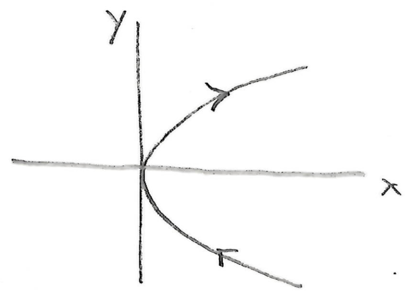
Worksheet #9 Solutions

I. Parametric Equations

a) $x = 2t^2$, $y = 3t$.

Solve for t from y since the expression is easier:

$$t = \frac{1}{3}y$$



So: $x = 2\left(\frac{1}{3}y\right)^2$

$$\boxed{x = \frac{2}{9}y^2}$$

As t increases, y increases, so the positive orientation is as indicated.

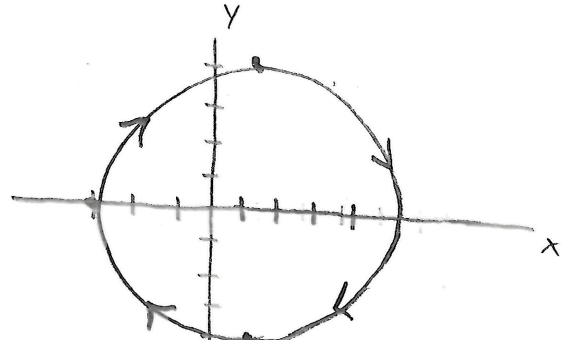
b) $x = 4\sin t + 1$, $y = 4\cos t$

$x - 1 = 4\sin t$ $\quad$ $y^2 = 16\cos^2 t$,

$(x-1)^2 = 16\sin^2 t$

So: $(x-1)^2 + y^2 = 16\sin^2 t + 16\cos^2 t$

$$\boxed{(x-1)^2 + y^2 = 16}$$



At $t=0$; start at $(1,4)$. As t increases, x increases, so the orientation is as shown.

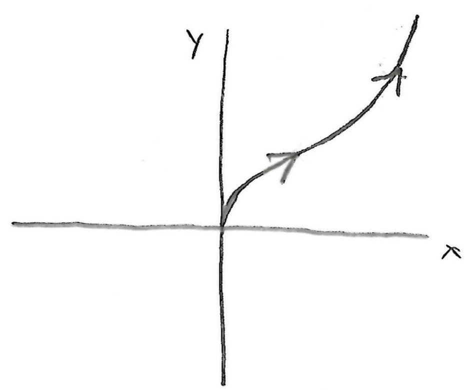
c) $x = 4e^t$, $y = e^{3t}$.

$\rightarrow e^t = \frac{1}{4}x$.

Since $y = e^{3t} = (e^t)^3$

$y = \left(\frac{1}{4}x\right)^3$

$$\boxed{y = \frac{1}{64}x^3}$$

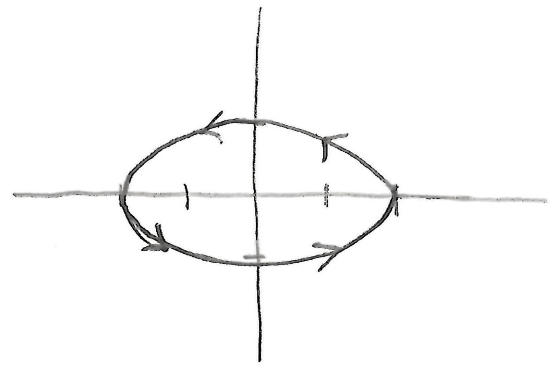


Note: x, y are always positive! As $t \to -\infty$ $(x,y) \to (0,0)$.

d) $x = 2\cos t \quad , \quad y = \sin t$

$\frac{1}{2}x = \cos t \qquad y = \sin t$

$\frac{1}{4}x^2 = \cos^2 t \qquad y^2 = \sin^2 t.$



$\frac{1}{4}x^2 + y^2 = \cos^2 t + \sin^2 t$

$\boxed{\frac{x^2}{4} + y^2 = 1}$ ← Ellipse

2. There are many different answers to these. Here are a few:

a) We can take $x = 2\cos t, \quad y - 1 = 2\sin t.$

But, when $t = 0$: $x = 2, \; y - 1 = 0$

So, try: $\boxed{x = 2\sin t, \qquad y = 1 + 2\cos t}$

A quick check shows $x(0) = 0, \quad y(0) = 3.$

b). The description above is traced out clockwise. Another would be

$\boxed{x = -2\cos t \quad , \qquad y = 1 + 2\sin t.}$

Note: The "−" changes the direction!

c) $x = 2\cos t \quad , \quad y = 1 + 2\sin t.$ will trace out the circle.

