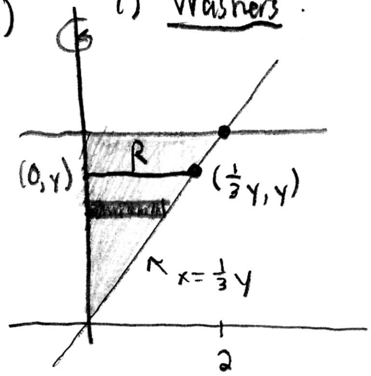
$$\left.\begin{aligned} A_{I} &= \int_{x=0}^{x=1} \left[(3-x)-(1-x)\right] dx \\ A_{II} &= \int_{x=1}^{x=3} (3-x)\, dx \end{aligned}\right\} A = A_{I} + A_{II}$$




Set up wrt y:

$$\left.\begin{aligned} A_{I} &= \int_{y=1}^{y=3} (3-y)\, dy \\ A_{II} &= \int_{y=0}^{y=1} \left[(3-y)-(1-y)\right] dy \end{aligned}\right\} A = A_{I} + A_{II}$$

II. 1a) i) Washers:




$3x = 6$

$x = 2$

R - dist from axis to outer = chng in x = $\frac{1}{3}y$

Washer: • rect are ⊥ to the axis of rotation

⇒ rect are horizontal

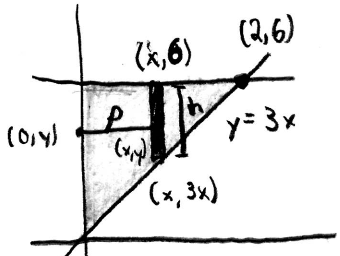
⇒ integrate wrt y

$$V = \pi \int_{y=c}^{y=d} (R^2 - r^2)\, dy$$

$$= \pi \int_{y=0}^{y=6} \left[\left(\tfrac{1}{3}y\right)^2 - 0^2\right] dy$$

$$\boxed{V = \pi \int_0^6 \left(\tfrac{1}{3}y\right)^2 dy}$$

ii) Shells.




• Shells ⇒ rectangles are parallel to the axis of rot.

⇒ rect. are vertical

⇒ int. wrt x.

$$V = 2\pi \int_{x=a}^{x=b} ph \, dx$$

$$\boxed{V = 2\pi \int_{x=0}^{x=2} x(6-3x) \, dx}$$

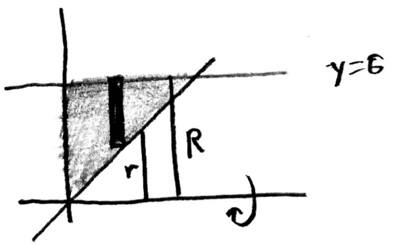
p: distance from axis to rect

= x - 0 = x

h: height of rect.

= 6 - 3x

b) i) Washers




• Washers ⇒ rect are ⊥ axis of rot

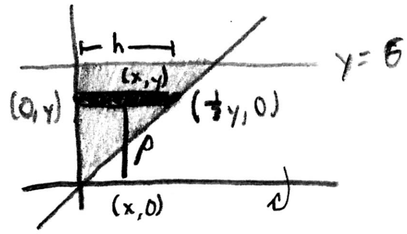
⇒ vertical strips

⇒ int wrt x.

$$V = \pi \int_{x=0}^{x=2} (R^2 - r^2) \, dx$$

$$\boxed{V = \pi \int_0^3 6^2 - (3x)^2 \, dx}$$

ii) Shells




• Shells ⇒ rect are ∥ axis of rot

⇒ use horizontal strips

⇒ int wrt y.

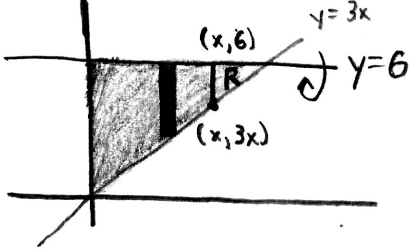
$$V = 2\pi \int_{y=0}^{y=6} ph \, dy$$

$$\boxed{V = 2\pi \int_{y=0}^{y=6} y\left(\tfrac{1}{3}y\right) dy}$$

$p = y - 0$

$h = \frac{1}{3}y - 0$

c) i) Washers




• Washers ⇒ rect are ⊥ axis of rot.

⇒ vertical strips

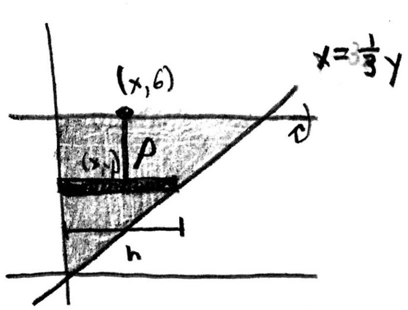
⇒ int wrt x.

$$\boxed{V = \pi \int_{x=0}^{x=3} (6-3x)^2 \, dx}$$

R = chng in y

= 6 - 3x

ii) Shells.




- Shells $\Rightarrow$ rect are $\parallel$ axis of rot
- $\Rightarrow$ horizontal strips
- $\Rightarrow$ int wrt $y$.

