

Worksheet #8 Solutions

I. 1.	$n$	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$	$a_n = \frac{f^{(n)}(\frac{\pi}{3})}{n!}$
	0	$\sin x$	$\frac{\sqrt{3}}{2}$	$\frac{\frac{\sqrt{3}}{2}}{0!} = \frac{\sqrt{3}}{2}$
	1	$\cos x$	$\frac{1}{2}$	$\frac{1/2}{1!} = \frac{1}{2}$
	2	$-\sin x$	$-\frac{\sqrt{3}}{2}$	$\frac{-\frac{\sqrt{3}}{2}}{2!} = -\frac{\sqrt{3}}{4}$
	3	$-\cos x$	$-\frac{1}{2}$	$\frac{-1/2}{3!} = -\frac{1}{12}$

So:  $\sin x = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$

$$\sin x = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{3}}{4}(x - \frac{\pi}{3})^2 - \frac{1}{12}(x - \frac{\pi}{3})^3 + \dots$$

2.

$n$	$f^{(n)}(x)$	$f^{(n)}(2)$	$a_n = \frac{f^{(n)}(2)}{n!}$
0	$\ln(1+3x)$	$\ln 7$	$\frac{\ln 7}{0!} = \ln 7$
1	$\frac{3}{1+3x}$	$\frac{3}{7}$	$\frac{3/7}{1!} = \frac{3}{7}$
2	$-\frac{9}{(1+3x)^2}$	$-\frac{9}{49}$	$\frac{-9/49}{2!} = -\frac{9}{98}$
3	$\frac{54}{(1+3x)^3}$	$\frac{54}{343}$	$\frac{54/343}{3!} = \frac{9}{343}$

So:  $\ln(1+3x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$

$$\ln(1+3x) = \ln 7 + \frac{3}{7}(x-2) - \frac{9}{98}(x-2)^2 + \frac{9}{343}(x-2)^3 + \dots$$

3.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = \frac{f^{(n)}(0)}{n!}$
0	$(1+x)^{1/2}$	1	$\frac{1}{0!} = 1$
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$	$\frac{1/2}{1!} = \frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$	$\frac{-1/4}{2!} = -\frac{1}{8}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$	$\frac{3/8}{3!} = \frac{1}{16}$

So:  $\sqrt{1+x} = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$

$$\boxed{\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots}$$

4.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = \frac{f^{(n)}(0)}{n!}$
0	$e^{3x}$	$e^3$	$\frac{e^3}{0!} = e^3$
1	$3e^{3x}$	$3e^3$	$\frac{3e^3}{1!} = 3e^3$
2	$9e^{3x}$	$9e^3$	$\frac{9e^3}{2!} = \frac{9}{2}e^3$
3	$27e^{3x}$	$27e^3$	$\frac{27e^3}{3!} = \frac{9}{2}e^3$

$$e^{3x} = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

$$\boxed{e^{3x} = e^3 + 3e^3(x-1) + \frac{9}{2}e^3(x-1)^2 + \frac{9}{2}e^3(x-1)^3 + \dots}$$

II. 5. Use Ratio Test:

$$\text{Let } L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} x^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{(-1)^{n+1} x^{2n}} \right| \quad \leftarrow \text{absolute value kills } (-1)^n$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{x^{2n}} \cdot \frac{3^n}{3^{n+1}} \right|$$

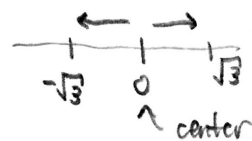
$$= \lim_{n \rightarrow \infty} \left| \frac{x^{2n} x^2}{x^{2n}} \cdot \frac{3^n}{3^n 3^1} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{x^2}{3} \right|$$

Ratio Test tells us the series converges if  $L = \left(\frac{x^2}{3}\right) < 1$

$$|x^2| < 3 \Rightarrow |x| < \sqrt{3}$$

The radius of convergence is  $\sqrt{3}$ .  
The IOC is  $-\sqrt{3} < x < \sqrt{3}$ .



6. Ratio Test: Let  $L = \lim_{n \rightarrow \infty} \left| \frac{[(n+1)x]^{n+1}}{(n+1)!} \cdot \frac{n!}{(nx)^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^n}{n^n} \cdot \frac{1}{n+1} \cdot x \right|$$

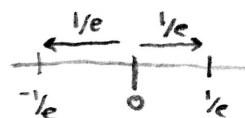
$$= |x| \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n$$

$$= |x| \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n \leftarrow \lim_{n \rightarrow \infty} \left( 1 + \frac{a}{n} \right)^{bn} = e^{ab}$$

$L = e|x|$

Hence, the series converges when  $L = e|x| < 1 \Rightarrow |x| < \frac{1}{e}$

ROC:  $\frac{1}{e}$   
IOC:  $-\frac{1}{e} < x < \frac{1}{e}$



7. Ratio Test: Let  $L = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| 2 \cdot \frac{1}{n+1} \cdot x \right| = 0$$

Hence  $L = 0 < 1$  regardless of which  $x$  we choose!

$$\boxed{\text{ROC: } \infty, \text{ IOC: } -\infty < x < \infty}$$

8. Ratio Test:

$$L = \lim_{n \rightarrow \infty} \left| \frac{[2(n+1)]! x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n)! x^n} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{n+1}}{(2n)! x^n} \cdot \frac{n^n}{(n+1)^{n+1}} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)2n(2n-1)\dots}{2n(2n-1)\dots} \cdot x \cdot \frac{n^n}{(n+1)^n (n+1)} \right|$$

$$L = \lim_{n \rightarrow \infty} \left| 2(n+1)(2n+1) \cdot x \cdot \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \right|$$

↓  
 $\infty$

↓  
 $e$

The limit is infinite!

Hence,  $L > 1$

no matter what  $x \neq 0$  we choose!

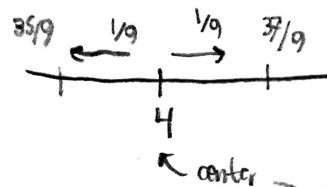
$$\boxed{\text{ROC: } 0 \quad \text{IOC: } x=0}$$

9. We could use Ratio test, but this is a geometric series!

$$\begin{aligned} 3^{2n+1} (x-4)^n &= 3^{2n} 3(x-4)^n \\ &= 9^n \cdot 3 \cdot (x-4)^n \\ &= 3 \cdot (9x-36)^n \end{aligned}$$

This converges when  $|9x-36| < 1 \Rightarrow 9|x-4| < 1$

$$|x-4| < \frac{1}{9}$$



The series is written in powers of  $(x-c) = (x-4)$  so  $c=4$ !

$$\boxed{\begin{aligned} \text{ROC: } & \frac{1}{9} \\ \text{IOC: } & \frac{35}{9} < x < \frac{37}{9} \end{aligned}}$$



10. Ratio Test:  $L = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{3(n+1)}}{8^{n+1}} \cdot \frac{8^n}{(n+1)x^{3n}} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \cdot \frac{x^{3n+3}}{x^{3n}} \cdot \frac{8^n}{8^{n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \cdot x^3 \cdot \frac{1}{8} \right|$$

$$L = \left| \frac{x^3}{8} \right| \lim_{n \rightarrow \infty} \frac{n+2}{n+1}$$

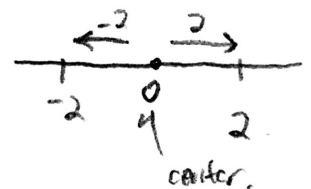
$$\Rightarrow L = \left| \frac{x^3}{8} \right|$$

Ratio Test guarantees convergence when  $L = \left| \frac{x^3}{8} \right| < 1$

$$|x^3| < 8$$

$$|x| < 2$$

ROC:  $\infty$   
 IOC:  $-2 < x < 2$



11. Ratio Test:  $L = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1} x^{2(n+1)+1}}{(n+1)!} \cdot \frac{n!}{4^n x^{2n+1}} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{4^n} \cdot \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| 4 \cdot x^2 \cdot \frac{1}{n+1} \right|$$

$$= |4x^2| \lim_{n \rightarrow \infty} \frac{1}{n+1}$$

$$L = 0$$

So  $L=0$  regardless of what  $x$  is!

