Worksheet #6 Answers

1. A, D, E, F.

2. C, D, E, F.

3. a) $\ln\frac{2}{9} - \pi$
   b) Diverges
   c) Diverges
   d) $-2\pi - \ln\frac{4}{26} + 4\arctan 5$

4. a) $\ln\frac{50}{27} - \frac{5}{3}\arctan\frac{1}{3}$
   b) Diverges
   c) Diverges
   d) Diverges.

5. Converges to $\ln 7$.

6. Diverges

7. Converges to $5\sqrt{5}$.

8. Converges to 3.

9. Converges to $\ln 10$

10. Diverges.

11. a) False
    b) True

12. <See solutions>

13. <see solutions>.

14. a) $-7, -2, 15, 44, 85, 138, \ldots$
    b) $1, 0, -1, 0, 1, 0, \ldots$
    c) $-1, 3/2, 7/3, 11/4, 3, 19/6, \ldots$
    d) $3, 10, 31, 94, 283, 850, \ldots$
    e) Cannot be determined without knowing a_1, a_2!
    f) $1, 1, 2, 3, 5, 8, \ldots$

15 a) 0

b) 1

c) $-\frac{1}{2}$

d) Diverges

e) $3/2$

f) 1

16 a) Converges to 21

b) Cannot be determined

c) Diverges

d) Converges to $\sqrt{7}$.

e) Converges to e^4

f) Diverges.

17 a) True

b) False; let $a_n = 1$, $c_n = \sin n$

c) False; let $a_n = 1$, $d_n = n$.

d) False; let $c_n = d_n = n$

e) False; let $a_n = b_n = 1$, $c_n = \sin n$, $d_n = \frac{1}{\sin n}$

f) True

g) False; let $c_n = n$

h) False; let $c_n = n$.

Worksheet #6 Solutions

1. A: Improper: $\frac{1}{x}$ is not cts. at $x=0$, which is a limit of integration.

   B. Proper: $\frac{1}{x}$ is not cts at $x=0$, but the limits of int are 2 to 3.

   C. Proper: $\frac{1}{x^2-6x} = \frac{1}{x(x-6)}$ is discont. at $x=0,6$. but neither are in the interval of integration.

   D. Improper: $\frac{1}{x^2+1}$ is cts, but ∞ is a limit of int.

   E. Improper: $\frac{1}{x^2-4x+3} = \frac{1}{(x-1)(x-3)}$ is discont. at $x=1,3$, which lie in the interval of integration.

   F. Improper: $-\infty$ is a limit

   G. Proper: $\frac{1}{x^2-5x-6} = \frac{1}{(x-6)(x+1)}$ is discont at $x=-1,6$ which are not in the interval of integration

   H. Proper: e^{-5x} is cts and $[0,2]$ is closed.

2. A. Proper: $\sin\frac{1}{x}$ is discont only at $x=0$

   B. Proper: $|x|$ is cts. and $[-1,1]$ is a closed interval.

   C. Improper: $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)}$ is discont at $x=0$, which is a limit of integration.

   D. Improper: ∞ is a limit of integration.

   E. Improper: $\frac{6x-1}{x^3-x} = \frac{6x-1}{x(x+1)(x-1)}$ is discont at $x=-1,0,1$, and $x=0,1$ are in the interval of integration

   F. Improper: ∞ is a limit of integration

   G. Proper: e^{-x^2} is cts. and $[0,1]$ is closed

   H. Proper: $\ln(5x-4)$ is discont when $5x-4=0$ or $x=\frac{4}{5}$. This does not lie in $[1,6]$

3. First, note $f(x) := \frac{2x+14}{(x-3)(x^2+1)}$ is discont at $x=3$. and also:

$$\int \frac{2x+14}{(x-3)(x^2+1)}\,dx = \int \left(\frac{2}{x-3} - \frac{2x+4}{x^2+1}\right) dx$$