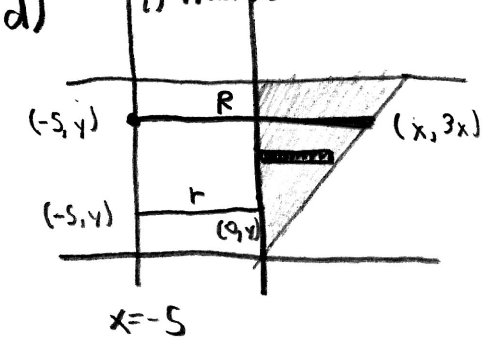
$$V = 2\pi \int_{y=0}^{y=6} \rho h \, dy$$

$$\boxed{V = 2\pi \int_{y=0}^{y=6} (6-y)\cdot\tfrac{1}{3}y \, dy}$$

$\rho = 6-y$

$h = \frac{1}{3}y - 0$

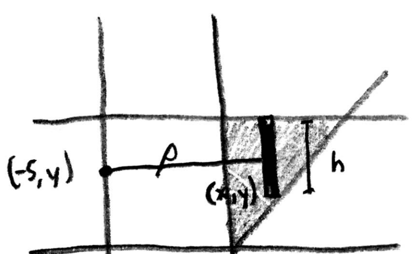
d) i) Washers




$$V = \pi \int_{y=0}^{y=6} (R^2 - r^2) \, dy$$

$$\boxed{V = \pi \int_{y=0}^{y=6} \left[ \left(\tfrac{1}{3}y-(-5)\right)^2 - (0-(-5))^2 \right] dy}$$

ii) Shells.

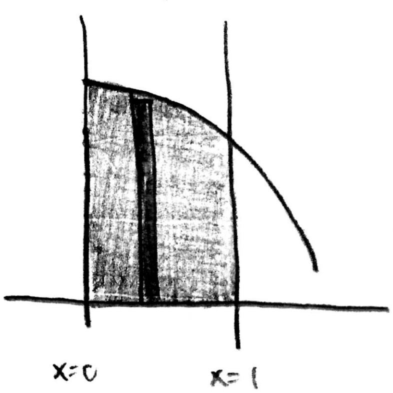



$$V = 2\pi \int_{x=0}^{x=3} \rho h \, dx$$

$$V = 2\pi \int_{x=0}^{x=3} (x-(-5))(6-3x) \, dx$$

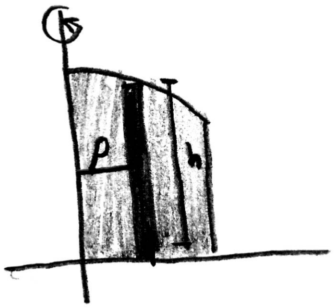
$$\boxed{V = 2\pi \int_{x=0}^{x=3} (x+5)(6-3x) \, dx}$$

2.




- We want to use vertical rectangles based on the region (we'd need 2 if we use horizontal).
- We'll determine the method in each case based on the axis, with the requirement that we must use vertical strips.

a) $x=0$.




- Vert rect are $\parallel$ to the axis of rot

$\Rightarrow$ Use Shell

- Vert rect $\Rightarrow$ integrate wrt x.

$$V = 2\pi \int_{x=0}^{x=1} ph\,dx$$

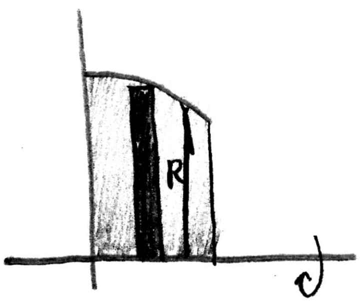
i) $$\boxed{V = 2\pi \int_{x=0}^{x=1} x(10-3x^2)\,dx}$$

$$= 2\pi \int_0^1 (10x - 3x^3)\,dx$$

$$= 2\pi \left[ 5x^2 - \frac{3}{4}x^4 \right|_0^1$$

ii) $$\boxed{V = \frac{17\pi}{2}}$$

b) $y=0$




- Vert rect are $\perp$ the axis of rot.

$\Rightarrow$ Washer method

$$V = \pi \int_0^1 (R^2 - r^2)\,dx$$

$$= \pi \int_0^1 \left[(10-3x^2)^2 - 0\right] dx$$

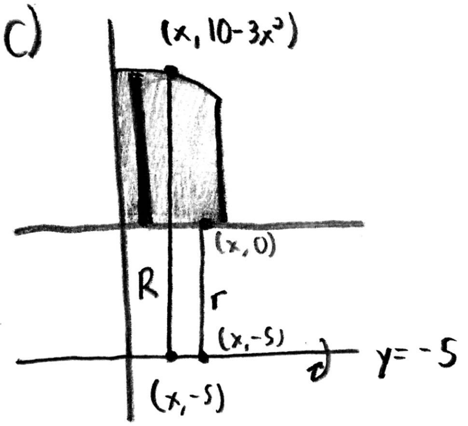
i) $$\boxed{V = \pi \int_0^1 (10-3x^2)^2\,dx}$$

$$= \pi \int_0^1 (100 - 60x^2 + 9x^4)\,dx \quad \text{(FOIL)}$$

$$= \pi \left[ 100x - 20x^3 + \frac{9}{5}x^5 \right|_0^1$$

ii) $$\boxed{V = \frac{409\pi}{5}}$$

c)




$R = (10-3x^2)-5 = 15-3x^2$

$r = 0-(-5) = 5$

- Vert rect are $\perp$ axis of rot.

