Worksheet #5 Answers

Make sure you look at the problems with "*" next to them! There is important information discussed in those solutions!!!

I.

1. $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$

2. $\frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+9}$

3. $\frac{Ax+B}{x^2+5} + \frac{C}{x+2} + \frac{D}{x-2}$

4.* $\frac{A}{x-4} + \frac{Bx+C}{x^2+8x+90} + \frac{D}{(x-3)^2} + \frac{E}{x-3}$

5. $\frac{A}{x-1} + \frac{B}{(2x-3)^3} + \frac{C}{(2x-3)^2} + \frac{D}{2x-3} + \frac{Ex+F}{4x^2+1} + \frac{Gx+H}{(x^2+36)^2} + \frac{Ix+J}{x^2+36}$

6. $\frac{A}{x+4} + \frac{B}{x-4} + \frac{Cx+D}{x^2+4}$

7. $\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+5x+9}$

8.* $\frac{1}{x+2}$

9. $\frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{(x-2)^2} + \frac{E}{x-2}$

10. $\frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+2} + \frac{Dx+E}{(x^2+3)^2} + \frac{Fx+G}{x^2+3}$.

II.

11. $\int \frac{x^2+2x+9}{x^3+9x}\,dx = \int\left(\frac{1}{x} + \frac{2}{x^2+9}\right)dx = \ln|x| + \frac{2}{3}\arctan\frac{x}{3} + C$

12. $\int \frac{x^4-x^2+3x+1}{(x-1)(x^2+1)^2}\,dx = \int\left[\frac{1}{x-1} - \frac{3}{(x^2+1)^2}\right]dx = \ln|x-1| - \frac{3}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C$

13.* $\int \frac{3x^2}{x^2-4}\,dx = \int\left(3 - \frac{3}{x+2} + \frac{3}{x-2}\right)dx = 3x + 3\ln|x+2| - 3\ln|x-2| + C$

14. $\int \frac{74x^2+3}{25x^4+x^2}\,dx = \int\left(\frac{3}{x^2} - \frac{1}{25x^2+1}\right)dx = -\frac{3}{x} - \frac{1}{5}\arctan 5x + C$

15. $\int \frac{2x^4+x^3+4x^2-3x+2}{x(x^2+1)^2}\,dx = \int\left(\frac{2}{x} - \frac{4}{(x^2+1)^2} + \frac{1}{x^2+1}\right)dx = 2\ln|x| - \arctan x - \frac{2x}{1+x^2} + C$

16-18: Note how changing the expression slightly affects the problem!

16. $-\frac{1}{8}\ln|x+4| + \frac{1}{8}\ln|x-4| + C$ -OR- $\frac{1}{8}\ln\left|\frac{x-4}{x+4}\right| + C$ ← Partial Fractions or Trig Sub!

17. $\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C$ ← Trig sub!

18. $\sqrt{x^2-16} + C$

19. $5\ln|x+1| + \frac{2}{x+2} + \ln|x+2| + C$

20. $\frac{1}{2}\ln|x| + \frac{1}{18}\arctan\frac{2x}{3} + C$

Worksheet #5 Solutions

- We want to factor the denominator as a product of:
  1) Linear factors
  2) Powers of linear factors
  3) Irreducible quadratics
  4) Powers of irreducible quadratics.

- Any terms in the decomposition that are powers of linear factors will have constants in the numerator
- Any terms in the decomposition that are powers of quadratic factors will have linear polynomials in the numerator.

I. 1. $$\frac{1+3x+4x^2}{x^3+4x^2+3x} = \frac{1+3x+4x^2}{x(x^2+4x+3)} = \frac{1+3x+4x^2}{x(x+1)(x+3)}$$ ← All linear factors

$$= \boxed{\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}}$$

2. $$\frac{1}{x^4+9x^2} = \frac{1}{x^2(x^2+9)} = \boxed{\frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+9}}$$

Irreducible quadratic

Repeated linear factor!

3. $$\frac{3x^3-7x}{(x^2+5)(x^2-4)} = \frac{3x^3-7x}{(x^2+5)(x+2)(x-2)} = \boxed{\frac{Ax+B}{x^2+5} + \frac{C}{x+2} + \frac{D}{x-2}}$$

Not irreducible!