$$= \int \left(\frac{2}{x-3} - \frac{2x}{x^2+1} - \frac{4}{x^2+1}\right) dx$$

$$= 2\ln|x-3| - \ln|x^2+1| - 4\arctan x + C$$

$$= \ln|x-3|^2 - \ln|x^2+1| - 4\arctan x + C$$

$$= \ln\left|\frac{(x-3)^2}{x^2+1}\right| - 4\arctan x + C.$$

a) On $[0,1]$, $f(x)$ is cts, so:

$$\int_0^1 \frac{2x+4}{(x-3)(x^2+1)}\,dx = \left[\ln\frac{(x-3)^2}{x^2+1} - 4\arctan x\right|_0^1$$

$$= \left[\ln 2 - 4\arctan 1\right] - \left[\ln 9 - 4\arctan 0\right].$$

$$= \boxed{\ln\frac{2}{9} - \pi} \quad \leftarrow \arctan 1 = \pi/4, \ \arctan 0 = 0.$$

b) $f(x)$ is discont at $x=3$, so:

$$\int_0^3 \frac{2x+14}{(x-3)(x^2+1)}\,dx = \lim_{b\to 3^-} \int_0^b \frac{2x+14}{(x-3)(x^2+1)}\,dx$$

$$= \lim_{b\to 3^-} \left[\ln\frac{(x-3)^2}{x^2+1} - 4\arctan x\right|_0^b$$

$$= \lim_{b\to 3^-} \left[\left(\ln\frac{b-3}{b^2+1} - 4\arctan b\right) - \left(\ln 9 - 4\arctan 0\right)\right]$$

$\uparrow$

$\ln 0 = -\infty$! The limit DNE!

The limit DNE $\Rightarrow$ $\boxed{\int_0^3 \frac{2x+14}{(x-3)(x^2+1)}\,dx \text{ diverges}}$

c) $f(x)$ is not cts at $x=3 \Rightarrow \int_0^5 f(x)\,dx = \lim_{a\to 3^-} \int_0^a f(x)\,dx + \lim_{b\to 3^+} \int_b^5 f(x)\,dx.$

- If either limit DNE, the integral diverges.
- In b) the first limit was found not to exist

$\Rightarrow$ $\boxed{\int_0^5 f(x)\,dx \quad \text{diverges}}$

d) $$\int_5^\infty \frac{2x+14}{(x-3)(x^2+1)}dx = \lim_{b\to\infty} \int_5^b \frac{2x+14}{(x-3)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty} \left[\ln \frac{(x-3)^2}{x^2+1} - 4\arctan x \right|_5^b$$

$$= \lim_{b\to\infty} \left[\left(\ln \frac{(b-3)^2}{b^2+1} - 4\arctan b\right) - \left(\ln \tfrac{4}{26} - 4\arctan 5\right)\right.$$

$$= \left[\left(\ln 1 - 4\cdot\tfrac{\pi}{2}\right) - \left(\ln \tfrac{4}{26} - 4\arctan 5\right)\right]$$

$$= \boxed{-2\pi - \ln \tfrac{4}{26} + 4\arctan 5}$$

4. First note $$\int \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx = \int \left(\frac{1}{x+2} + \frac{4x-5}{x^2+9}\right)dx$$

$$= \int\left(\frac{1}{x+2} + \frac{4x}{x^2+9} - \frac{5}{x^2+9}\right)dx$$

$$= \ln|x+2| + 2\ln|x^2+9| - \tfrac{5}{3}\arctan\tfrac{x}{3} + C$$

$$= \ln|x+2| + \ln(x^2+9)^2 - \tfrac{5}{3}\arctan\tfrac{x}{3} + C$$

$$= \ln\left|(x+2)(x^2+9)^2\right| - \tfrac{5}{3}\arctan\tfrac{x}{3} + C$$

a) $\frac{5x^2+3x-1}{(x+2)(x^2+9)}$ is cts on $[0,1]$ so:

$\ln 300 - \ln 162 = \ln \frac{300}{162} = \ln \frac{50}{27}$

$$\int_0^1 \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx = \left[\ln\left|(x+2)(x^2+9)^2\right| - \tfrac{5}{3}\arctan\tfrac{x}{3}\right|_0^1 = \left(\ln 300 - \tfrac{5}{3}\arctan\tfrac{1}{3}\right) - \ln 162 - \tfrac{5}{3}\arctan 0.$$

b) $\dfrac{5x^2+3x-1}{(x+2)(x^2+9)}$ is discontinuous at $x=-2$, so

$$\int_{-2}^{1} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx = \lim_{b\to -2^+} \int \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$$

$$= \lim_{b\to -2^+} \left[\ln\left|(x+2)(x^2+9)^2\right| - \frac{5}{3}\arctan\frac{x}{3} \right|_b^1$$

$$= \lim_{b\to -2^+} \left[\left(\ln 300 - \frac{5}{3}\arctan\frac{1}{3}\right) - \left(\ln\left|(b+2)(b^2+9)^2\right| - \frac{5}{3}\arctan\frac{b}{3}\right)\right]$$

$\uparrow$

$\ln 0 = -\infty$!

The limit DNE, so $\boxed{\int_{-2}^{1} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx \text{ diverges}}$

c) Diverges for the same reason as 3c).

d) $\dfrac{5x^2+3x-1}{(x+2)(x^2+9)}$ is cts on $(-\infty,-4)$ so

$$\int_{-\infty}^{-4} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx = \lim_{b\to -\infty} \int_{b}^{-4} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx$$

$$= \lim_{b\to -\infty} \left[\ln\left|(x+2)(x^2+9)^2\right| - \frac{5}{3}\arctan\frac{x}{3} \right|_b^{-4}$$

$\uparrow$

limit won't exist as $b\to-\infty$! $\ln(\infty)=\infty$!

The limit DNE $\Rightarrow$ $\boxed{\int_{-\infty}^{-4} \frac{5x^2+3x-1}{(x+2)(x^2+9)}\,dx \text{ diverges.}}$

5. Use partial fractions:

$$\frac{4}{x^2-4} = \frac{4}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$4 = A(x-2) + B(x+2)$$

$\underline{x=2:}$ $4 = 4B \rightarrow \underline{B=1}$

$\underline{x=-2:}$ $4 = -4A \rightarrow \underline{A=-1}$

So: $\dfrac{4}{x^2-4} = -\dfrac{1}{x+3} + \dfrac{1}{x-3}$

Since $\dfrac{4}{x^2-4}$ is cts on $[4,\infty)$,

$$\int_4^\infty \frac{4}{x^2-4}\,dx = \lim_{b\to\infty} \int_4^b \left(-\frac{1}{x+3} + \frac{1}{x-3}\right) dx$$

$$= \lim_{b\to\infty} -\ln|x+3| + \ln|x-3| \Big|_4^b$$

$$= \lim_{b\to\infty} \ln\left|\frac{x-3}{x+3}\right|$$

$$= \lim_{b\to\infty} \ln\frac{b-3}{b+3} - \ln\frac{1}{7}$$

$$= \ln 1 - \ln\frac{1}{7}$$

$$= -\ln\frac{1}{7} \text{ or } \boxed{\ln 7}$$

$-\ln\frac{1}{7} = \ln\left(\frac{1}{7}\right)^{-1}$
$= \ln 7.$

6. $\dfrac{3x}{x^2-9}$ is cts on $[10,\infty)$. Also, via a u-sub, $\displaystyle\int \frac{3x}{x^2-9}\,dx = \frac{3}{2}\ln|x^2-9| + C$

So: $$\int_{10}^\infty \frac{3x}{x^2-9}\,dx = \lim_{b\to\infty}\int_{10}^b \frac{3x}{x^2-9}\,dx =$$

$$= \lim_{b\to\infty} \frac{3}{2}\ln|x^2-9|\Big|_{10}^b$$

$$= \lim_{b\to\infty} \frac{3}{2}\ln(b^2-9) - \cancel{\frac{3}{2}\ln 1}^{\,0} \leftarrow \boxed{\text{Diverges!}}$$

limit DNE!