$\Rightarrow$ Use Washers.

i) $\boxed{V = \pi \int_{x=0}^{x=1} \left[(15-3x^2)^2 - 5^2\right] dx}$

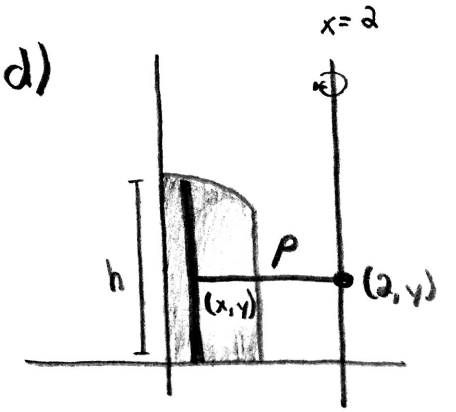
$= \pi \int_0^1 (225 - 90x^2 + 9x^4 - 25)\, dx$

$= \pi \int_0^1 (200 - 90x^2 + 9x^4)\, dx$

$= \pi \left[ 200x - 30x^3 + \frac{9}{5}x^5 \right|_0^1$

ii) $\boxed{V = \frac{859\pi}{5}}$

d)




$p = 2-x$

$h = 10-3x^2$

- Vert rect are $\parallel$ axis of rot

$\Rightarrow$ Use shells.

$V = 2\pi \int_{x=0}^{x=1} ph\, dx$

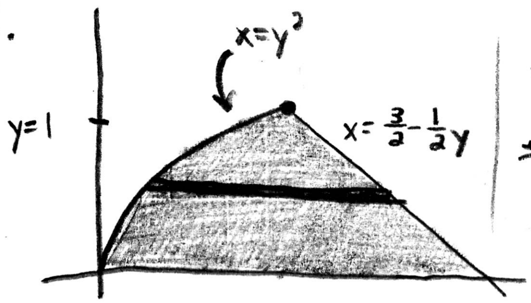
i) $\boxed{V = 2\pi \int_{x=0}^{x=1} (2-x)(10-3x^2)\, dx}$

$= 2\pi \int_0^1 \left[20 - 10x - 6x^2 + 3x^3\right] dx$

$= 2\pi \left[ 20x - 5x^2 - 2x^3 + \frac{3}{4}x^4 \right|_0^1$

ii) $\boxed{V = \frac{89\pi}{2}}$

3.




- We want to use horizontal strips.

$\Rightarrow$ We'll need everything in terms of $y$.

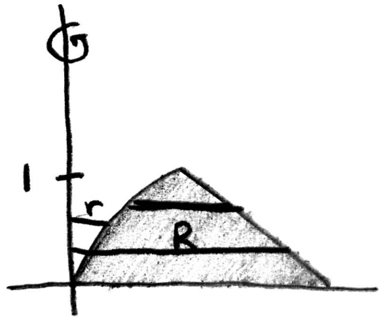
- $y = \sqrt{x} \to x = y^2$
- $y = 3-2x \to x = \frac{3}{2} - \frac{1}{2}y$

Upper y limit: $y^2 = \frac{3}{2} - \frac{1}{2}y$

$2y^2 = 3-y$

$2y^2+y-3=0 \to (2y+3)(y-1)=0 \Rightarrow y = 1, \cancel{-\frac{3}{2}}$

a) $x = 0$




• Rect are $\perp$ axis of rot.

$\Rightarrow$ Use washers.

$$V = \pi \int_{y=0}^{y=1} (R^2 - r^2)\, dy$$

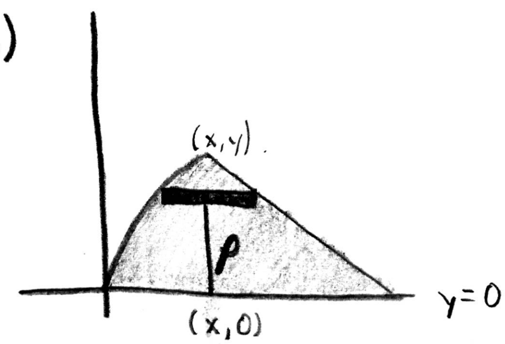
i) $\boxed{V = \pi \int_{y=0}^{y=1} \left[ \left(\frac{3}{2} - \frac{1}{2}y\right)^2 - \left(y^2\right)^2 \right] dy}$

$$= \pi \int_{y=0}^{y=1} \left(\frac{9}{4} - \frac{3}{2}y + \frac{1}{4}y^2 - y^4\right) dy$$

$$= \pi \left[ \frac{9}{4}y - \frac{3}{4}y^2 + \frac{1}{12}y^3 - \frac{1}{5}y^5 \right|_0^1$$

ii) $\boxed{V = \frac{83\pi}{60}}$

b)




$\rho = y - 0$

$h = \left(\frac{3}{2} - \frac{1}{2}y\right) - y^2$

• Rect are $\parallel$ axis of rot

$\Rightarrow$ Use shells

$$V = 2\pi \int_{y=0}^{y=1} \rho h\, dy$$

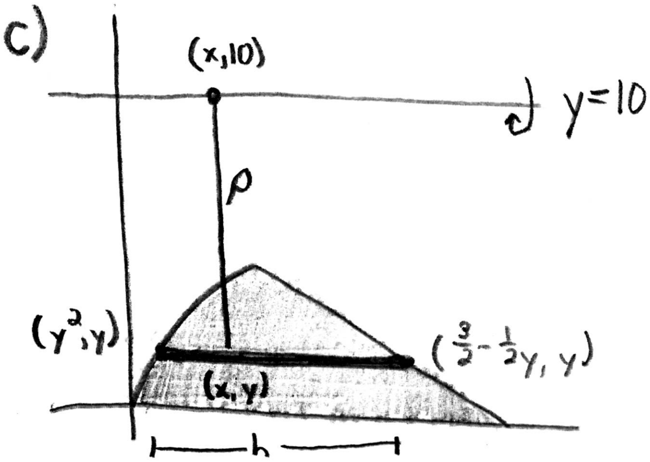
i) $\boxed{V = 2\pi \int_{y=0}^{y=1} y\left(\frac{3}{2} - \frac{1}{2}y - y^2\right) dy}$