It starts at $(2, 0)$. We need 2 full rotations to occur between $t = 0$ and $t = 1$, so we need the argument of cos to be 0 at $t = 0$ and 4π when $t = 1$:

$\Rightarrow \quad \boxed{x = 2\cos 4\pi t \quad , \quad y = 1 + 2\sin 4\pi t}$

3a) $x = at \rightarrow t = \frac{x}{a}.$

Since $y = bt$, $y = b\left(\frac{x}{a}\right)$ or $\boxed{y = \frac{b}{a}x}$

b) When $t=0$, $y=b$, so try $\boxed{y=bt+b}$

we need $y=\frac{b}{a}x$

$$bt+b=\frac{b}{a}x.$$

$$\cancel{b}(t+1)=\frac{\cancel{b}}{a}x$$

$$\boxed{a(t+1)=x}$$

Note that we can simply replace $t\to t+1$ in the original expression!

4 a) $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\boxed{\frac{2t}{3t^2-3}}$

b) • Horizontal Tangents: occur when the numerator is 0:

$$2t=0\to \underline{t=0}$$

When $t=0$: $y=0^2=0$, $x=0^3-3(0)$ so $\boxed{\text{the horizontal tangent is } y=0 \text{ and occurs at } (0,0)}$

• Vertical Tangents: occur when the denom is 0:

$$3t^3-3=0$$
$$3(t^2-1)=0$$
$$3(t+1)(t-1)=0$$
$$\underline{t=-1,1.}$$

When $t=1$: $y=(1)^2=1$, $x=1^3-3(1)=-2$ so $\boxed{\text{the vertical tangent is } x=-2 \text{ and occurs at } (-2,1)}$

When $t=-1$: $y=(-1)^2=1$, $x=(-1)^3-3(-1)=2$ so $\boxed{\text{the vertical tangent is } x=2 \text{ and occurs at } (2,1)}$

c) When $t=2$:

- $\frac{dy}{dx} = \frac{2(2)}{3(2)^2-3} = \frac{4}{9}$. ← slope
- $x = (2)^3 - 3(2) = 2$
- $y = (2)^2 = 4$.

So:

$$y - y_0 = m(x - x_0) \leftarrow \text{eqn of a line}$$

$$\boxed{y - 4 = \frac{4}{9}(x-2)}$$

d) When $(x,y) = (2,1)$, we need to find t.

$$x = 2 = t^3 - 3t \leftarrow \text{Hard to solve!}$$

- Try y instead:

$$y = 1 = t^2 \Rightarrow t = 1 \text{ or } -1.$$

- Need to find which makes $x = 2$:

$t=1$: $x = 1^3 - 3(1) = -2$ ✗

$t=-1$ $x = (-1)^3 - 3(-1) = 2$ ✓

So: $t=-1$!

This corresponds to the solution found in 2b):

The tangent line there was found to be $\boxed{x=2}$

5. a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{t-1}{2t-6}}$

b) The horizontal tangents: Set $t-1=0$

$t=1$.

When $t=1$:

$x = 1^2 - 6(1) + 1 = -4$

$y = \frac{1}{2}(1)^2 - 1 = -\frac{1}{2}$

⇒ The horizontal tangent is $y = -\frac{1}{2}$ and occurs at $(-4, -\frac{1}{2})$

Vertical tangents: $2t-6=0 \rightarrow \underline{t=3}$

When $t=3$: $x = 3^2-6(3)+1 = -8$, $y = \frac{1}{2}(3)^2-3 = \frac{3}{2}$ $\rightarrow$ $\boxed{\text{The vertical tangent line is } x=-8 \text{ and occurs at } (-8, \tfrac{3}{2})}$

c). This was just found in b)!

d) When $(x,y) = (1,12)$, we need to find t.

$$x = 1 = t^2-6t+1$$
$$t^2-6t=0$$
$$t(t-6)=0 \rightarrow \underline{t=0}, \underline{t=6}.$$

- We need to find which t-value gives $y=12$:

$\underline{t=0}$: $y = \frac{1}{2}(0)^2-0=0$ ✗

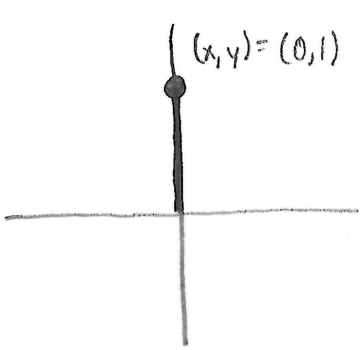
$\underline{t=6}$: $y=\frac{1}{2}(6)^2-6=12$ ✓.