ROC:  $\infty$   
 IOC:  $-\infty < x < \infty$

12. Ratio Test:

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x+1)^{n+1}}{4(n+1)+1} \cdot \frac{4n+1}{(-1)^n (x+1)^n} \right|$$

↑ absolute kills  $(-1)^n$

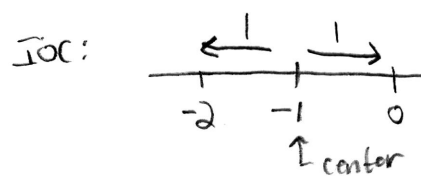
$$= \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \cdot \frac{4n+1}{4n+5} \right|$$

$$= |x+1| \lim_{n \rightarrow \infty} \frac{4n+1}{4n+5}$$

$$L = |x+1|$$

Ratio Test guarantees the series converges when  $L = |x+1| < 1$ .

ROC: 1  
IOC:  $-2 < x < 0$



13.  $f(x) = xe^{5x}$ .

We know  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , all  $x$

$\rightarrow e^{5x} = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$ , all  $x$ .

So:  $xe^{5x} = x + 5x^2 + \frac{25}{2}x^3 + \frac{125}{6}x^4 + \dots = \sum_{n=0}^{\infty} \frac{5^n x^{n+1}}{n!}$

ROC:  $\infty$

14.  $f(x) = \frac{4}{3-2x}$  ← Need to look like  $\frac{1}{1-r}$ .

$$\frac{4}{3-2x} = \frac{4}{3(1-\frac{2}{3}x)} = \frac{4}{3} \frac{1}{1-\frac{2}{3}x} \leftarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

$$= \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}x\right)^n, \left|\frac{2}{3}x\right| < 1$$

$$= \frac{4}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n, |x| < \frac{3}{2}$$

Writing out the first 4 terms:

$$\frac{4}{3-2x} = \frac{4}{3} + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{32}{81}x^3 + \dots$$

15.  $f(x) = 5 \sin x^2$ .

We know  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ , all  $x$

$$\rightarrow \sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$$

So:

$$5 \sin x^2 = 5x^2 - \frac{5}{6}x^6 + \frac{1}{24}x^{10} - \frac{5}{7!}x^{14} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

for all  $x$   
(Roc =  $\infty$ )

16.  $\frac{2x}{1+x^2} = 2x \left( \frac{1}{1+x^2} \right)$

↑  
already in powers of  $x$

↖ only need this Taylor series!

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Since  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$ ,  $|x| < 1$ .

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots = \sum_{n=0}^{\infty} (-x^2)^n, \quad |x^2| < 1$$

$|x|^2 < 1$   
 $|x| < 1$

$$= 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

Hence:

$$2x \left( \frac{1}{1+x^2} \right) = 2x - 2x^3 + 2x^5 - 2x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}, \quad |x| < 1$$

$$17. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ all } x.$$

$$\cos 5x = 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \frac{(5x)^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n}}{(2n)!}, \text{ all } x.$$

$$\cos 5x = 1 - \frac{25}{2} x^2 + \frac{625}{24} x^4 - \frac{3125}{144} x^6 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} x^{2n}, \text{ all } x$$

$$18. \frac{1}{1+4x} = \frac{1}{1-(-4x)}$$

$$\text{Since } \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1-(-4x)} = 1 + (-4x) + (-4x)^2 + (-4x)^3 + \dots = \sum_{n=0}^{\infty} (-4x)^n, \quad |-4x| < 1$$

$$\frac{1}{1+4x} = 1 - 4x + 16x^2 - 64x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n 4^n x^n, \quad |x| < \frac{1}{4}$$

$$19. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ all } x.$$

$$\rightarrow \sin 4x = (4x) - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} - \frac{(4x)^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!}, \text{ all } x.$$

$$\sin 4x = 4x - \frac{32}{3} x^3 + \frac{128}{15} x^5 - \frac{1024}{315} x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$\rightarrow 4x^3 \sin 4x = 16x^4 - \frac{128}{3} x^6 + \frac{512}{15} x^8 - \frac{4096}{315} x^{10} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{4^{2n+4}}{(2n+1)!} x^{2n+4}, \text{ all } x$$

$$20. \frac{4}{8x-1} = -4 \cdot \frac{1}{1-8x}$$

$$= -4 \left[ 1 + 8x + (8x)^2 + (8x)^3 + \dots \right] = -4 \sum_{n=0}^{\infty} (8x)^n, \quad |8x| < 1$$

$$\frac{4}{8x-1} = -4 - 32x - 256x^2 - 2048x^3 - \dots = \sum_{n=0}^{\infty} -4 \cdot 8^n x^n, \quad |x| < \frac{1}{8}$$

IV.

21. Note:  $\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$  on one hand.

On the other:

$$\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + x^4 + \dots) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right), |x| < 1$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} \frac{d}{dx} (x^n), |x| < 1$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} n x^{n-1}, |x| < 1$$

Hence,  $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} n x^{n-1}, |x| < 1$

22.  $\frac{3x}{(1+x)^2} = 3x \cdot \frac{1}{(1+x)^2}$  So we just need the series for  $\frac{1}{(1+x)^2}$ .

We could immediately obtain this from above:

$$\frac{1}{(1+x)^2} = \frac{1}{[1-(-x)]^2} = \sum_{n=0}^{\infty} n(-x)^{n-1} = \sum_{n=0}^{\infty} (-1)^{n-1} n \cdot x^{n-1}$$

or derive it (since you may not have the series for  $\frac{1}{(1-x)^2}$  given on the exam):

Note:  $\therefore \frac{d}{dx} \left( \frac{1}{1+x} \right) = \frac{d}{dx} (1+x)^{-1} = -1(1+x)^{-2} (1+x)' = -\frac{1}{(1+x)^2}$ .

