Worksheet #7 Solutions

1. $\frac{4}{3^n} = 4 \cdot \frac{1}{3^n} = 4 \cdot \left(\frac{1}{3}\right)^n$ ← Geometric w/ $r<1$!

So $\boxed{\sum_{n=1}^{\infty} \frac{4}{3^n} \text{ converges}}$. It converges to $\frac{ar^{n_0}}{1-r} = \frac{4\left(\frac{1}{3}\right)^1}{1-\frac{1}{3}}$

↑ $a=4$, $r=\frac{1}{3}$, $n_0=1$

$$\boxed{\sum_{n=1}^{\infty} \frac{4}{3^n} = 2}$$

2. $3^{2-3n} = \frac{3^2}{3^{3n}} = \frac{9}{(3^3)^n} = \frac{9}{(27)^n} = 9 \cdot \left(\frac{1}{27}\right)^n$. ← Geometric w/ $r<1$!

By the same argument in 1., $\boxed{\sum_{n=3}^{\infty} 3^{2-3n} \text{ converges}}$

It converges to: $\frac{ar^{n_0}}{1-r} = \boxed{\frac{9\left(\frac{1}{27}\right)^3}{1-\frac{1}{27}}}$ ← $a=9$, $r=\frac{1}{27}$, $n_0=3$.

3. $\frac{2^{3n+1}}{7^{n+100}} = \frac{2^{3n}2}{7^n 7^{100}} = \frac{2}{7^{100}} \cdot \frac{8^n}{7^n} = \frac{2}{7^{100}}\left(\frac{8}{7}\right)^n$

This is geometric w/ $r>1$, so $\boxed{\sum_{n=5}^{\infty} \frac{2^{3n+1}}{7^{n+100}} \text{ diverges}}$

4. $\sum_{n=0}^{\infty} 4(r^2)^n$ converges if $r^2<1 \Rightarrow |r|<1$

In this case, it converges to $\frac{a\tilde{r}^{n_0}}{1-\tilde{r}} = \frac{a\cancel{(r^2)^0}^{\,1}}{1-r^2} = \frac{a}{1-r^2}$

So: $$\boxed{\sum_{n=0}^{\infty} 4r^{2n} \begin{cases} \text{converges to } \frac{a}{1-r^2} \text{ if } |r|<1 \\ \text{diverges, otherwise} \end{cases}}$$

5. $\frac{3^{2n+1}}{10^{n+2}} = \frac{3^{2n}3^1}{10^n 10^2} = \frac{3}{100}\frac{9^n}{10^n} = \frac{3}{100}\left(\frac{9}{10}\right)^n$. ← Geometric w/ $r<1$!

So, $\boxed{\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^{n+2}} \text{ converges}}$. It converges to: $\frac{ar^{n_0}}{1-r} = \frac{\frac{3}{100}\cancel{\left(\frac{9}{10}\right)^0}^{\,1}}{1-\frac{9}{10}} = \boxed{\frac{3}{10}}$

6. $\underline{\text{Note:}}$ $\dfrac{(-5)^n}{6^{2n}} = \dfrac{(-5)^n}{(6^2)^n} = \dfrac{(-5)^n}{36^n} = \left(-\dfrac{5}{36}\right)^n \leftarrow$ Geometric w/ $r<1$!

So $\boxed{\sum_{n=0}^{\infty} \frac{(-5)^n}{6^{2n}} \text{ converges}}$ It converges to: $\dfrac{ar^{n_0}}{1-r} = \dfrac{1\left(-\frac{5}{36}\right)^0}{1-\left(-\frac{5}{36}\right)}$

$= \boxed{\dfrac{36}{41}}$

7. $\underline{\text{Note:}}$ $4(-1.75)^{n/3} = 4\left[(-1.75)^{1/3}\right]^n \leftarrow$ This is geometric w/ $r = (-1.75)^{1/3}$ so $|r|>1$!

Hence $\boxed{\sum_{n=0}^{\infty} 4(-1.75)^{n/3} \text{ diverges}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{60000} = \sum_{n=1}^{\infty} \frac{1}{60000} 8^n \leftarrow$ geometric w $r>1$!