$$= 2\pi \int_{y=0}^{y=1} \left(\frac{3}{2}y - \frac{1}{2}y^2 - y^3\right) dy$$

$$= 2\pi \left[ \frac{3}{4}y^2 - \frac{1}{6}y^3 - \frac{1}{4}y^4 \right|_0^1$$

ii) $\boxed{V = \frac{2\pi}{3}}$

c)




$\rho = 10 - y$

$h = (\frac{3}{2} - \frac{1}{2}y) - y^2$

- Rectangles are ∥ to axis of rot.

⇒ Use shell method.

$$V = 2\pi \int_{y=0}^{y=1} \rho h \, dy$$

i) $$\boxed{V = 2\pi \int_{y=0}^{y=1} (10-y)\left[\frac{3}{2} - \frac{1}{2}y - y^2\right] dy}$$

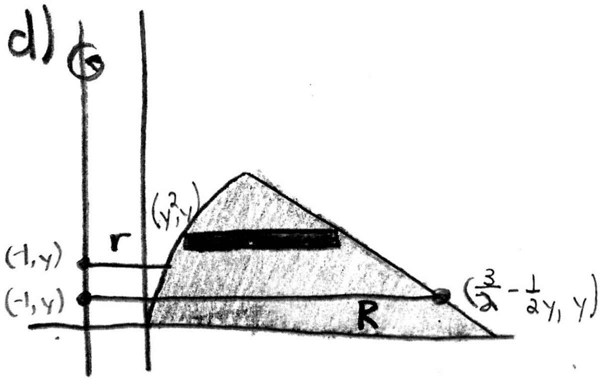
$$= 2\pi \int_0^1 \left[15 - 5y - 10y^2 - \frac{3}{2}y + \frac{1}{2}y^2 + y^3\right] dy$$

$$= 2\pi \int_0^1 \left(15 - \frac{13}{2}y - \frac{19}{2}y^2 + y^3\right) dy$$

$$= 2\pi \left[15y - \frac{13}{4}y^2 - \frac{19}{6}y^3 + \frac{1}{4}y^4\right|_0^1$$

ii) $$\boxed{V = \frac{89\pi}{2}}$$

d)




$x = -1$

$R = (\frac{3}{2} - \frac{1}{2}y) - (-1)$

$R = \frac{5}{2} - \frac{1}{2}y$

$r = y^2 - (-1)$

$= y^2 + 1$

- Rectangles are ⊥ to axis of rot.

⇒ Use washers

$$V = \pi \int_{y=0}^{y=1} (R^2 - r^2) \, dy$$

i) $$\boxed{V = \pi \int_{y=0}^{y=1} \left[\left(\frac{5}{2} - \frac{1}{2}y\right)^2 - \left(y^2 + 1\right)^2\right] dy}$$

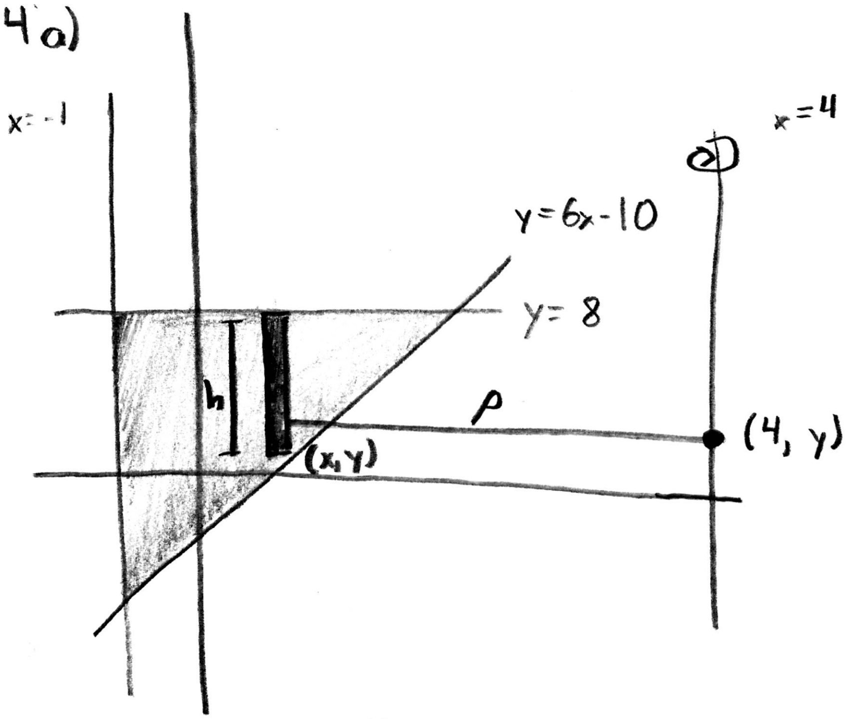
$$= \pi \int_0^1 \left[\frac{25}{4} - \frac{5}{2}y + \frac{1}{4}y^2 - y^4 - 2y^2 - 1\right] dy$$

$$= \pi \int_0^1 \left(\frac{19}{4} - \frac{5}{2}y - \frac{7}{4}y^2 - y^4\right) dy$$

$$= \pi \left[\frac{19}{4}y - \frac{5}{4}y^2 - \frac{7}{12}y^3 - \frac{1}{5}y^5\right|_0^1$$

ii) $$\boxed{V = \frac{193\pi}{60}}$$

4 a)




- We have y in terms of x, so int wrt x

$\Rightarrow$ Vertical Strips.