4. $$\frac{5x^3+2x^2}{(x-4)(x^2+8x+90)(x-3)^2}$$

* Note: $x^2+8x+90$ is irreducible! Why? Recall that if a quadratic polynomial has roots $x=r_1$ and $x=r_2$, then

the polynomial has the form:

$$p(x) = a(x-r_1)(x-r_2).$$

For example: $p(x) = 2x^2 - 3x$ has roots $x=0$ and $x=\frac{3}{2}$. One readily checks: $p(x) = 2(x-0)(x-\frac{3}{2})$.

Thus, a quadratic cannot be factored over the reals if and only if it has no real roots. We can use the quadratic formula to determine this!

For $x^2+8x+90$: roots: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-8 \pm \sqrt{64-4(1)(90)}}{2}$

The expression under $\sqrt{}$ (called the discriminant) is negative! There are thus no real roots! Hence, $x^2+8x+90$ has no real roots and is thus irreducible!

So: $$\frac{5x^3+2x^2}{(x-4)(x^2+8x+90)(x-3)^2} = \boxed{\frac{A}{x-4} + \frac{Bx+C}{x^2+8x+90} + \frac{D}{(x-3)^2} + \frac{E}{x-3}}$$

Irr. quadratic

power of linear factor

5. $$\frac{x^7-6x^5+1}{(x-1)(2x-3)^3(4x^2+1)(x^2+36)^2} = \boxed{\frac{A}{x-1} + \underbrace{\frac{B}{(2x-3)^3} + \frac{C}{(2x-3)^2} + \frac{D}{2x-3}} + \frac{Ex+F}{4x^2+1} + \underbrace{\frac{Gx+H}{(x^2+36)^2} + \frac{Ix+J}{x^2+36}}}$$

power of linear

power of irr quadratic

6. $$\frac{3x^2+1}{x^4-16} = \frac{3x^2+1}{(x^2-4)(x^2+4)} = \frac{3x^2+1}{(x+4)(x-4)(x^2+4)}$$

$$= \boxed{\frac{A}{x+4} + \frac{B}{x-4} + \frac{Cx+D}{x^2+4}}$$

7. $$\frac{6x+7}{(x^2-2x+1)(x^2+5x+9)}$$

We should check if the factors are irreducible.

* **Note:** $x^2-2x+1 = (x-1)^2$

For x^2+5x+9: if we try to find roots (cf description preceeding solution to #4):

$$\text{roots: } x = \frac{-5 \pm \sqrt{25-4(1)(9)}}{2}$$

The discriminant (the expr under the radical) is negative! So, there are no real roots! This is an irreducible quadratic!

Hence:
$$\frac{6x+7}{(x^2-2x+1)(x^2+5x+9)} = \frac{6x+7}{(x-1)^2(x^2+5x+9)} = \boxed{\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+5x+9}}$$

Repeated linear

8. $$\frac{x^2+x}{x^3+3x^2+2x} = \frac{\cancel{x}(x+1)}{\cancel{x}(x^2+3x+2)} = \frac{\cancel{x+1}}{(x+2)\cancel{(x+1)}} = \boxed{\frac{1}{x+2}}$$

* **Note:** Normally, we don't care about the numerator, but if things cancel nicely, we should do the algebra!

If we didn't; the result is consistent, as seen below.

$$\frac{x^2+x}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$x^2+x = A(x+1)(x+2) + Bx(x+2) + Cx(x+1).$$

$x=0$: $0 = 2A \Rightarrow A=0$

$x=-1$: $0 = -B \Rightarrow B=0$

$x=-2$: $2 = +2C \Rightarrow C=1$

So: $$\frac{x^2+x}{x(x+1)(x+2)} = \frac{0}{x} + \frac{0}{x+1} + \frac{1}{x+2}.$$ This is consistent!