Also:  $-\frac{d}{dx} \left( \frac{1}{1+x} \right) = -\frac{d}{dx} \left( \frac{1}{1-(-x)} \right) \downarrow$  use Taylor series for  $\frac{1}{1-u}$

$$= -\frac{d}{dx} (1 - x + x^2 - x^3 + x^4 - \dots) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} (-1)^n x^n \right), |x| < 1$$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$$

Note:  $(-1)^{n-1} = (-1)^{n+1}$  since  $(-1)^{n+1} = (-1)^{n-1} (-1)^2 = (-1)^{n-1}$

Hence:  $3x \frac{1}{(1+x)^2} = 3x - 6x^2 + 9x^3 - 12x^4 + \dots = \sum_{n=0}^{\infty} (-1)^{n+1} 3n x^n, |x| < 1$

$$23. \quad \frac{2x}{(3+4x^3)^2} = 2x \frac{1}{(3+4x^3)^2}$$

We know we must differentiate  $\frac{1}{3+4x^3}$  to get  $(3+4x^3)^2$  in the denom:

$$\begin{aligned} \bullet \quad \frac{d}{dx} \left( \frac{1}{3+4x^3} \right) &= \frac{d}{dx} (3+4x^3)^{-1} = -1 (3+4x^3)^{-2} (3+4x^3)' \\ &= -\frac{12x^2}{(3+4x^3)^2} \end{aligned}$$

We'll worry about the  $-12x^2$  later. For now:

$$\begin{aligned} \bullet \quad \frac{d}{dx} \left( \frac{1}{3+4x^3} \right) &= \frac{d}{dx} \left( \frac{1}{3(1+\frac{4}{3}x^3)} \right) = \frac{1}{3} \frac{d}{dx} \left( \frac{1}{1-(\frac{4}{3}x^3)} \right) \\ &= \frac{1}{3} \frac{d}{dx} \sum_{n=0}^{\infty} \left( -\frac{4}{3}x^3 \right)^n = \frac{1}{3} \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} x^{3n}, \quad \left| -\frac{4}{3}x^3 \right| < 1 \\ &= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} \frac{d}{dx} (x^{3n}), \quad |x^3| < \frac{3}{4} \\ &= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} \cdot 3n x^{3n-1}, \quad |x| < \sqrt[3]{\frac{3}{4}} \end{aligned}$$

$$\text{Hence:} \quad \frac{-12x^2}{(3+4x^3)^2} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} \cdot 3n x^{3n-1}$$

We want  $2x$  on the LHS to recover  $f(x)$ ; so divide both sides by  $-6x$ :

$$-\frac{1}{6x} \left[ \frac{-12x^2}{(3+4x^3)^2} \right] = -\frac{1}{6x} \left[ \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} \cdot 3n x^{3n-1} \right]$$

$$\boxed{\frac{2x}{(3+4x^3)^2} = -\frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^n} \cdot n x^{3n-2}, \quad |x| < \sqrt[3]{\frac{3}{4}}}$$

24. We know  $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$ .

so  $a = \frac{1}{3}$  here and:

$$\int \frac{1}{x^2 + \frac{1}{9}} dx = 3 \arctan 3x + C.$$

$$\int \frac{1}{\frac{1}{9}(1+9x^2)} dx = 3 \arctan 3x + C$$

Use series for  $\frac{1}{1-u}$

$$9 \int \frac{1}{1-(-9x^2)} dx = 3 \arctan 3x + C$$

So:  $3 \int \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n} dx = \arctan 3x + \tilde{C}, | -9x^2 | < 1.$

$$3 \sum_{n=0}^{\infty} (-1)^n 9^n \int x^{2n} dx = \arctan 3x + \tilde{C}, |x^2| < \frac{1}{9}.$$

$$3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan 3x + \tilde{C}, |x| < \frac{1}{3}.$$

Letting  $x=0 \Rightarrow C=0$  so.

$$\boxed{\arctan 3x = 3 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, |x| < \frac{1}{3}}$$

25. We know  $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$  so  $\ln(1+x) = \int \frac{1}{1+x} dx$ .

$$\begin{aligned} \text{Also: } \int \frac{1}{1+x} dx &= \int \frac{1}{1-(-x)} dx = \int \sum_{n=0}^{\infty} (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx, |x| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C. \end{aligned}$$

So:  $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C, |x| < 1.$

Letting  $x=0 \Rightarrow C=0$ , so:

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, |x| < 1}$$

$$26. f(x) = x^2 \ln(1-3x^2).$$

Already a polynomial!

Note:  $\frac{d}{dx} \ln(1-3x^2) = \frac{1}{1-3x^2} (1-3x^2)' = -\frac{6x}{1-3x^2}.$

So:  $\ln(1-3x^2) = \int -\frac{6x}{1-3x^2} dx \quad (1)$

Also:

$$-\frac{6x}{1-3x^2} = -6x \frac{1}{1-(-3x^2)}$$

$$= -6x \sum_{n=0}^{\infty} (-3x^2)^n, \quad |-3x^2| < 1$$

$$= -6x \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n}, \quad |x^2| < \frac{1}{3}$$

$$= -6 \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n+1}, \quad |x| < \frac{1}{\sqrt{3}}. \quad (2)$$

So:  $\ln(1-3x^2) = \int -6 \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n+1} dx \leftarrow \text{Plug (2) into (1)}$

$$= -6 \sum_{n=0}^{\infty} (-1)^n 3^n \int x^{2n+1} dx$$

$$\ln(1-3x^2) = -6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+2}}{2n+2} + C$$

Letting  $x=0 \Rightarrow C=0.$

So:  $x^2 \ln(1-3x^2) = x^2 \left[ -6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+2}}{2n+2} \right]$

$$x^2 \ln(1-3x^2) = -6 \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{2n+4}}{2n+2}$$



V.

$$27. (x^2+1) \sin x = (x^2+1) \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$$

Collect like powers:

$$= x + \left(1 - \frac{1}{3!}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{3!}\right)x^5 + \left(\frac{1}{5!} - \frac{1}{7!}\right)x^7 + \dots$$

$$= \boxed{x + \frac{5}{6}x^3 - \frac{19}{120}x^5 + \frac{41}{5040}x^7 + \dots}$$

$$28. f(x) = e^x \cos x$$

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

Collect like powers:

$$= 1 + x + \left(-\frac{1}{2!} + \frac{1}{2!}\right)x^2 + \left(-\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{2!} + \frac{1}{4!}\right)x^4 + \dots$$

$$= \boxed{1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots}$$

Note: Here, if we identify  $e^x \cos x = \sum_{n=0}^{\infty} a_n x^n$ , note  $a_2 = 0$ .

We also have  $a_2 = \frac{f^{(2)}(0)}{2!} \Rightarrow \underline{f''(0) = 0}$ .

We could of course use this to compute other derivatives at  $x=0$ ;

$$a_4 = -\frac{1}{6} = \frac{f^{(4)}(0)}{4!} \Rightarrow \underline{f^{(4)}(0) = -4}$$
, etc!

$$29. f(x) = \frac{e^x}{1-x} = e^x \cdot \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(1 + x + x^2 + x^3 + \dots\right)$$

Collect like powers:

$$= 1 + (1+1)x + \left(1+1+\frac{1}{2!}\right)x^2 + \left(1+1+\frac{1}{2!}+\frac{1}{3!}\right)x^3 + \dots$$

$$= \boxed{1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots}$$

$$30. f(x) = \sin 2x + 3 \cos x.$$

Note:  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots$$

So:  $\sin 2x + 3 \cos x = \left(2x - \frac{8x^3}{3!} + \frac{2^5 x^5}{5!} + \dots\right) + \left(3 - 3\frac{x^2}{2!} + 3\frac{x^4}{4!} - \dots\right)$

$$= \boxed{3 + 2x - \frac{3}{2}x^2 - \frac{4}{3}x^3 + \dots}$$

$$31.a) \lim_{x \rightarrow 0} \frac{x \sin x - x^2}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - x^2}{\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right) - 1}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x^2} - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \cancel{x^2}}{\frac{1}{2}x^2 + \dots}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x^2} \left(-\frac{1}{6} + \frac{1}{120}x^2 + \dots\right)}{\cancel{x^2} \left(\frac{1}{2} - \frac{1}{4}x^2 + \dots\right)}$$