- Vert Strips are $\parallel$ axis of rot

$\rightarrow$ Shell Method.

$$V = 2\pi \int_{x=-1}^{x=3} (4-x)(18-6x)\,dx.$$

$$= 2\pi \int_{x=-1}^{x=3} (72 - 42x + 6x^2)\,dx$$

$$= 2\pi \left[ 72x - 21x^2 + 2x^3 \right|_{-1}^{3}$$

$$\boxed{V = 352\pi}$$

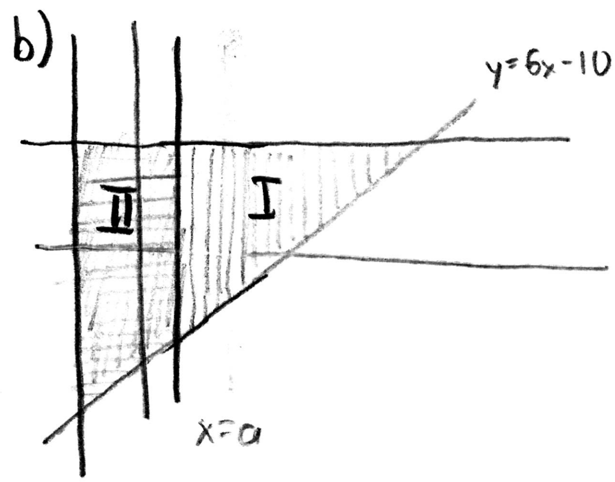
$\rho = 4 - x$

$h = 8 - (6x - 10)$

$h = 18 - 6x$

<u>Int pt</u>: $6x - 10 = 8 \Rightarrow \underline{x = 3}$

b)




We want to choose a so

$$V_I = V_{II} = \frac{1}{2} V$$

$\downarrow$

$$V_I = 2\pi \int_a^3 (4-x)(18-6x)\,dx = 352\pi$$

$$2\cancel{\pi} \left[ 72x - 21x^2 + 2x^3 \right|_a^3 = 352\cancel{\pi}$$

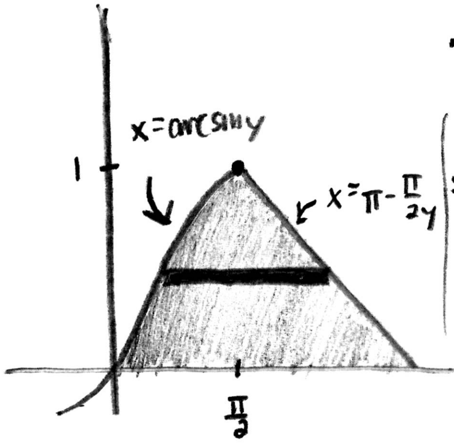
$\uparrow$ from a)

$$72(3) - 21(3)^2 + 2(3)^3 - 72a + 21a^2 - 2a^3 = 176$$

$$\underline{2a^3 - 21a^2 + 72a - 95 = 0}$$

Using a graphing utility (or Wolfram Alpha): $\boxed{a = -.095}$

5.




- The region is best set up using horizontal strips

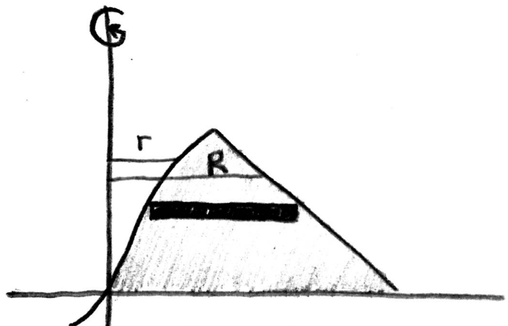
$\Rightarrow$ Integrate wrt $y$

(Note: We're only asked to set this up here! If we had to find the volume, we'd have to integrate wrt $x$ since we can't integrate $\arcsin x$ yet!).

- $y = \sin x \rightarrow x = \arcsin y$
- $y = 2 - \frac{2}{\pi}x \rightarrow \frac{2}{\pi}x = 2 - y$

$$\underline{x = \pi - \frac{\pi}{2}y}$$

a)




$R = (\pi - \frac{\pi}{2}y) - 0 = \pi - \frac{\pi}{2}y$

$r = \arcsin y - 0 = \arcsin y$

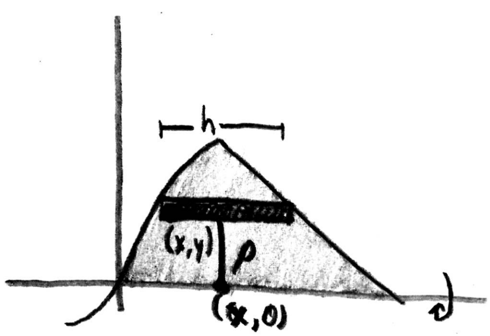
- Strips are $\perp$ the axis of rot.