→ **LESSON: Do algebra first!!!** ☆

9. $$\frac{2x^2+1}{x^5-4x^4+4x^3} = \frac{2x^2+1}{x^3(x^2-4x+4)} = \frac{2x^2+1}{x^3(x-2)^2}$$

$$= \boxed{\frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{(x-2)^2}+\frac{E}{x-2}}$$

10. $$\frac{1-8x}{x^2(x+2)(x^2+3)^2} = \boxed{\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}+\frac{Dx+E}{(x^2+3)^2}+\frac{Fx+G}{x^2+3}}$$

II.

11. $$\int \frac{x^2+2x+9}{x^3+9x}\,dx$$

After checking for the prior techniques, nothing will work! (algebra, u-sub, int by parts).! But we can factor the denominator:

$$\frac{x^2+2x+9}{x^3+9x} = \frac{x^2+2x+9}{x(x^2+9)} = \frac{A}{x}+\frac{Bx+C}{x^2+9}$$

$$x^2+2x+9 = A(x^2+9)+x(Bx+C)$$

- No convenient values for x but x=0 so try writing RHS as a polynomial

$$= Ax^2+9A+Bx^2+Cx$$

$$x^2+2x+9 = (A+B)x^2+Cx+9A$$

- Equate coefficients:

$\underline{x^2}$: $\cancel{1} = A+B$ → $1 = 1+B$, $\underline{B=0}$

$\underline{x}$: $2 = C \to \underline{C=2}$

$\underline{\text{const}}$: $9 = 9A \to \underline{A=1}$

So: $$\frac{x^2+2x+9}{x^3+9x} = \frac{A}{x}+\frac{Bx+C}{x^2+9} = \frac{1}{x}+\frac{2}{x^2+9}$$

and $$\int \frac{x^2+2x+9}{x^3+9x}\,dx = \int\left(\frac{1}{x}+\frac{2}{x^2+9}\right)dx$$

$$= \boxed{\ln|x| + \frac{2}{3}\arctan\frac{x}{3} + C}$$

Recall: $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a}+C$

12. $\int \frac{x^4 - x^2 + 3x + 1}{(x-1)(x^2+1)^2}\,dx$

$$\frac{x^4 - x^2 + 3x + 1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}$$

$$x^4 - x^2 + 3x + 1 = A(x^2+1)^2 + (Bx+C)(x-1) + (Dx+E)(x-1)(x^2+1)$$

Let $x=1$: $\quad 4 = 4A \Rightarrow \underline{A=1}$:

So: $\quad x^4 - x^2 + 3x + 1 = (x^2+1)^2 + (Bx+C)(x-1) + (Dx+E)(x-1)(x^2+1)$

Let's try the equating coefficients method: After a bit of FOILing:

$$x^4 - x^2 + 3x + 1 = (x^4 + 2x^2 + 1) + (Bx^2 - Bx + Cx - C) + (Dx^4 - Dx^3 + Ex^3 + Dx^2 - Ex^2 - Dx + Ex - E).$$

$$-3x^2 + 3x = Dx^4 + (-D+E)x^3 + ((B+D-E)x^2 + (-B+C-D+E)x + (-C-E).$$

Comparing coefficients:

$\underline{x^4}$: $\quad 0 = D$

$\underline{x^3}$: $\quad 0 = -D+E \Rightarrow \underline{E=0}$

$\underline{x^2}$: $\quad -3 = B+D-E \Rightarrow \underline{B=-3}$

$\underline{x}$: $\quad 3 = -B+C-D+E \Rightarrow C=0$

$3 = -(-3)+C-0+0$

$\underline{C=0}$

So: $\quad \frac{x^4 - x^2 + 3x + 1}{(x-1)(x^2+1)^2} = \frac{1}{x-1} + \frac{-3}{(x^2+1)^2}$

and $\displaystyle\int \frac{x^4 - x^2 + 3x + 1}{(x-1)(x^2+1)^2}\, dx = \int \left[\frac{1}{x-1} - \frac{3}{(x^2+1)^2}\right] dx$

$$= \frac{1}{x-1} - 3\int \frac{1}{(x^2+1)^2}\, dx \qquad (1)$$

We must compute $\displaystyle\int \frac{1}{(x^2+1)^2}\, dx$. To compute $\displaystyle\int \frac{1}{(u^2+a^2)^n}\, du$ do a trig sub:

$$x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$

$$\int \frac{1}{(x^2+1)^2}\, dx = \int \frac{1}{(\tan^2\theta+1)^2} \sec^2\theta\, d\theta$$