$\boxed{\sum_{n=1}^{\infty} \frac{2^{3n}}{60000} \text{ diverges}}$

9. $\displaystyle\sum_{n=3}^{\infty} \frac{2-4n}{6+n}$. $\underline{\text{Note:}}$ $\displaystyle\lim_{n\to\infty} \frac{6-4n}{6+n} = -4 \neq 0.$

$\boxed{\text{The series diverges by the divergence test}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!} \leftarrow$ $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = 0$ so div test fails! Use Ratio Test:

$$\text{Let } L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}\,2^1}{\cancel{2^n}} \cdot \frac{\cancel{n(n-1)(n-2)\cdots}}{(n+1)\cancel{(n)(n-1)(n-2)}}$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = \underline{0}$$

$\boxed{\text{The series converges by Ratio Test since } L = 0 < 1.}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+1)\,3^{2n}}{(2n)!}$

Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{[(n+1)^2+1]\;3^{2(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{(n^2+1)\,3^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+2}{n^2+1} \cdot \frac{3^{2n+2}}{3^{2n}} \cdot \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+2}{n^2+1} \cdot \frac{\cancel{3^{2n}}\,3^2}{\cancel{3^{2n}}} \cdot \frac{\cancel{2n(2n-1)\ldots}}{(2n+2)(2n+1)\cancel{2n(2n-1)\ldots}} \right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{n^2+2n+2}{n^2+1}}_{\downarrow\, 1} \cdot 3^2 \cdot \underbrace{\frac{1}{(2n+2)(2n+1)}}_{\downarrow\, 0} \right|$$

$$\underline{L = 0.}$$

$\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+1)\,3^{2n}}{(2n)!} \text{ converges by the ratio test since } L = 0 < 1}$

12. $$\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2^n}{4^n} + \frac{3^n}{4^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Both sums on the RHS are $\boxed{\text{geometric w/ } r < 1 \text{ so the series converges}}$

13. $$\lim_{n\to\infty} \frac{6n^2-1}{\sqrt{n^4+7}} = \lim_{n\to\infty} \frac{6n^2-1}{\sqrt{n^4\left(1+\frac{7}{n^4}\right)}} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(6-\frac{1}{n^2}\right)}{\cancel{n^2}\sqrt{1+\frac{7}{n^4}}} = 6.$$

Hence, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{6n^2-1}{\sqrt{n^4+7}} \text{ diverges by the divergence test.}}$

14. Recall that $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$

(this can be shown by letting $L = \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn}$, taking $\ln(\ )$ of both sides, then manipulating the expression so as to use L'Hopital's Rule)

Hence, $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n} = e^{-2} \neq 0$, so $\boxed{\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n} \text{ diverges by divergence test}}$

15. $\lim_{n\to\infty} \cos\left(1-\frac{1}{n}\right) = \cos(1-0) = \cos 1 \neq 0.$

Hence, $\boxed{\sum_{n=1}^{\infty} \cos\left(1-\frac{1}{n}\right) \text{ diverges by divergence test}}$

16. Way 1: To compute $\lim_{n\to\infty} \frac{n^n}{n!}$, note for all n:

$$\frac{n!}{n^n} = \underbrace{\frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2}, \ldots, \frac{n}{1}}_{\text{each is } > 1} > 1 \cdot n$$

So, $\lim_{n\to\infty} \frac{n!}{n^n} \geq \lim_{n\to\infty} n = \infty.$

Hence, $\boxed{\sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges by divergence test!}}$

Way 2: Ratio Test:

$$\text{Let } L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{1}{\cancel{n+1}} \cdot \frac{(n+1)^n \cdot \cancel{(n+1)}}{n^n}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n =$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \leftarrow \text{Recall: } \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$$

$$= e$$

So, $L = e > 1$, so $\boxed{\text{the series diverges by Ratio Test}}$

17. Use ratio test.

$$L = \lim_{n\to\infty} \left| \frac{[p(n+1)]^{n+1}}{(n+1)!} \cdot \frac{n!}{(pn)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{p^{n+1}}{p^n} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= p \underbrace{\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \right|}_{\text{We computed this in \#16!}}$$

$$= pe.$$

This converges if $L = pe < 1$

$$\boxed{p < \frac{1}{e}}$$

<u>Note:</u> To check when $p=e$ requires more in-depth asymptotic tools. We won't concern ourself with this, but if you're curious, look up "Stirling's Formula"!