7. $\dfrac{5x}{\sqrt{x^2-4}} = \dfrac{5x}{\sqrt{(x+2)(x-2)}}$ is discont at $x = \pm 2$, so:

$$\int_a^3 \frac{5x}{\sqrt{x^2-4}}\,dx = \lim_{a \to 2^+} \int_0^a \frac{5x}{\sqrt{x^2-4}}\,dx$$

Since $\displaystyle\int \frac{5x}{(x^2-4)^{1/2}}\,dx = 5\sqrt{x^2-4} + C$ (can be done via a u-sub w/ $u = x^2-4$).

$$\lim_{a \to 2^+} \int_a^3 \frac{5x}{\sqrt{x^2-4}}\,dx = \lim_{a \to 2^+} 5\sqrt{x^2-4}\,\Big|_a^3 = \lim_{a \to 2^-} \left(5\sqrt{5} - 5\sqrt{a^2-4}\right)$$

$$= \boxed{5\sqrt{5}}$$

$\lim_{a \to 2^-}() = 0$

8. $\dfrac{5x+1}{4\sqrt[3]{x}}$ is not continuous at $x = 0$, so:

$$\int_{-1}^1 \frac{5x+1}{4x^{1/3}}\,dx = \lim_{a \to 0^-} \int_{-1}^a \frac{5x+1}{4x^{1/3}}\,dx + \lim_{b \to 0^+} \int_b^1 \frac{5x+1}{4x^{1/3}}\,dx$$

Since $\displaystyle\int \frac{5x+1}{4x^{1/3}}\,dx = \int \left(\frac{5}{4}\frac{x}{x^{1/3}} + \frac{1}{x^{1/3}}\right) dx = \int \left(\frac{5}{4}x^{2/3} + x^{-1/3}\right) dx$

$$= \frac{\cancel{5}}{4}\frac{3}{\cancel{5}}x^{5/3} + \frac{3}{2}x^{2/3} + C$$

- $\displaystyle\lim_{a \to 0^-} \int_{-1}^a \frac{5x+1}{4x^{1/3}}\,dx = \lim_{a \to 0^-} \left[\frac{3}{4}x^{5/3} + \frac{3}{2}x^{2/3}\,\Big|_{-1}^a\right.$

$$= \lim_{a \to 0^-} \left[\left(\frac{3}{4}a^{5/3} + \frac{3}{2}a^{2/3}\right) + \left(-\frac{3}{4} + \frac{3}{2}\right)\right]$$

$$= \frac{3}{4}.$$

- Similarly: $\displaystyle\lim_{b \to 0^+} \int_b^1 \frac{5x+1}{4x^{1/3}}\,dx = \frac{9}{4}.$

So: $\displaystyle\int_{-1}^1 \frac{5x+1}{4x^{1/3}}\,dx = \frac{3}{4} + \frac{9}{4} = \boxed{3}$

9. • $\dfrac{4x^2+18}{x^3+9x} = \dfrac{4x^2+18}{x(x^2+9)}$ is continuous on $[1,\infty)$.