$\Rightarrow$ Washer Method

$$V = \pi \int_{y=0}^{y=1} (R^2 - r^2)\, dy$$

$$\boxed{V = \pi \int_{y=0}^{y=1} \left[\left(\pi - \frac{\pi}{2}y\right)^2 - \arcsin^2 y\right] dy}$$

b)




$\rho = y$

$h = (\pi - \frac{\pi}{2}y) - \arcsin y$

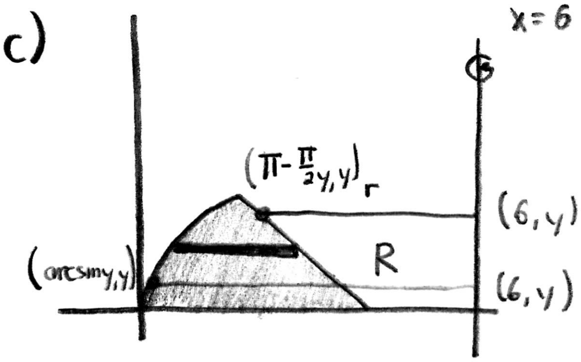
- Strips are $\parallel$ axis of rot.

$\Rightarrow$ Shells

$$V = \pi \int_{y=0}^{y=1} \rho h\, dy$$

$$\boxed{V = \pi \int_{y=0}^{y=1} y\left(\pi - \frac{\pi}{2}y - \arcsin y\right) dy}$$

c)




- Strips are ⊥ axis of rot
$\Rightarrow$ Washer

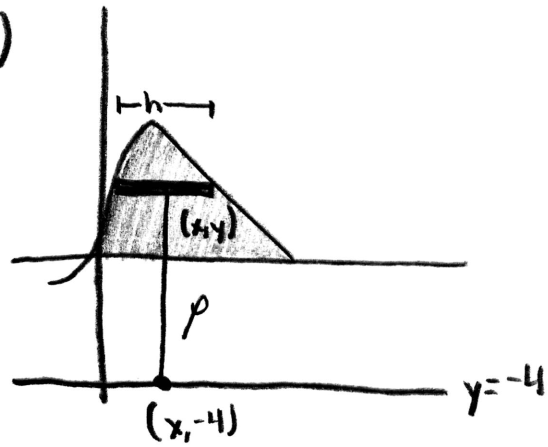
$$V = \pi \int_{y=0}^{y=1} (R^2 - r^2)\, dy$$

$$\boxed{V = \pi \int_{y=0}^{y=1} \left[ (6 - \arcsin y)^2 - \left(6 - \pi + \frac{\pi}{2} y\right)^2 \right]}$$

$R = 6 - \arcsin y$

$r = 6 - \left(\pi - \frac{\pi}{2} y\right)$

d)




- Strips are ∥ to the axis of rotation
$\Rightarrow$ Shells.

$$V = 2\pi \int_{y=0}^{y=1} \rho h \, dy$$

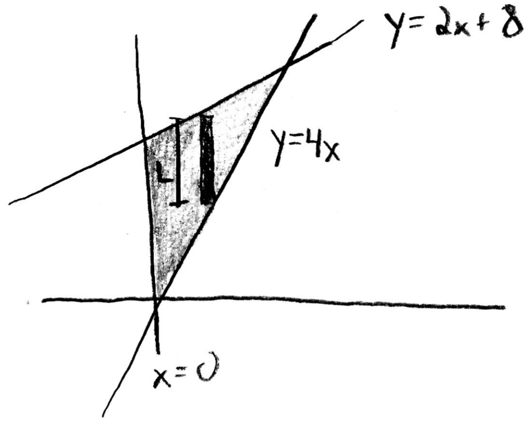
$$\boxed{V = 2\pi \int_{y=0}^{y=1} (y+4)\left(\pi - \frac{\pi}{2} y - \arcsin y\right) dy}$$

$\rho = y - (-4) = \underline{y+4}$

$h = \underline{\pi - \frac{\pi}{2} y - \arcsin y}$

# Worksheet #2 Addendum Solutions

5. The base of the solid is shown below:




To find the limits of integration, note the lower limit is clearly $x=0$ and the upper limit is the x-value where $y = 2x+8$ and $y=4x$ intersect

$$2x+8 = 4x$$
$$8 = 2x$$
$$\underline{x = 4}.$$

We now just need to evaluate $V = \int_{x=0}^{x=4} A(x)\,dx$ for the different types of cross-sections:

↑ integrate with respect to x because cross-sections are ⊥ x-axis!

a) Squares

· For a square, $A(x) = [L(x)]^2$. $L(x)$ is the vertical distance between $y = 2x+8$ and $y=4x$, so we find $L(x)$ by subtracting the bottom y-value from the top:

$$L(x) = y_{top} - y_{bottom}$$
$$= 2x+8-4x$$
$$\underline{L(x) = 8-2x.}$$

Hence, $V = \int_{x=0}^{x=4} (8-2x)^2\,dx$.

$$= \int_0^4 (64 - 32x + 4x^2)\,dx$$

$$= \left[64x - 16x^2 + \frac{4}{3}x^3\right|_0^4$$

$$\boxed{V = \frac{256}{3}}$$

b) For an equilateral triangle, $A(x) = \frac{\sqrt{3}}{4}[L(x)]^2$.

$$= \frac{\sqrt{3}}{4}(8-2x)^2$$

Hence, $V = \int_{x=0}^{x=4} \frac{\sqrt{3}}{4}(8-2x)^2\,dx$.

