Worksheet #8

I. Find the first 4 terms in the Taylor Series centered at $x=c$ for the following functions.

1. $f(x) = \sin x$, $c = \pi/3$
2. $f(x) = \ln(1+3x)$, $c = 2$
3. $f(x) = \sqrt{1+x}$, $c = 0$
4. $f(x) = e^{3x}$, $c = 1$.

II. Find the radius and interval of convergence for the following

5. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{3^n}$
6. $f(x) = \sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$
7. $f(x) = \sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$
8. $f(x) = \sum_{n=1}^{\infty} \frac{(2n)!\, x^n}{n^n}$
9. $f(x) = \sum_{n=1}^{\infty} 3^{2n+1}(x-4)^n$
10. $f(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{3n}}{8^n}$
11. $f(x) = \sum_{n=1}^{\infty} \frac{4^n x^{2n+1}}{n!}$
12. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{4n+1}$

III. Write down the Taylor series centered at $x=0$ for the following functions both in summation notation and by writing out the first 4 terms. In each case, give the radius of convergence.

13. $f(x) = xe^{5x}$
14. $f(x) = \frac{4}{3-2x}$
15. $f(x) = 5 \sin x^2$
16. $f(x) = \frac{2x}{1+x^2}$
17. $f(x) = \cos 5x$
18. $f(x) = \frac{1}{1+4x}$
19. $f(x) = 4x^3 \sin 4x$
20. $f(x) = \frac{4}{8x-1}$

IV. Integrating and Differentiating Taylor Series.

Repeat the directions from the previous section.

21. $f(x) = \dfrac{1}{(1-x)^2}$

22. $f(x) = \dfrac{3x}{(1+x)^2}$

23. $f(x) = \dfrac{2x}{(3+4x^3)^2}$

24. $f(x) = \arctan 3x$

25. $f(x) = \ln(1+x)$

26. $f(x) = x^2 \ln(1-3x^2)$

V. Write out the first 4 terms in the following Taylor Series.

27. $f(x) = (x^2+1)\sin x$

28. $f(x) = e^x \cos x$

29. $f(x) = \dfrac{e^x}{1-x}$

30. $f(x) = \sin 2x + 3\cos x.$

VI. Applications

31. Find the following limits using Taylor series.

a) $\displaystyle\lim_{x\to 0} \frac{x\sin x - x^2}{\cos x - 1}$

b) $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2}{\sin 2x}$

c) $\displaystyle\lim_{x\to 0} \frac{\cos(x^2) - 1}{x^3(e^x - 1)}$

d) $\displaystyle\lim_{x\to 0} \frac{\sin 3x - 3xe^x}{4\cos 4x - 4}.$

32. Using the Taylor series for $\arctan x$, show $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ given that the series converges at $x = 1$ (which we haven't shown).

33. Using the Taylor series for $\ln(1+x)$, show $\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \dots$

BE CAREFUL.

34. Use the first 3 terms in the Taylor series of the following integrands to approximate their values. Compare with the answers given by a computer.

a) $\displaystyle\int_0^{.1} \cos x^2 \, dx$

b) $\displaystyle\int_0^{.1} e^{x^2} \, dx.$

Taylor Series Additional Questions

35. Suppose the power series for an infinitely differentiable function $f(x)$ is given by:

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4k} (x-3)^{2k}$$

a) Find the radius of convergence.

b) Find the open interval of convergence (do not consider convergence at the endpoints).

c) Determine if the series for $f(6)$ converges

d) Find $f(3)$, $f'(3)$, $f''(3)$, and $f'''(3)$

36. Suppose the power series for an infinitely differentiable function $f(x)$ is given by:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^{k+1}}{2k}$$

a) Find the radius of convergence

b) Find the first 4 nonzero terms in the power series for $g(x) = x^2 f(3x)$.

c) Determine if the series for $f'(3)$ converges or diverges.

d) Find $f''(0)$ by:

i) Using the definition $a_k = \dfrac{f^{(k)}(c)}{k!}$

ii) By writing out the first 4 nonzero terms in the series for $f(x)$, differentiating twice, and plugging in $x = 0$.

iii) By computing the series for $f''(x)$ in summation notation then plugging in $x=0$.

37. Suppose $f(x) = \int_0^x te^{-t^2}\,dt$. Find the first 3 nonzero terms in the Taylor series for $f(x)$ centered at $x=0$ by:

a) Using the definition.

b) Evaluating the definite integral, then finding the Taylor series (integrate, then find a Taylor series)

c) Writing out the first 3 nonzero terms in the Taylor series for te^{-t^2}, then integrate

38. Suppose $f(x) = \int_0^x \frac{2}{1-t^2}\,dt$, $x \geq 0$.

a) For which values of x is the above integral improper?

b) Suppose $0 < x < 1$. Compute the Taylor series centered at 0 for $f(x)$ by:

i) Evaluating the definite integral, then finding the Taylor series.
(Hint: $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$)

ii) Writing out the Taylor series for $\frac{2}{1-t^2}$, substituting it into the integral, and evaluating the result.

39. Suppose $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^k$. Find:

a) $f'''(0)$

b) $f^{(10)}(0)$.

40. Suppose $f(x) = \sum_{k=0}^{\infty} \frac{k^2+1}{3k+1} x^{2k+1}$. Find:

a) $f''(0)$

b) $f^{(17)}(0)$

c) $f^{(18)}(0)$.

41. Using Taylor series, compute $\frac{d^{40}}{dx^{40}}\left(e^{x^2}\right)$ at $x=0$.

42. Suppose $f(x) = \sum_{k=1}^{\infty} a_k (x-1)^k$ and it is known that $\sum_{k=1}^{\infty} 4^k a_k$ converges.

a) Determine if the series given by $f(3)$ must converge or diverge or if there is not enough information to determine this.

b) Determine if $\sum_{k=1}^{\infty} a_k$ must converge or diverge or state there is not sufficient information to determine this.

43. Suppose that $f(x) = \sum_{k=1}^{\infty} a_k (2x+3)^k$.

a) What is the center of the series?

b) Given that $\sum_{k=1}^{\infty} a_k$ converges and that $\sum_{k=1}^{\infty} 6^k a_k$ diverges, determine if the series for $f(0)$, $f(1)$, and $f(2)$ must converge, could converge, or must diverge.

c) Must the series for $f'(-\frac{5}{4})$ converge? Why?

Series and Taylor Series

44. Suppose $\{a_k\}$ is a sequence for which $\sum_{k=0}^{\infty} a_k$ converges.

a) What is the minimum radius of convergence for $\sum_{k=0}^{\infty} a_k x^k$?

b) Deduce $\sum_{k=0}^{\infty} \frac{a_k}{2^k}$ converges.

c) Is there a maximal radius of convergence for $\sum_{k=0}^{\infty} a_k x^k$?

45. Suppose $\{a_k\}$ is a sequence for which $\sum_{k=0}^{\infty} a_k$ diverges.

a) Is there a minimal non-zero radius of convergence for $\sum_{k=0}^{\infty} a_k x^k$?

Is there a maximal radius of convergence?

b) Argue why $\sum_{k=0}^{\infty} 2^k a_k$ must diverge.

c) What can be said about $\sum_{k=0}^{\infty} \frac{a_k}{2^k}$?