0 in the denom.

$$= \boxed{0}$$

$$b) \lim_{x \rightarrow 0} \frac{e^{x^2} - 1 - x^2}{\sin 2x} = \lim_{x \rightarrow 0} \frac{\left(\cancel{1} + \cancel{x^2} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots\right) - 1 - x^2}{2x - \frac{(2x)^3}{3!} + \dots}$$

$$= \lim_{x \rightarrow 0} \frac{x^4 \left(\frac{1}{2!} + \frac{1}{3!}x^2 + \dots\right)}{x \left(2 - \frac{8}{6}x^2 + \dots\right)}$$

$$= \lim_{x \rightarrow 0} x^3 \frac{\frac{1}{2!} + \frac{1}{3!}x^2 + \dots}{2 - \frac{8}{6}x^2 + \dots} \leftarrow \text{Goes to } \frac{1}{2!} \text{ when } x \rightarrow 0!$$

$$= \boxed{0}$$

$$\begin{aligned}
 c) \quad \lim_{x \rightarrow 0} \frac{\cos x^2 - 1}{x^3(e^x - 1)} &= \lim_{x \rightarrow 0} \frac{\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \dots\right) - 1}{x^3 \left(1 + x + \frac{x^2}{2!} + \dots\right)} \\
 &= \lim_{x \rightarrow 0} \frac{-\frac{1}{2!} x^4 + \frac{1}{4!} x^8 - \dots}{x^4 + \frac{1}{2} x^5 + \dots} \\
 &= \lim_{x \rightarrow 0} \frac{-\frac{1}{2!} + \frac{1}{4!} x^4 - \dots}{1 + \frac{1}{2} x + \dots}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \lim_{x \rightarrow 0} \frac{\sin 3x - 3xe^x}{4 \cos 4x - 4} &= \lim_{x \rightarrow 0} \frac{\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots\right) - 3x \left(1 + x + \frac{x^2}{2!} + \dots\right)}{4 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 4} \\
 &= \lim_{x \rightarrow 0} \frac{3x - \frac{9}{2} x^3 + \frac{81}{40} x^5 - \dots - 3x - 3x^2 - \frac{3}{2} x^3 + \dots}{4 - 2x^2 + \frac{1}{6} x^4 - \dots - 4} \\
 &= \lim_{x \rightarrow 0} \frac{-3 - 6x + \dots}{-2 + \frac{1}{6} x^2 + \dots} \\
 &= \boxed{\frac{3}{2}}
 \end{aligned}$$

32. We know  $\arctan x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-x^2)^n = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$

Letting  $x=0 \Rightarrow \underline{C=0}$  so:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + C$$

IF the series converges at  $x=1$ , which we are told it does.

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Note that  $x=1$  is an endpoint of the IOC!

33. We showed in #25:

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \begin{array}{l} \swarrow \\ \text{write out the sum!} \end{array} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

and that the ROC is  $|x| < 1$ . We do not know if the series converges at  $x=1$  (It actually does for the same reason as in #32, but we don't discuss this!). To get around this,

let  $x = -\frac{1}{2}$ :

$$\ln\left(1 - \frac{1}{2}\right) = -\frac{1}{2} - \frac{\left(-\frac{1}{2}\right)^2}{2} + \frac{\left(-\frac{1}{2}\right)^3}{3} - \frac{\left(-\frac{1}{2}\right)^4}{4} + \dots$$

$$\ln \frac{1}{2} = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} + \frac{1}{64} + \dots$$

$$-\ln 2 = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} + \frac{1}{64} + \dots$$

$$\boxed{\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \dots}$$

$$34. \int_0^1 \cos x^2 dx \approx \int_0^1 \left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!}\right) dx$$

$$\approx \int_0^1 \left(1 - \frac{1}{2}x^4 + \frac{1}{24}x^8\right) dx$$

$$= \left[ x - \frac{1}{10}x^5 + \frac{1}{216}x^9 \right]_0^1$$

$$= \left[ 1 - \frac{1}{10}(1)^5 + \frac{1}{216}(1)^9 \right] - 0$$

$$= \boxed{.09999900463}$$

A computer shows that to 11 decimal places:  $\int_0^1 \cos x^2 dx = .09999900000$

$$\begin{aligned} b) \int_0^{.1} e^{x^2} dx &\approx \int_0^{.1} \left[ 1 + (x^2) + \frac{(x^2)^2}{2} \right] dx \\ &= \int_0^{.1} \left( 1 + x^2 + \frac{1}{2} x^4 + \dots \right) dx \\ &= x + \frac{1}{3} x^3 + \frac{1}{10} x^5 \Big|_0^{.1} \\ &= \left[ (.1) + \frac{1}{3} (.1)^3 + \frac{1}{10} (.1)^5 \right] - 0 \\ &= \boxed{.1003334333} \end{aligned}$$

A computer gives to 11 decimal places:

$$\int_0^{.1} e^{x^2} dx \approx \underline{\underline{.1003343357}}$$



# Additional Questions Solutions

35. 
$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} (x-3)^{2k}$$

a) To find the radius of convergence, use Ratio Test:

$$L(x) = \lim_{k \rightarrow \infty} \left| \frac{(x-3)^{2(k+1)}}{4^{k+1}} \cdot \frac{4^k}{(x-3)^{2k}} \right| \leftarrow \begin{array}{l} \text{Note: } \left| \frac{(-1)^k (x-3)^k}{4^k} \right| = \frac{|(-1)^k| |x-3|^k}{|4^k|} \\ = \frac{|x-3|^k}{|4^k|} \end{array}$$

$$= \lim_{k \rightarrow \infty} \left| \frac{4^k}{4^{k+1}} \cdot \frac{(x-3)^{2k+2}}{(x-3)^{2k}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{4^k}{4^k 4} \cdot \frac{(x-3)^{2k} (x-3)^2}{(x-3)^{2k}} \right|$$

$$L(x) = \frac{|x-3|^2}{4}$$

( $|ab| = |a||b|$ , so we can ignore the  $(-1)^k$  terms!)

Ratio Test guarantees the series will converge for all value of  $x$  for which  $L(x) < 1$ .  $\rightarrow$

$$L(x) = \frac{|x-3|^2}{4} < 1$$

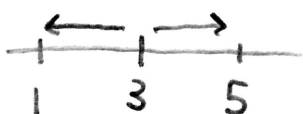
$$|x-3|^2 < 4$$

$$\underline{|x-3| < 2.} \leftarrow$$

Do NOT say  $|x|-3 < 2$ !  
 $|a+b| \neq |a|+|b|!!!$   
(ex: let  $a=-2, b=2$ )

The radius of convergence is 2.

b) To find the open interval, note the series is centered at  $x=3$ :



The radius is the maximum distance we can go from the center, so the **IOC is:  $(1, 5)$** .

c) The point  $x=6$  is not in the interval of convergence, so the series for  **$f(6)$  diverges**.

d) Write out the first several terms:

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} (x-3)^{2k}$$

Compute deriv:

$$f(x) = 1 - \frac{1}{16}(x-3)^2 + \frac{1}{64}(x-3)^4 - \dots \rightarrow \boxed{f(3) = 1}$$

$$f'(x) = -\frac{1}{8}(x-3) + \frac{1}{16}(x-3)^3 - \dots \rightarrow \boxed{f'(3) = 0}$$

$$f''(x) = -\frac{1}{8} + \frac{3}{16}(x-3)^2 - \dots \rightarrow \boxed{f''(3) = -\frac{1}{8}}$$

$$f'''(x) = \frac{3}{8}(x-3) - \dots \rightarrow \boxed{f'''(3) = 0}$$

\* Note: We can also use  $a_k = \frac{f^{(k)}(c)}{k!}$  to write:

$$f^{(k)}(3) = k! \cdot a_k$$

and extract the  $a_k$  from the Taylor series  $f(x) = \sum_{k=0}^{\infty} a_k (x-3)^k$

Note that the index  $k$  in  $a_k$  must match the power  $k$  on  $x^k$

Here,  $a_0 = 1$ ,  $a_1 = 0$ ,  $a_2 = -\frac{1}{16}$ ,  $a_3 = 0$ ,  $a_4 = \frac{1}{64}$ , etc

36.  $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$

a) To find the ROC, use ratio test.

$$L(x) = \lim_{k \rightarrow \infty} \left| \frac{x^{(k+1)+1}}{2(k+1)} \cdot \frac{2k}{x^{k+1}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{x^{k+2}}{x^{k+1}} \cdot \frac{2k}{2k+2} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{\cancel{x^{k+1}} \cdot x^1}{\cancel{x^{k+1}}} \cdot \frac{2k}{2k+2} \right|$$

$$= |x| \lim_{k \rightarrow \infty} \left( \frac{2k}{2k+2} \right)$$



$$L(x) = |x|$$

The Ratio Test ensures that the series will converge for all  $x$  such that  $L(x) < 1$  and diverge for all  $x$  where  $L(x) > 1$ .

$$\rightarrow L(x) = |x| < 1$$

The ROC is  $|$

b) MAKE SURE YOU KNOW THAT "the first 4 nonzero terms" means that you need to write the sum of the first 4 powers of  $(x-c)$  whose coeff is not 0!

ex: "First 4 nonzero terms" in the series of  $x=0$  for  $\sin x$  is  $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

$$\text{Here, } f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k} = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{6}x^4 + \frac{1}{8}x^5 + \dots$$

$$\begin{aligned} f(3x) &= \frac{1}{2}(3x)^2 + \frac{1}{4}(3x)^3 + \frac{1}{6}(3x)^4 + \frac{1}{8}(3x)^5 + \dots \\ &= \frac{9}{2}x + \frac{27}{4}x^2 + \frac{27}{2}x^3 + \frac{243}{8}x^4 + \dots \end{aligned}$$

$$g(x) = x^2 f(3x) = x^2 \left[ \frac{9}{2}x + \frac{27}{4}x^2 + \frac{27}{2}x^3 + \frac{243}{8}x^4 + \dots \right]$$

$$g(x) = \frac{9}{2}x^3 + \frac{27}{4}x^4 + \frac{27}{2}x^5 + \frac{243}{8}x^6 + \dots$$

c) Differentiating does NOT change the ROC. Since the ROC for  $f(x)$  is  $|$ , the ROC for  $f'(x)$  is  $|$ . The series  $f(x)$  is centered at  $x=0$ , so the open IOC is  $(-1, 1)$ . Hence, <sup>the series for</sup>  $f(x)$  (and  $f'(x)$ ) will converge for any  $-1 < x < 1$  and diverge if  $|x| > 1$ .

$\rightarrow$  The series for  $f'(3)$  diverges

d)  $f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$  ← centered at  $x=0$ .

i) Recall: The Taylor series <sup>centered at  $x=c$</sup>  for  $f(x)$  is given by  $\sum_{k=0}^{\infty} a_k (x-c)^k$   
 where  $a_k = \frac{f^{(k)}(c)}{k!}$ .

Implicit to this is that  $a_k$  is the coeff of  $x^k$   
 (the index and powers should match!).

Since we want to find  $f''(0)$ , we use:

$$a_2 = \frac{f''(0)}{2!}$$

$$f''(0) = 2! a_2.$$

We find  $a_2$  by inspecting the Taylor series:

- Way 1: Write out  $f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \dots$   
 $\swarrow$   
 $a_2$  is the coeff of  $x^2 \rightarrow \underline{a_2 = \frac{1}{2}}$
- Way 2: We need to find the index  $k$  so  $x^{k+1} = x^2 \rightarrow k=1$ .  
 Thus  $a_2 = \frac{1}{2(1)} = \underline{\frac{1}{2}}$ .

Using  $f''(0) = 2! a_2$  gives:  $f''(0) = 2! \cdot \frac{1}{2} \rightarrow \boxed{f''(0) = 1}$

ii). Writing out:

Differentiate:  
 $f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{6}x^4 + \frac{1}{8}x^5 + \dots$

$$f'(x) = x + \frac{3}{4}x^2 + \frac{2}{3}x^3 + \frac{5}{8}x^4 + \dots$$

$$f''(x) = 1 + \frac{3}{2}x + 2x^2 + \frac{5}{2}x^3 + \dots$$

Plug in  $x=0$ :

$$f''(0) = 1 + 0 \rightarrow \boxed{f''(0) = 1}$$

$$c) f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$$

$$f'(x) = \frac{d}{dx} \left[ \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k} \right]$$

Differentiate term-by-term  
(interchange  $\frac{d}{dx}$  and  $\Sigma$ ).

$$= \sum_{k=1}^{\infty} \frac{d}{dx} \left[ \frac{x^{k+1}}{2k} \right]$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k} \frac{d}{dx} (x^{k+1})$$

$$f'(x) = \sum_{k=1}^{\infty} \frac{k+1}{2k} x^k$$

$$f''(x) = \frac{d}{dx} \left[ \sum_{k=1}^{\infty} \frac{k+1}{2k} x^k \right]$$

$$= \sum_{k=1}^{\infty} \frac{k+1}{2k} \frac{d}{dx} [x^k]$$

$$f''(x) = \sum_{k=1}^{\infty} \frac{k+1}{2k} x^{k-1}$$

Note when  $k=1$ ,  $x^{k-1} = x^0$   
So when you plug in  $x=0$ , you  
do NOT get 0 on the RHS!

$$f''(x) = 1 + \frac{3}{2}x + 2x^2 + \frac{5}{2}x^3 + \dots$$

$$\rightarrow \boxed{f''(0) = 1}$$

This matches what we found on  
the previous page!

$$37. f(x) = \int_0^x te^{-t^2} dt.$$

a) To use the definition, we note we are looking for the first 3 nonzero coefficients in the expansion:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

where  $a_k = \frac{f^{(k)}(0)}{k}$  (since  $c=0$  here).