$$= \frac{\sqrt{3}}{4}\int_{x=0}^{x=4}(8-2x)^2\,dx.$$

This is the same integral as before!

$$= \frac{\sqrt{3}}{4}\left[\frac{256}{3}\right]$$

From a)

$$\boxed{V = \frac{64\sqrt{3}}{3}}$$

c) For a semicircle, $A(x) = \frac{1}{2}\pi[r(x)]^2$. Here, $L(x)$ is the diameter of a semicircle, so:

$$r(x) = \frac{1}{2}L(x) = \frac{1}{2}(8-2x) = 4-x.$$

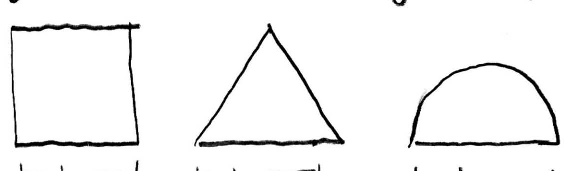
and $V = \int_{x=0}^{x=4} \frac{1}{2}\pi(4-x)^2\,dx$.

$$= \int_{x=0}^{x=4} \frac{1}{2}\pi\,(16-8x+x^2)\,dx$$

$$= \frac{1}{2}\pi\left[16x - 4x^2 + \frac{4}{3}x^3\right|_0^4$$

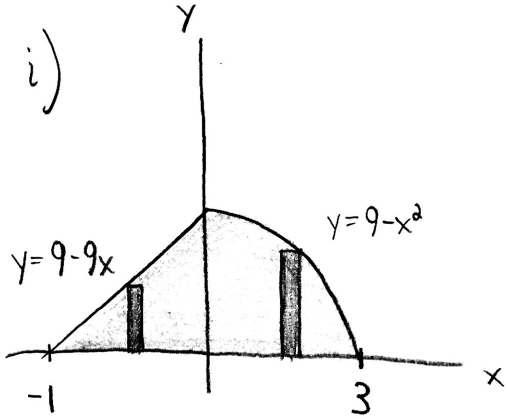
$$\boxed{V = \frac{32\pi}{3}}$$

d). Intuitively, the solid whose cross-sections are squares should have the largest volume. Since, for each fixed x value, the square has the largest cross-sectional area!

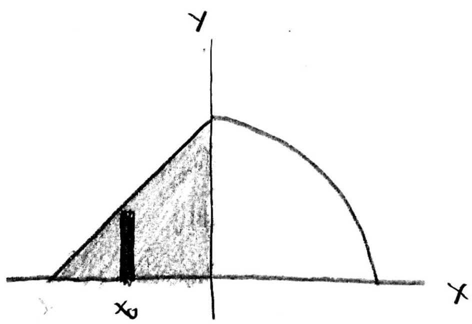


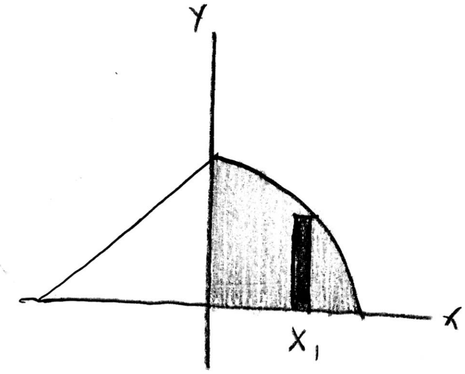

Computationally, we have verified this is the case as well!

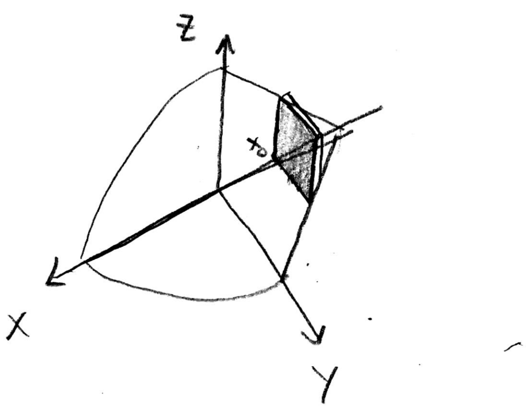
7a) i)




Note that the length of the side of the square is found using a different function in different parts of the region!



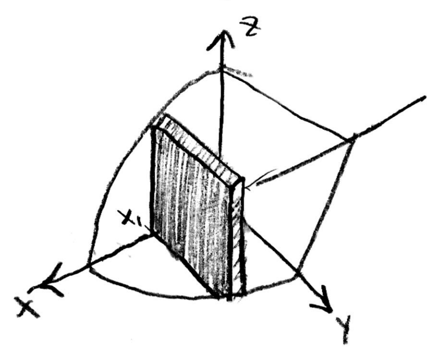
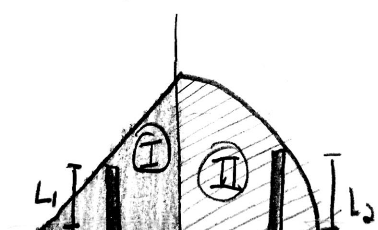




Fig 1: The square corresponding to the sidelength at $x_0$ has its length determined by the curve $y = 9+9x$

Fig 2: The square corresponding to the side length at $x_1$ has its length determined by the curve $y = 9-x^2$.

We thus split the region into two parts about $x=0$ (where $y = 9+9x$ and $y = 9-x^2$ intersect).