• Via partial fractions, one can show:

$$\frac{4x^2+18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9} = \frac{2}{x} + \frac{2x}{x^2+9}.$$

So: $\displaystyle\int_1^\infty \frac{4x^2+18}{x^3+9x}\,dx = \lim_{b\to\infty} \int_1^b \left(\frac{2}{x} + \frac{2x}{x^2+9}\right) dx$

$\downarrow u = x^2+9$

$$= \lim_{b\to\infty} \left[2\ln|x| + \ln|x^2+9|\right|_1^b$$

$$= \lim_{b\to\infty} \ln x^2 + \ln(x^2+9) \Big|_1^b$$

← No "| |" since each arg is positive!

$$= \lim_{b\to\infty} \ln \frac{x^2}{x^2+9} \Big|_1^b$$

$$= \lim_{b\to\infty} \left(\ln \frac{b^2}{b^2+9} - \ln \frac{1}{10}\right)$$

$$= \cancelto{0}{\ln 1} - \ln \frac{1}{10}$$

$$= -\ln \frac{1}{10} \text{ or } \boxed{\ln 10}$$

10. • This is cts on $[1,\infty)$

• Via partial fractions, one can show:

$$\frac{60x^2-4x+100}{(x+1)(16x^2+25)} = \frac{4}{x+1} - \frac{4x}{16x^2+25}.$$

So $\displaystyle\int_1^\infty \frac{4x^2+18}{x^3+9x}\,dx = \lim_{b\to\infty} \int_1^b \left(\frac{4}{x+1} - \frac{4x}{16x^2+25}\right) dx$

$$= \lim_{b\to\infty} \left[4\ln|x+1| - \frac{1}{8}\ln(16x^2+25)\right|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{8}\left[32\ln|x+1| - \ln(16x^2+25)\right|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{8}\left[\ln (x+1)^{32} - \ln(16x^2+25)\right]\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{8} \ln \frac{(x+1)^{32}}{16x^2+25}\Big|_1^b$$

limit DNE as $b\to\infty$ since $\frac{(b+1)^{32}}{16b^2+25} \to \infty$ as $b\to\infty$.

$\boxed{\text{Diverges}}$

11. a) <u>False</u>; You can only use antiderivatives to compute $\int_a^b f(x)\,dx$ if $f(x)$ is cts on the closed interval $[a,b]$.

- Here $\frac{1}{x^2}$ is <u>NOT</u> cts at $x=0$, so:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x^2}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}\,dx.$$

- The incorrect step is circled below:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx \overset{!}{=} -x^{-1}\Big|_{-1}^{1}$$

<u>CANNOT</u> use antideriv since $\frac{1}{x^2}$ is NOT cts at $x=0$!

b) <u>True</u>; $\frac{1}{(x+2)^2}$ is cts on $[-1,1]$ so Fund. Thm. of Calc. Applies!

12. a) $\int_a^b f(x)\,dx$ represents the area between $y=f(x)$ and the x-axis from $x=a$ to $x=b$.

b) In the special case where $[a,b]$ is closed and $f(x)$ is cts on $[a,b]$, the Fund. Thm. of Calculus says we may use antiderivatives to evaluate $\int_a^b f(x)\,dx = F(b)-F(a)$, where $F(x)$ is an antideriv of $f(x)$. This only holds in this special case!

13a) $\int_0^b \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\frac{x}{3}\Big|_0^b$

$= \frac{1}{3}\arctan\frac{b}{3} - \frac{1}{3}\cancel{\arctan 0}^{\,0}$

So: $\int_0^\infty \frac{1}{x^2+9}\,dx = \lim_{b\to\infty} \frac{1}{3}\arctan\frac{b}{3}$

$= \frac{1}{3}\cdot\frac{\pi}{2}$

$= \boxed{\frac{\pi}{6}}$

b). An Excel sheet or other program numerically suggests as b gets larger, $\frac{1}{3}\arctan\frac{b}{3} \to \frac{\pi}{6}$.

IT DOES **NOT** PROVE IT THOUGH!!!