Note:  $f(x) = \int_0^x te^{-t^2} dt$  so  $f(0) = \int_0^0 te^{-t^2} dt = 0$ .

$$f'(x) = \frac{d}{dx} \left[ \int_0^x te^{-t^2} dt \right] = xe^{-x^2} \text{ by the Fund Thm of Calculus}$$

$$\rightarrow f'(0) = 0 \rightarrow \underline{a_1 = 0}$$

$$\bullet f''(x) = e^{-x^2} - 2x^2e^{-x^2} \rightarrow \underline{f''(0) = 1} \text{ Hence } a_2 = \frac{f''(0)}{2!}$$

$$\underline{\underline{a_2 = \frac{1}{2}}}$$

$$\bullet f'''(x) = -2xe^{-x^2} - 4xe^{-x^2} + 4x^3e^{-x^2} \rightarrow \underline{f'''(0) = 0} \rightarrow \underline{a_3 = 0}$$

$$f'''(x) = -6xe^{-x^2} + 4x^3e^{-x^2}$$

$$\bullet f^{(4)}(x) = -6e^{-x^2} + 12x^2e^{-x^2} + 12x^2e^{-x^2} - 8x^4e^{-x^2}$$
$$= -6e^{-x^2} + 24x^2e^{-x^2} - 8x^4e^{-x^2}$$

$$\rightarrow f^{(4)}(0) = -6 \text{ so } a_4 = \frac{f^{(4)}(0)}{4!} = \frac{-6}{24}$$

$$\underline{\underline{a_4 = -\frac{1}{4}}}$$

$$\bullet f^{(5)}(x) = 12xe^{-x^2} + 48xe^{-x^2} - 48x^3e^{-x^2} - 32x^3e^{-x^2} + 16x^5e^{-x^2}$$
$$= 60xe^{-x^2} - 80x^3e^{-x^2} + 16x^5e^{-x^2}$$

$$f^{(5)}(0) = 0 \rightarrow a_5 = \frac{f^{(5)}(0)}{5!} \text{ so } \underline{a_5 = 0}$$

$$\bullet f^{(6)}(x) = 60e^{-x^2} - 120x^2e^{-x^2} - 240x^2e^{-x^2} + 160x^4e^{-x^2}$$
$$+ 80x^4e^{-x^2} - 32x^6e^{-x^2}$$

$$f^{(6)}(0) = 60 \text{ so } a_6 = \frac{f^{(6)}(0)}{6!} = \frac{60}{720}$$

$$\underline{\underline{a_6 = \frac{1}{12}}}$$

Thus, the first 3 nonzero terms in the Taylor series centered at  $x=0$  for  $f(x) = \int_0^x t e^{-t^2} dt$  are:  $f(x) = a_2 x^2 + a_4 x^4 + a_6 x^6$

$$f(x) = \frac{1}{2} x^2 - \frac{1}{4} x^4 + \frac{1}{12} x^6 + \dots$$

Remark: This is PAINFUL! We really never want to use the definition unless we have to!

b) Evaluate the definite integral:

$$\begin{aligned} \int_0^x t e^{-t^2} dt &= \int_{t=0}^{t=x} \cancel{\phi} \cdot e^u \frac{du}{\cancel{-2\phi}} \\ u &= -t^2 \\ du &= -2t dt \quad \Bigg| \quad = -\frac{1}{2} e^u \Big|_{t=0}^{t=x} \\ &= -\frac{1}{2} e^{-t^2} \Big|_0^x \end{aligned}$$

$$f(x) = -\frac{1}{2} e^{-x^2} + \frac{1}{2}$$

Find the T.S. using rules:

$$e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \dots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{1}{2} (-x^2)^2 + \frac{1}{6} (-x^2)^3$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6$$

$$-\frac{1}{2} e^{-x^2} = -\frac{1}{2} \left[ 1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6 + \dots \right]$$

$$= -\frac{1}{2} + \frac{1}{2} x^2 - \frac{1}{4} x^4 + \frac{1}{12} x^6 - \dots$$

$$f(x) = -\frac{1}{2} e^{-x^2} + \frac{1}{2} = \frac{1}{2} x^2 - \frac{1}{4} x^4 + \frac{1}{12} x^6 - \dots$$

c) · Write out the Taylor series first:

$$e^t = 1 + t + \frac{1}{2}t^2 + \dots$$

$$e^{-t^2} = 1 + (-t^2) + \frac{1}{2}(-t^2)^2 + \dots$$

$$= 1 - t^2 + \frac{1}{2}t^4 - \dots$$

$$te^{-t^2} = t - t^3 + \frac{1}{2}t^5 - \dots$$

· Integrate:

$$f(x) = \int_0^x te^{-t^2} dt = \int_0^x (t - t^3 + \frac{1}{2}t^5 - \dots) dt$$

$$= \left. \frac{1}{2}t^2 - \frac{1}{4}t^4 + \frac{1}{12}t^6 - \dots \right|_0^x$$

$$= \left( \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 - \dots \right) - 0.$$

$$f(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 - \dots$$

\* You get the same result in all 3 cases! This is not an accident!

LESSON: Choose the most efficient way to compute the T.S!

38.  $f(x) = \int_0^x \frac{2}{1-t^2} dt, x \geq 0.$

a) The integrand is unbounded when  $t = \pm 1$ , so if  $x \geq 1$ ,  $\frac{2}{1-t^2}$  will be unbounded on  $[0, x]$ .

The integral is improper if  $x \geq 1$

b). i) Compute  $\int_0^x \frac{2}{1-t^2} dt$ . ← **WARNING:**  $\int \frac{1}{1-t^2} dt \neq \arctan t + C$  !!!

Note:  $\frac{2}{1-t^2} = \frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$

Use partial fractions!

$$2 = (1-t)A + (1+t)B.$$

$t=1: 2 = 0A + 2B \rightarrow \underline{B=1}$

$t=-1: 2 = 2A + 0B$

$\underline{A=1}$

So:  $\int_0^x \frac{2}{1-t^2} dt = \int_0^x \left( \frac{1}{1+t} + \frac{1}{1-t} \right) dt$   
 $= \left[ \ln|1+t| - \ln|1-t| \right]_0^x$

$f(x) = \ln(1+x) - \ln(1-x)$

No "1" necessary since  $0 < x < 1$

We're given  $\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$  so  $\ln(1+x) = -\sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1}$

$f(x) = \ln(1+x) - \ln(1-x) = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$

$f(x) = \sum_{k=0}^{\infty} [(-1)^{k+1} + 1] \frac{1}{k+1} x^{k+1}$  ← lower indices match so add

Note: IF  $k$  is odd,  $-(-1)^{k+1} - 1 = 0$  so we only need sum over even indices; letting  $k=2m$  and noting  $-(-1)^{2m+1} + 1 = 2$  gives:

$f(x) = \sum_{m=0}^{\infty} \frac{2}{2m+1} x^{2m+1}$

ii) Find the Taylor series for  $\frac{2}{1-t^2}$  first:

$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$

$\frac{1}{1-t^2} = \sum_{k=0}^{\infty} (t^2)^k$

$\frac{2}{1-t^2} = 2 \sum_{k=0}^{\infty} t^{2k} = \sum_{k=0}^{\infty} 2t^{2k}$

Integrate:  $f(x) = \int_0^x \sum_{k=0}^{\infty} 2t^{2k} dt = \sum_{k=0}^{\infty} \int_0^x 2t^{2k} dt$  (int term-by-term (switch  $\int$  and  $\sum$ ))

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} t^{2k+1} \Big|_0^x$$

$$= \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1} \quad \leftarrow \text{Note: This is exactly } \sum_{m=0}^{\infty} \frac{2}{2m+1} x^{2m+1}$$

\* An easier variant of this question would be to ask for the first 3 or 4 nonzero terms in the Taylor series using each method rather than working in summation notation.

39.  $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$

a) We have two options for  $f'''(0)$ .