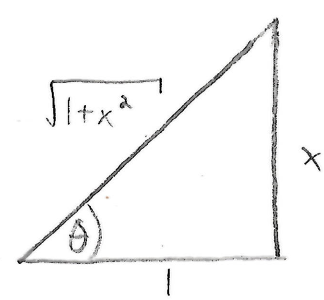
$$= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \int \cos^2\theta\, d\theta \longleftarrow \quad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$



$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$\frac{x}{1} = \tan\theta$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\,\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + C$$

Substituting into (1):

$$\boxed{\int \frac{x^4 - x^2 + 3x + 1}{(x-1)(x^2+1)^2}\, dx = \ln|x-1| - 3\cdot\left(\frac{1}{2}\arctan x + \frac{x}{1+x^2}\right) + C}$$

13. $\displaystyle\int \frac{3x^2}{x^2-4}\,dx$

The degree of the numerator is the same as the denominator! Do long division first!

$$\begin{array}{r} 3 - \frac{12}{x^2-4} \\ x^2-4 \;\big)\; 3x^2 + 0x + 0 \\ -(3x^2 - 12) \\ \hline -12 \end{array}$$

* If the degree of the numerator is equal to, or exceeds, the degree of the denominator, do long division first!

So $\displaystyle\int \frac{3x^2}{x^2-4}\,dx = \int \left(3 - \frac{12}{x^2-4}\right)dx = \underline{3x - \int \frac{12}{x^2-4}\,dx}$ (2)

We will compute $\displaystyle\int \frac{12}{x^2-4}\,dx$ and sub back into (2).

$$\frac{12}{x^2-4} = \frac{12}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$12 = A(x-2) + B(x+2).$$

$\underline{x=2:}$ $12 = 4B \Rightarrow \underline{B=3}$

$\underline{x=-2:}$ $12 = -4A \Rightarrow \underline{A=-3}$

So; $\displaystyle\int \frac{12}{x^2-4}\,dx = \int -\frac{3}{x+2} + \frac{3}{x-2}\,dx$

$$= \underline{-3\ln|x+2| + 3\ln|x-2| + C}$$

Substituting into (2):

$$\int \frac{3x^2}{x^2-4}\,dx = 3x - \int \frac{12}{x^2-4}\,dx = 3x - \left[-3\ln|x+2| + 3\ln|x-2|\right] + C$$

$$= \boxed{3x + 3\ln|x+2| - 3\ln|x-2| + C}$$

14. $\displaystyle\int \frac{74x^2+3}{25x^4+x^2}\,dx$

$$\frac{74x^2+3}{25x^4+x^2} = \frac{74x^2+3}{x^2(25x^2+1)} = \frac{A}{x^2}+\frac{B}{x}+\frac{Cx+D}{25x^2+1}$$

$$74x^2+3 = A(25x^2+1)+Bx(25x^2+1)+x^2(Cx+D).$$

$\underline{x=0:}$ $\quad \underline{3 = A}$

So: $\quad 74x^2+3 = 3(25x^2+1)+Bx(25x^2+1)+x^2(Cx+D)$

Let's equate coefficients since there are no more convenient x values

$$74x^2+\cancel{3} = 75x^2+\cancel{3} + 25Bx^3+Bx+Cx^3+Dx^2$$

$$-x^2 = (25B+C)x^3+Dx^2+Bx$$

Equate $\underline{x^3}$: $\quad 0 = 25B+C \quad 25(0)+C=0$

$\underline{C=0}$.

$\underline{x^2}$: $\quad \underline{-1 = D}$

$\underline{x}$: $\quad \underline{0 = B}$

need x^2+a^2 on bottom → factor out 25!