14a) $a_1 = 6(1)^2 - 13(1) = -7$

$a_2 = 6(2)^2 - 13(2) = -2$

$a_3 = 6(3)^2 - 13(3) = 15$

etc.

b) $a_1 = \sin\frac{(1)\pi}{2} = 1$

$a_2 = \sin\frac{2\pi}{2} = 0$

$a_3 = \sin\frac{3\pi}{2} = -1$

etc

c) $a_1 = 4 - \frac{5}{1} = -1$

$a_2 = 4 - \frac{5}{2} = 3/2$

$a_3 = 4 - \frac{5}{3} = 7/3$

etc

d) $a_2 = 3a_1 + 1 = 3(3)+1 = 10$

$a_3 = 3a_2 + 1 = 3(10)+1 = 31$

$a_4 = 3a_3 + 1 = 3(31)+1 = 94$

etc.

e) Cannot specify since we don't know what a_1, a_2 are!

f) $a_3 = a_1 + a_2 = 1+1 = 2$

$a_4 = a_2 + a_3 = 1+2 = 3$

$a_5 = a_3 + a_4 = 2+3 = 5$

$a_6 = a_4 + a_5 = 3+5 = 8$

This is the Fibonnacci sequence!

15 a) $\boxed{\text{Converges to } 0}$ since $\dfrac{3n^2-1}{\sqrt{n}+n+n^3} \simeq \dfrac{3n^2}{n^3} = \dfrac{3}{n}$ for large n

b) $\frac{1}{n^3} \to 0$ as $n \to \infty$ so $\cos\left(\frac{1}{n^3}\right) \to \cos(0) = \boxed{1}$ as $n \to \infty$

c) Use squeeze thm since the limit of the numerator DNE:

$$-1 \leq \sin n^2 \leq 1 \text{ for all } n.$$

$$-3 \leq 3\sin n^2 \leq 3$$

$$-3-n \leq 3\sin n^2 - n \leq 3-n$$

$$\frac{-3-n}{2n+1} \leq \frac{3\sin n^2 - n}{2n+1} \leq \frac{3-n}{2n}$$

Since $\lim\limits_{n\to\infty} \dfrac{-3-n}{2n+1} = \lim\limits_{n\to\infty} \dfrac{3-n}{2n} = -\dfrac{1}{2}$, by Squeeze Thm:

$$\boxed{\lim_{n\to\infty} \frac{3\sin n^2 - n}{2n+1} = -\frac{1}{2}}$$

d) $\boxed{\text{Diverges}}$; $\dfrac{6n^2\cos n + 1}{5n-4n^2} = \dfrac{6n^2}{5n-4n^2}\cos n + \dfrac{1}{5n-4n^2}$

As $n \to \infty$, $\dfrac{6n^2}{5n-4n^2} \to -\dfrac{6}{4}$ and $\dfrac{1}{5n-4n^2} \to 0$.

but $\cos n$ oscillates, so $\lim\limits_{n\to\infty} \dfrac{6n^2}{5n-4n^2}\cos n$ DNE

e) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sqrt{9n^2\left(1+\frac{1}{9n^2}\right)}}{n\left(2-\frac{3}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{3\cancel{n}\sqrt{1+\frac{1}{9n^2}}}{\cancel{n}\left(2-3/n\right)}$

$$= \boxed{\frac{3}{2}}$$

f). $n^{\sin\frac{1}{n}} \to \infty^0$ which is indeterminate!

Let $L = \lim\limits_{n\to\infty} n^{\sin\frac{1}{n}}$

$\ln L = \lim\limits_{n\to\infty} \ln n^{\sin\frac{1}{n}} = \lim\limits_{n\to\infty} \underbrace{\sin\tfrac{1}{n} \cdot \ln n}_{0\cdot\infty}$

$$\ln L = \lim_{n\to\infty} \frac{\ln n}{\csc \frac{1}{n}} \leftarrow \frac{\infty}{\infty}$$

$$\ln L \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{+\csc\frac{1}{n}\cot\frac{1}{n} \cdot -\frac{1}{n^2}}$$ by L'Hopitals rule