Way 1: Write out several terms in the series for  $f(x)$ :

$$f(x) = 1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \frac{1}{9}x^4 - \dots$$

Differentiate:

$$f'(x) = -\frac{1}{3} + \frac{2}{5}x - \frac{3}{7}x^2 + \frac{4}{9}x^3 - \dots$$

$$f''(x) = \frac{2}{5} - \frac{6}{7}x + \frac{4}{3}x^2 - \dots$$

$$f'''(x) = -\frac{6}{7} + \frac{8}{3}x - \dots$$

Plug in  $x=0$ :

$$f'''(0) = -\frac{6}{7}$$

This formula is derived using a generalization of the above procedure.

Way 2: Use  $a_3 = \frac{f'''(0)}{3!}$  to find  $f'''(0) = 3! a_3$ .

Note:  $a_3$  is the coefficient of the  $x^3$  power. From above, we see  $a_3 = -\frac{1}{7}$ . Alternatively, the  $x^k$  in  $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$  is  $x^3$  when  $k=3$ . The coeff is  $\frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$ .



So,  $f'''(0) = 3! a_3 = 3! \left(-\frac{1}{7}\right)$

$f'''(0) = -\frac{6}{7}$

b) To find  $f^{(10)}(0)$ , it would be awful to try to compute the series for  $f^{(10)}(x)$ . Instead, we'll use

$a_{10} = \frac{f^{(10)}(0)}{10!} \rightarrow f^{(10)}(0) = 10! a_{10}$

The power of  $x$  in  $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$  is  $x^{10}$  when  $k=10$ .

The coefficient is  $a_{10} = \frac{(-1)^{10}}{2(10)+1} = \frac{1}{21}$ . Thus,  $f^{(10)}(0) = \frac{10!}{21}$

40 a)  $f(x) = \sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$

a) We have 2 reasonable options to compute  $f''(0)$ .

Way 1: • Write out several terms in the series for  $f(x)$ .

$f(x) = x + \frac{1}{2} x^3 + \frac{5}{7} x^5 + \dots$

• Take 2 derivatives

$f'(x) = 1 + \frac{3}{2} x^2 + \frac{25}{7} x^4 + \dots$

$f''(x) = 3x + \frac{100}{7} x^3 + \dots$

• Plug in  $x=0$ :

$f''(0) = 0$

Way 2: Use  $a_2 = \frac{f''(0)}{2!} \rightarrow f''(0) = 2! a_2$ .

There is no " $x^2$ " term in the series, which means  $a_2 = 0$

(if a power is "missing", this tells us the corresponding coefficient is 0)

Alternatively, no integer index  $k$  makes the  $x^{2k+1}$  term be  $x^2$ !

Hence,  $a_2 = 0$  and  $f''(0) = 2! a_2 \rightarrow \boxed{f''(0) = 0}$

b) To find  $f^{(17)}(0)$ , we use the fact  $a_{17} = \frac{f^{(17)}(0)}{17!}$  to find

$$f^{(17)}(0) = 17! a_{17}.$$

To find  $a_{17}$ , we need to find which <sup>integer</sup> value of  $k$  makes the power of  $x$  in  $\sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$  be  $x^{17}$ .

$$\begin{aligned} \rightarrow 2k+1 &= 17 \\ \underline{k} &= 8. \end{aligned}$$

Thus, the coefficient in front of  $x^{17}$  is  $\frac{(8)^2+1}{3(8)+1} = \frac{65}{25} = \frac{13}{5}$

$$\rightarrow \underline{a_{17} = \frac{13}{5}}$$

Hence,  $f^{(17)}(0) = 17! a_{17}$

$$\boxed{f^{(17)}(0) = 17! \cdot \frac{13}{5}}$$

c) To find  $f^{(18)}(0)$ , we use the fact  $a_{18} = \frac{f^{(18)}(0)}{18!}$  to find

$$f^{(18)}(0) = 18! a_{18}$$

To find  $a_{18}$ , we need to find which <sup>integer</sup> value of  $k$  makes the power of  $x$  in  $\sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$  be  $x^{18}$ .

$$\begin{aligned} \rightarrow 2k+1 &= 18 \\ k &= \frac{17}{2} \end{aligned}$$

For a better understanding, actually write out to  $x^{19}$  in the series

There is no integer value of  $k$  to make there be an  $x^{18}$  term,

so  $a_{18} = 0$ . Hence,  $f^{(18)}(0) = 18! a_{18} \rightarrow \boxed{f^{(18)}(0) = 0}$

41. Computing  $\frac{d^{40}}{dx^{40}}(e^{x^2})$  would be a pain!

To find this derivative at  $x=0$ , we write the Taylor series for  $e^{x^2}$  at  $x=0$ :

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k$$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$

Note when  $k=20$ ,  $x^{2k} = x^{40}$ , so  $a_{40}$  (which is the coefficient in front of  $x^{40}$ ) is  $\frac{1}{20!}$ .

Since  $a_{40} = \frac{f^{(40)}(0)}{40!}$ ,  $f^{(40)}(0) = 40! a_{40}$

$$f^{(40)}(0) = 40! \cdot \frac{1}{20!}$$

42.  $f(x) = \sum_{k=1}^{\infty} a_k (x-1)^k$ .

We're given  $\sum_{k=1}^{\infty} 4^k a_k$  converges. To relate this to the function we're

given, note that if  $(x-1)^k = 4^k$ ,  $x=5$ . Plugging this into  $f(x)$ :

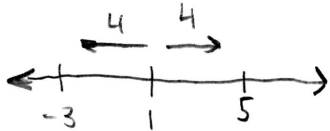
$$f(5) = \sum_{k=1}^{\infty} a_k (5-1)^k = \sum_{k=1}^{\infty} 4^k a_k.$$

Since  $f(5)$  converges, and  $f(x)$  is centered at  $x=1$ , we know the radius of convergence for  $f(x)$  is at least 4 (since  $x=5$  is a distance of 4 away from the center  $x=1$ ).

Note: The series of the form  $\sum_{k=1}^{\infty} a_k (x-c)^k$  is centered at  $x=c$ .

Hence,  $\sum_{k=1}^{\infty} a_k (x-1)^k$  is centered at  $\boxed{x=1}$

- a) As mentioned earlier, since the series is centered at  $x=1$ , and converges for  $x=5$ , the minimum ROC is 4, so the minimal open interval of convergence is  $(-3, 5)$ .



Since  $x=3$  is within this interval,

$\boxed{\text{the series for } f(3) \text{ converges}}$ .

- b) To determine if  $\sum_{k=1}^{\infty} a_k$  converges, we must relate this to

$$f(x) = \sum_{k=1}^{\infty} a_k (x-1)^k. \quad \text{Note when } x=2,$$

$$f(2) = \sum_{k=1}^{\infty} a_k (2-1)^k = \sum_{k=1}^{\infty} a_k (1)^k = \sum_{k=1}^{\infty} a_k.$$

Since the series converges for all  $x$  in  $(-3, 5)$ , the

series for  $f(2)$  converges so  $\boxed{f(2) = \sum_{k=1}^{\infty} a_k \text{ converges.}}$

43.  $f(x) = \sum_{k=1}^{\infty} a_k (2x+3)^k = \sum_{k=1}^{\infty} a_k \left[2\left(x+\frac{3}{2}\right)\right]^k = \sum_{k=1}^{\infty} 2^k a_k \left(x+\frac{3}{2}\right)^k$

- a) The series is centered at  $x = -\frac{3}{2}$

- b) It's more useful now to think of  $f(x)$  as  $\sum_{k=1}^{\infty} a_k (2x+3)^k$ .