So: $\displaystyle\int \frac{74x^2+3}{25x^4+x^2}\,dx = \int\left(\frac{3}{x^2}-\frac{1}{25x^2+1}\right)dx$

$$= \int 3x^{-2}\,dx - \frac{1}{25}\int \frac{1}{x^2+\frac{1}{25}}\,dx$$

$a = 1/5$!

$$= -3x^{-1} - \frac{1}{25}\cdot 5\arctan 5x + C$$

$$= \boxed{-\frac{3}{x} - \frac{1}{5}\arctan 5x + C}$$

15. $\displaystyle\int \frac{2x^4+x^3+4x^2-3x+2}{x(x^2+1)^2}\,dx$

$$\frac{2x^4+x^3+4x^2-3x+2}{x(x^2+1)^2} = \frac{A}{x}+\frac{Bx+C}{(x^2+1)^2}+\frac{Dx+E}{x^2+1}$$

$$2x^4+x^3+4x^2-3x+2 = A(x^2+1)^2 + x(Bx+C) + x(x^2+1)(Dx+E)$$

$\underline{x=0:}$ $\quad \underline{2 = A}$

Write out RHS as a polynomial:

$$2x^4+x^3+4x^2-3x+2 = 2(x^2+1)^2 + Bx^2+Cx+(x^3+x)(Dx+E)$$

$$\cancel{2x^4}+x^3+\cancel{4x^2}-3x+\cancel{2} = \cancel{2x^4}+\cancel{4x^2}+\cancel{2}+Dx^4+Ex^3+(B+D)x^2+(C+E)x$$

Equate coeff:

$\underline{x^4:}$ $\quad \underline{0 = D}$

$\underline{x^3:}$ $\quad \underline{1 = E}$

$\underline{x^2:}$ $\quad 0 = B+D \Rightarrow \underline{B=0}$

$\underline{x:}$ $\quad -3 = C+E \Rightarrow \underline{C=-4}$

So:
$$\int \frac{2x^4+x^3+4x^2-3x+2}{x(x^2+1)^2}\,dx = \int\left(\frac{2}{x} - \frac{4}{(x^2+1)^2} + \frac{1}{x^2+1}\right)dx$$

↓ computed in #12!

$$= 2\ln|x| - 4\cdot\frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + \arctan x + C$$

$$= \boxed{2\ln|x| - \arctan x - \frac{2x}{1+x^2} + C}$$

16-18: These are meant to show how slight changes to the expression leads to different techniques!

16. $\displaystyle\int \frac{1}{x^2-16}\,dx$ ← Trig Sub or partial fractions! I'll do PF.

$$\frac{1}{x^2-16} = \frac{1}{(x+4)(x-4)} = \frac{A}{x+4} + \frac{B}{x-4}$$

$$1 = A(x-4) + B(x+4).$$

$\underline{x=4:}$ $\quad 1 = 8B \Rightarrow \underline{B=\frac{1}{8}}$

$\underline{x=-4:}$ $\quad 1 = -8A \Rightarrow \underline{A=-\frac{1}{8}}$

So: $\int \frac{1}{x^2-16}\,dx = \int \left(-\frac{1}{8}\frac{1}{x+4} + \frac{1}{8}\frac{1}{x-4}\right) dx$

$$= \boxed{-\frac{1}{8}\ln|x+4| + \frac{1}{8}\ln|x-4| + C}$$

(or $\boxed{\frac{1}{8}\ln\left|\frac{x-4}{x+4}\right| + C}$)

17. $\int \frac{1}{\sqrt{x^2-16}}\,dx$ ← "$\sqrt{\ }$" suggests trig sub instead of PF.

PF works when there are no $\sqrt{\ }$!

- We have u^2-a^2: $u^2=x^2 \to u=x$
  $a^2=16 \to a=4$.
- "$-$" in front of const ⇒ $u = a\sec\theta$
  $\Rightarrow x = 4\sec\theta$
  $dx = 4\sec\theta\tan\theta\,d\theta$

So: $\int \frac{1}{\sqrt{x^2-16}}\,dx = \int \frac{1}{\sqrt{16\sec^2\theta-16}} \cdot 4\sec\theta\tan\theta\,d\theta$

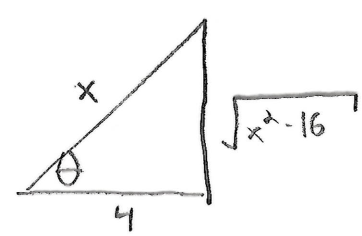
$$= \int \frac{1}{\sqrt{16\tan^2\theta}}\,4\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{\cancel{4}\cancel{\tan\theta}} \cdot \cancel{4}\sec\theta\cancel{\tan\theta}\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C}$$



$\cancel{x=4\sec\theta}$

$\frac{x}{4} = \sec\theta$

18. $\int \frac{x}{\sqrt{x^2-16}}\,dx$ ← u-sub or trig sub.