$$\ln L = \lim_{n\to\infty} \frac{n}{\csc\frac{1}{n}\cot\frac{1}{n}} \leftarrow \frac{\infty}{\infty}$$

$$\ln L = \lim_{n\to\infty} n \sin\frac{1}{n} \tan\frac{1}{n}$$

$$\ln L = \lim_{n\to\infty} \frac{n\left(\sin\frac{1}{n}\right)^2}{\cos\frac{1}{n}} \quad \cos\frac{1}{n} \to 1 \text{ as } n\to\infty.$$

$$\ln L = \lim_{n\to\infty} \frac{\left(\sin\frac{1}{n}\right)^2}{\frac{1}{n}}$$

$$\ln L \overset{L'H}{=} \lim_{n\to\infty} \frac{2\sin\frac{1}{n}\cos\frac{1}{n} \cdot \cancel{-\frac{1}{n^2}}}{\cancel{-\frac{1}{n^2}}}$$

$$\ln L = 2 \cdot \sin 0 \cdot \cos 0$$

$$\ln L = 0$$

$$L = e^0 = \boxed{1}$$

16a) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} a_n + 3b_n) = 3 + 3(6) = \boxed{21}$

b) Cannot be determined;

Case 1: Let $c_n = \sin n$. Then, $\lim_{n\to\infty} \frac{3a_n}{c_n}$ DNE.

Case 2: Let $c_n = n$. Then, $\lim_{n\to\infty} \frac{3a_n}{c_n} = \frac{\lim_{n\to\infty} 3a_n}{\lim_{n\to\infty} n} = 0.$

c) $\boxed{\text{Diverges}}$; if it converged, say to L:

$$\lim_{n\to\infty} \sqrt{c_n} = L \Rightarrow \lim_{n\to\infty} c_n = L^2$$

d) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \sqrt{a_n+4} = \sqrt{\lim_{n\to\infty} (a_n+4)}$ since $\sqrt{\cdot}$ is cts

$$= \sqrt{3+4} = \boxed{\sqrt{7}}$$

e) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} e^{b_n-2} = e^{\lim_{n\to\infty}(b_n-2)}$ since $e^{(\cdot)}$ is cts.

$$= e^{6-2} = \boxed{e^4}$$

f) $\lim_{n\to\infty} (a_n - 4c_n)$ DNE since $\{c_n\}$ diverges; if $\lim_{n\to\infty} A_n = L$,

then: $\lim_{n\to\infty} A_n = \lim_{n\to\infty} (a_n - 4c_n)$

$$L = 3 - 4\lim_{n\to\infty} c_n \Rightarrow \lim_{n\to\infty} c_n = \frac{3-L}{4}.$$

So, $\boxed{\lim_{n\to\infty} A_n \text{ diverges}}$

17. a) True

b) False; let $a_n = 1$, $c_n = \sin n$. Then $a_n c_n = \sin n$ which diverges.

c) False; let $a_n = 1$, $d_n = n$.

d) False; let $c_n = d_n = n$ so $c_n - d_n = 0$ for all n (so it converges to 0) even though c_n, d_n diverge.

e) False; let $a_n = b_n = 1$, $c_n = \sin n$, $d_n = \frac{1}{\sin n}$

f) True; see 16c).

g) False; let $c_n = n$.

$$\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$$

We can compute this:

let $L = \lim_{n\to\infty} n^{1/n}$

$$\ln L = \lim_{n\to\infty} \ln n^{1/n}$$

$$\ln L = \lim_{n\to\infty} \frac{1}{n} \ln n$$

$$\ln L = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\ln L \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$\ln L = 0$$

$$\Rightarrow L = e^0 = 1.$$

So, in this case $\sqrt[n]{c_n}$ converges!

h) False; let $c_n = n$. As discussed in class,

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$$

so $$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{c_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

\!!!/ You should know how to compute this \!!!/