We know  $\sum_{k=1}^{\infty} a_k$  converges. We need to relate this to  $f(x)$ , which we can do by noting  $|r|^k = 1$ , so set  $2x+3 = 1$ , then

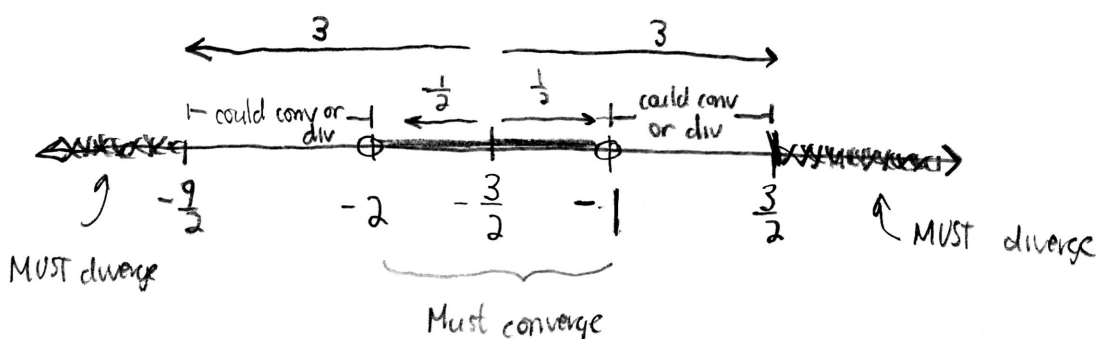
$$\text{Hence } f(-1) = \sum_{k=1}^{\infty} a_k (2(-1)+3)^k = \sum_{k=1}^{\infty} a_k 1^k = \sum_{k=1}^{\infty} a_k \quad \boxed{x = -1}.$$

The series is centered at  $x = -\frac{3}{2}$  and converges when  $x = -1$

which is a distance of  $\frac{1}{2}$  from the center, so the ROC of  $f(x)$  is at least  $\frac{1}{2}$ .

Similarly,  $f(\frac{3}{2}) = \sum_{k=1}^{\infty} a_k [2(\frac{3}{2})+3]^k = \sum_{k=1}^{\infty} 6^k a_k$

and since we know this diverges (and  $x = \frac{3}{2}$  is 3 units from the center  $x = -\frac{3}{2}$ ), the ROC of  $f(x)$  is at most 3.



Note, we have no information to determine what happens if  $-\frac{9}{2} < x < -2$  or  $-1 < x < \frac{3}{2}$ !

Hence, the series for  $f(0)$  and  $f(1)$  could converge or diverge however, the series for  $f(2)$  must diverge!

c) The series for  $f(-\frac{5}{4})$  must converge since  $-\frac{5}{4}$  is  $\frac{1}{4}$  from the center of the series and the ROC is at least  $\frac{1}{2}$ .

(Alternatively, the minimal IOC is  $(-1, -2)$  and  $x = -\frac{5}{4}$  is in this interval)

Since differentiating does not change the ROC, the minimal IOC for  $f'(x)$  must be  $(-1, -2)$ , so  $f'(-\frac{5}{4})$  must converge.

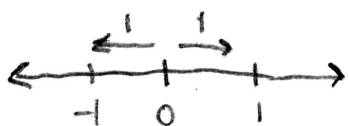
44. We know the (numeric) series  $\sum_{k=0}^{\infty} a_k$  converges.

a) Note that when  $x=1$ , the power series  $\sum_{k=0}^{\infty} a_k x^k$  becomes  $\sum_{k=0}^{\infty} a_k$ . So,

the power series is centered at  $x=0$  and converges at  $x=1$ . Thus the

minimal ROC is 1

b) We found above the minimal ROC is 1, so the minimal interval of convergence is  $(-1, 1)$ , meaning that



if  $-1 < x < 1$ , the series  $\sum_{k=0}^{\infty} a_k x^k$  must converge. In particular, if  $x = \frac{1}{2}$ , the series converges and.

$$\text{When } x = \frac{1}{2}, \quad \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k \left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} a_k \frac{1^k}{2^k} = \sum_{k=0}^{\infty} \frac{a_k}{2^k}$$

so  $\sum_{k=0}^{\infty} \frac{a_k}{2^k}$  converges.

(Note: If you take a convergent series and "speed up" its rate of convergence by multiplying <sup>each</sup>  $a_k$  by  $r^k$  for  $|r| < 1$ , the resulting series not surprisingly converges!)

c) There is no maximal ROC if we aren't given more information about  $a_k$ :

• If  $a_k = \frac{1}{k^2+k}$ ,  $\sum_{k=1}^{\infty} \frac{1}{k^2+k} \stackrel{\text{use partial fractions!}}{=} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ . This is a telescoping

series and, converges (you should be able to give details; if this is difficult, ask your instructors!)

Using the ratio test,  $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k^2+k}$  can be shown to have ROC=1

In this case the minimal and maximal ROC are 1.

• If  $a_k = \frac{1}{k!}$ , ratio test guarantees  $\sum \frac{1}{k!}$  converges, and  $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$  which is the series for  $e^x$ . We know this series has an infinite ROC so in

this case, the minimal ROC is 1 and the maximal ROC is infinite.

45.  $\sum_{k=0}^{\infty} a_k$  diverges.

a) There is no minimal ROC for  $\sum_{k=0}^{\infty} a_k x^k$ ; if  $a_k = k!$ ,

divergence test ensures  $\sum k!$  diverges. Using ratio test on

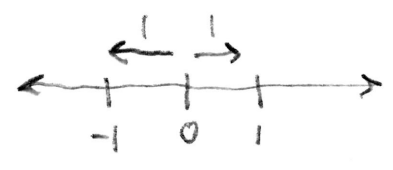
$\sum a_k x^k = \sum k! x^k$  shows the Taylor series has ROC 0.

However, if  $a_k = \frac{1}{k}$ ,  $\sum a_k = \sum \frac{1}{k}$  is the harmonic series and thus diverges, but  $\sum a_k x^k = \sum \frac{x^k}{k}$  has ROC 1 (use Ratio Test)

⇒ There is no minimal non-zero ROC; we need more info about  $a_k$

There is a maximal ROC of 1 however; the series  $\sum a_k x^k$  is centered at  $x=0$  and when  $x=1$ ,  $\sum a_k x^k = \sum a_k$ , which diverges. Hence, the largest possibility for the ROC is 1

b) Since the maximal ROC is 1, the maximal interval of convergence is  $(-1, 1)$ , meaning if  $x > 1$  or  $x < -1$ , the series  $\sum a_k x^k$  MUST diverge!



Note when  $x=2$ , the resulting series thus must diverge. When  $x=2$

$\sum a_k x^k = \sum a_k \cdot 2^k$ , so this series must diverge!

c) Nothing can be said about  $\sum \frac{a_k}{2^k}$ ;  
• If  $a_k = k$ , Ratio test will ensure  $\sum \frac{k}{2^k}$  converges  
• If  $a_k = 2^k$ , divergence test ensures  $\sum \frac{a_k}{2^k} = \sum 1$  diverges!  
In both cases  $\sum a_k$  diverges!