This is best taken care of with a u-sub! You should <u>ALWAYS</u> check this before trying trig sub or PF!

$u = x^2 - 16$

$du = 2x\,dx \quad \rightarrow \quad \int \frac{x}{\sqrt{x^2-16}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\,\frac{du}{2\cancel{x}}$

$\frac{du}{2x} = dx$

$= \int \frac{1}{2} u^{-1/2}\,du$

$= \frac{1}{\cancel{2}} \cdot \cancel{2} u^{1/2} + C$

$= \boxed{\sqrt{x^2-16} + C}$

19. $\int \frac{6x^2+21x+20}{(x+1)(x+2)^2}\,dx$

Only option is PF!

$$\frac{6x^2+21x+20}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{x+2}$$

$$6x^2+21x+20 = (x+2)^2 A + B(x+1) + C(x+1)(x+2)$$

$\underline{x=-2}$: $\quad 2 = -B \Rightarrow \underline{B=-2}$

$\underline{x=-1}$: $\quad \underline{5 = A}$

You could also do the polynomial coeff trick, but this is easier here!

Picking $x=0$ (or another value to find C):

$20 = 4A + B + 2C$

$20 = 4(5) + (-2) + 2C$

$\underline{C=1}$.

So: $\int \frac{6x^2+21x+20}{(x+1)(x+2)^2}\,dx = \int \left[\frac{5}{x+1} - \frac{2}{(x+2)^2} + \frac{1}{x+2}\right] dx$

$= \int \left(5 \cdot \frac{1}{x+1} - 2(x+2)^{-2} + \frac{1}{x+2}\right) dx$

$= 5\ln|x+1| + 2(x+2)^{-1} + \ln|x+2| + C$

$= \boxed{5\ln|x+1| + \frac{2}{x+2} + \ln|x+2| + C}$

20. $\int \frac{12x^2+2x+27}{6x(4x^2+9)}\,dx$

$$\frac{12x^2+2x+27}{6x(4x^2+9)} = \frac{A}{6x} + \frac{Bx+C}{4x^2+9}$$

$$12x^2+2x+27 = (4x^2+9)A + 6x\,(Bx+C)$$

$\underline{x=0:}$ $\quad 27 = 9A \Rightarrow \underline{A=3}$

$$12x^2+2x+27 = 3(4x^2+9) + 6x(Bx+C)$$

Write RHS as a polynomial and compare coeff.

$$\cancel{12x^2} + 2x + \cancel{27} = \cancel{12x^2} + \cancel{27} + 6Bx^2 + 6Cx$$

$\underline{\text{Equating:}}$ $\underline{x^2}$: $0 = 6B \Rightarrow \underline{B=0}$

$\underline{x}$: $2 = 6C \Rightarrow \underline{C = \tfrac{1}{3}}$

factor out $x=4$ to get $\int \frac{1}{x^2+a^2}$

So: $\int \frac{12x^2+2x+27}{6x(4x^2+9)}\,dx = \int \left(\frac{3}{6x} + \frac{1}{3}\,\frac{1}{4x^2+9}\right) dx$

$$= \int \left(\frac{1}{2}\,\frac{1}{x} + \frac{1}{3}\,\frac{1}{4}\,\frac{1}{x^2+\frac{9}{4}}\right) dx$$

$a = 3/2$

$$= \frac{1}{2}\ln|x| + \frac{1}{12}\cdot\frac{2}{3}\arctan\frac{2x}{3} + C$$

$$= \boxed{\frac{1}{2}\ln|x| + \frac{1}{18}\arctan\frac{2x}{3} + C}$$

* Recall: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$!