18. a) We know $s_N = \sum_{n=1}^{N} a_n = \frac{4N-1}{3N^2}$, and

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{4N-1}{3N^2} = \boxed{0}.$$

b) Since $\lim_{N\to\infty} s_N$ exists, the series converges and thus $\boxed{\lim_{n\to\infty} a_n = 0}$

19. Know: $s_N = \sum_{n=1}^{N} a_n = \frac{4+N}{3-2N}.$

a) $a_1 + a_2 + a_3 = s_3 = \frac{4+3}{3-2(3)} = \boxed{-\frac{7}{3}}$

b) $a_5 + a_6 = (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) - (a_1 + a_2 + a_3 + a_4)$

$= s_6 - s_4$

$= \frac{4+6}{3-2(6)} - \frac{4+4}{3-2(4)}.$

$= \boxed{-\frac{10}{9} + \frac{8}{5}}$

c) $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{4+N}{3-2N} = \boxed{-\frac{1}{2}}$

d) The series converges since $\lim_{N\to\infty} s_N$ exists so $\boxed{\lim_{n\to\infty} a_n = 0}$

20 a) $a_1 = 1$ $\quad s_1 = a_1 = 1$

$a_2 = \frac{3}{4}$ $\quad s_2 = a_1 + a_2 = 1 + \frac{3}{4}$

$a_3 = \frac{5}{9}$ $\quad s_3 = a_1 + a_2 + a_3 = 1 + \frac{3}{4} + \frac{5}{9}$

$a_4 = \frac{7}{16}$ $\quad s_4 = a_1 + a_2 + a_3 + a_4 = 1 + \frac{3}{4} + \frac{5}{9} + \frac{7}{16}$

So: $s_1 =$, $s_2 =$, $s_3 =$, $s_4 =$

Following this template:

b) $s_1 =$, $s_2 =$, $s_3 =$, $s_4 =$

c) $s_1 =$, $s_2 =$, $s_3 =$, $s_4 =$

d) $s_1 =$, $s_2 =$, $s_3 =$, $s_4 =$

21. We know $s_N = \sum_{n=1}^{N} a_n$, so:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N.$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges iff $\lim_{N\to\infty} s_N$ exists. In this case $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N$.

22. a) False; $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

b) True

c) True; $\lim_{n\to\infty} s_N = \sum_{n=1}^{\infty} a_n = L$ by assumption.

d) False $L \neq 0$ by assumption!

e) False; $\lim_{N\to\infty} s_N = L \neq 0$ by assumption. Hence $\sum s_n$ diverges by the divergence test!

f) Cannot be determined: $\sum_{n=8}^{\infty} a_n = L$ iff $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 0$ (so $\sum_{n=1}^{\infty} a_n = \sum_{n=8}^{\infty} a_n$).

g) False; $\lim_{n\to\infty} (a_n + 1) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 1 = 0 + 1 = 1$.

Hence, $\sum (a_n + 1)$ diverges by divergence test!

h) False; $\lim_{n\to\infty} (s_n - s_{n-1}) = L - L = 0$.

23. a) CBD; $\lim_{n\to\infty} a_n$ may not exist or could be finite!

b) CBD; s_n may oscillate!

c) CBD; $\sum a_n$ could diverge even if $\lim_{n\to\infty} a_n = 0$! ex: $\sum_{n=1}^{\infty} \frac{1}{n}$.

d) CBD; $s_n - s_{n+1} = a_{n+1}$ (Why?). and we don't know anything about $\lim_{n\to\infty} a_{n+1}$!

e) True; let $a_n = 1$ for all n

f) True; let $a_n = 2^n$ for all n so $\sum \frac{2}{a_n} = \sum \frac{2}{2^n} = \sum 2\left(\frac{1}{2}\right)^n$

g) False; let $a_n = -1$ for all n so $a_n + 1 = 0$ for all n and $\sum_{n=1}^{\infty} (a_n + 1) = \sum 0 = 0$.

h). True; since $\sum a_n$ diverges, $\lim_{n\to\infty} s_n$ DNE.

By div. test $\sum_{n=1}^{\infty} s_n$ must diverge.

24. a) True; the first finitely many terms do NOT affect whether a series converges!

b) FALSE; divergence test NEVER guarantees convergence!

c) False; $\lim_{n\to\infty} (a_n + b_n) = \underbrace{\lim_{n\to\infty} a_n}_{=0 \text{ since } \sum a_n \text{ conv!}} + \lim_{n\to\infty} b_n = 0 + 5$

Hence, by div. test, $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

d) False; same argument as above.

e) False; the Ratio test is inconclusive here! The series could do anything!

f) False; unless $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, $\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n}$ diverges by div. test.