Clearly, $L_1(x) = (9-9x) - 0 = 9-9x$

$L_2(x) = (9-9x^2) - 0 = 9-9x^2$

and $A_1(x) = [L_1(x)]^2$, $A_2(x) = [L_2(x)]^2$

Thus, $V = V_{I} + V_{II} = \int_{x=-1}^{x=0} A_1(x)\,dx + \int_{x=0}^{x=3} A_2(x)\,dx$

$$= \int_{x=-1}^{x=0} (9+9x)^2\,dx + \int_{x=0}^{x=3} (9-9x^2)^2\,dx.$$

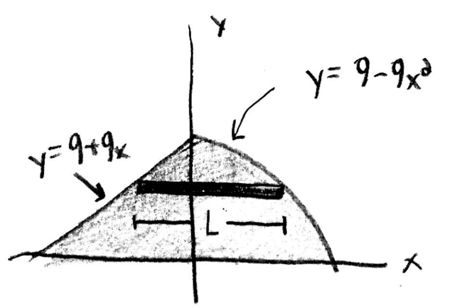
$$= \int_{x=-1}^{x=0} (81+162x+81x^2)\,dx + \int_{x=0}^{x=3} (81-162x^2+81x^4)\,dx$$

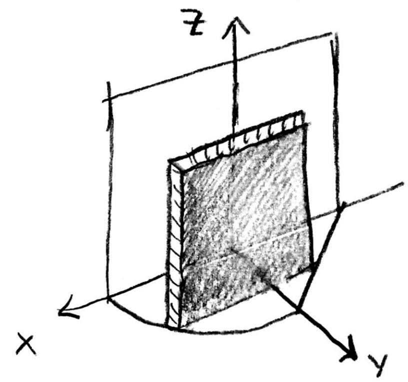
$$= \left[81x + 81x^2 + 27x^3\right|_{-1}^{0} + \left[81x - 54x^3 + \frac{81}{5}x^5\right|_{0}^{3}$$

$$= 27 + \frac{13608}{5}$$

$$\boxed{V = \frac{13743}{5}}$$

ii) The length $L(y)$ of the side of the square is given by the same curves here!




Since the cross-sections are perpendicular to the y-axis, we integrate with respect to y. We thus need L as a function of y (i.e. $L = L(y)$) and y-limits of integration.

- $L(y)$ is the horizontal distance between the parabola and the line:

Parabola: $y = 9-9x^2$
$9x^2 = 9-y$
$x^2 = 1-\frac{1}{9}y$

$x = \sqrt{1-\frac{1}{9}y}$

Line: $y = 9+9x$
$9x = 9-y$
$x = 1-\frac{1}{9}y$

Thus, $L(y) = x_{right} - x_{left}$

$$\underline{L(y) = \sqrt{1-\tfrac{1}{9}y} - \left(1-\tfrac{1}{9}y\right)}$$

and $A(y) = [L(y)]^2 = \left[\sqrt{1-\tfrac{1}{9}y} - \left(1-\tfrac{1}{9}y\right)\right]^2$

$$\underline{A(y) = \left(1-\tfrac{1}{9}y\right) - 2\left(1-\tfrac{1}{9}y\right)^{3/2} + \left(1-\tfrac{1}{9}y\right)^2}$$

So, $V = \int_{y=0}^{y=9} A(y)\,dy$

$$V = \int_{y=0}^{y=9} \left[\left(1-\tfrac{1}{9}y\right) - 2\left(1-\tfrac{1}{9}y\right)^{3/2} + \left(1-\tfrac{1}{9}y\right)^2\right] dy.$$

Let $u = 1-\tfrac{1}{9}y$ — Limits: $\underline{y=0} \to u = 1-\tfrac{1}{9}(0) = 1$

$du = -\tfrac{1}{9}\,dy$ — $\underline{y=9} \to u = 1-\tfrac{1}{9}(9) = 0$

$\underline{dy = -9\,du}$

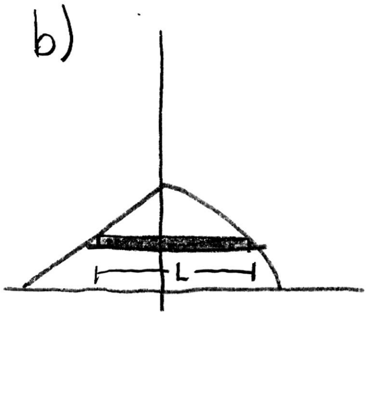
So $V = \int_{u=1}^{u=0} \left(u - 2u^{3/2} + u^2\right)(-9\,du)$

$= \int_{u=1}^{u=0} \left(-9u + 18u^{3/2} - 9u^2\right) du$

$= -\tfrac{9}{2}u^2 + \tfrac{36}{5}u^{5/2} - 3u^3 \Big|_1^0$

$$\boxed{V = \tfrac{3}{10}}$$

b)




For semicircles, $A(y) = \tfrac{1}{2}\pi[r(y)]^2$.

Since $L(y)$ is the diameter,

$r(y) = \tfrac{1}{2}L(y) = \tfrac{1}{2}\left[\sqrt{1-\tfrac{1}{9}y} - \left(1-\tfrac{1}{9}y\right)\right].$

So: $$\boxed{V = \int_{y=0}^{y=9} \tfrac{1}{2}\pi \cdot \left(\tfrac{1}{2}\left[\sqrt{1-\tfrac{1}{9}y} - \left(1-\tfrac{1}{9}y\right)\right]\right)^2 dy.}$$