So: $\left.\frac{dy}{dx}\right|_{t=6} = \frac{6-1}{2(6)-6} = \underline{\frac{5}{6}}.$ ← slope

- Tan line: $y-y_0 = m(x-x_0)$

$$\boxed{y-12=\frac{5}{6}(x-1)}$$

6. Consider this visually:



Way 1:	$r=1, \theta=\frac{\pi}{2}$	(rotate $r=1$ CCW by $\frac{\pi}{2}$).
Way 2:	$r=1, \theta=-\frac{3\pi}{2}$	(rotate $r=1$ CW by $\frac{3\pi}{2}$).
Way 3:	$r=-1, \theta=\frac{3\pi}{2}$	(rotate $r=-1$ CCW by $\frac{3\pi}{2}$).
Way 4:	$r=-1, \theta=-\frac{\pi}{2}$	(rotate $r=-1$, CW by $\frac{\pi}{2}$)

$* \begin{cases} \theta>0 \leftrightarrow \text{CCW rotation} \\ \theta<0 \leftrightarrow \text{CW rotation} \end{cases}$

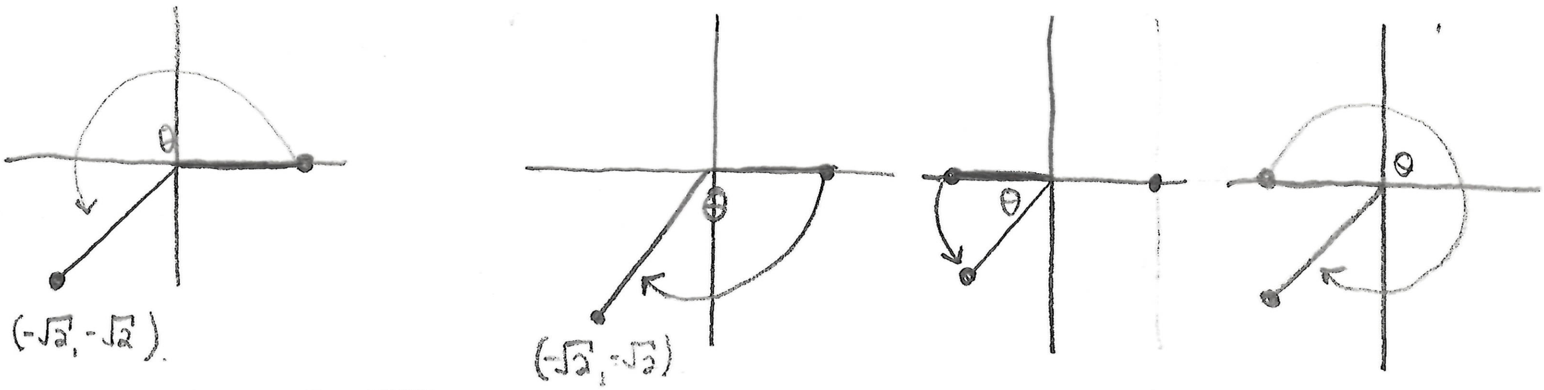
7. First note $r^2 = x^2+y^2 \rightarrow r^2 = (-\sqrt{2})^2 + (\sqrt{2})^2$

$$r^2 = 2+2$$

$$r = \pm 2.$$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$ So $\theta = \cancel{\tfrac{\pi}{4}}, \frac{5\pi}{4}$

Once again, think visually:



Way 1: $\boxed{r=2,\ \theta=\frac{5\pi}{4}}$ Way 2: $\boxed{r=2,\ \theta=-\frac{3\pi}{4}}$ Way 3: $\boxed{r=-2,\ \theta=\frac{\pi}{4}}$ Way 4 $\boxed{r=-2,\ \theta=-\frac{7\pi}{4}}$

8. $x = r\cos\theta = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = \underline{2}$

$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = \underline{2\sqrt{3}}$

So $\boxed{(x,y) = (2, 2\sqrt{3})}$

9.a) $r = 4\sec\theta$ ← convert to $\sin\theta$, $\cos\theta$

$$r = \frac{4}{\cos\theta}$$

$$\underbrace{r\cos\theta}_{} = 4$$

$$\boxed{x = 4}$$

b) $r^2 = \tan\theta$ ← convert to $\sin\theta$, $\cos\theta$

$r^2 = \frac{\sin\theta}{\cos\theta}$ ← make $r\sin\theta$, $r\cos\theta$ appear

$$r^2 = \frac{r\sin\theta}{r\cos\theta}$$

$$\boxed{x^2+y^2 = \frac{y}{x}}$$

c) $1 = \sin\theta\cos\theta.$

$$r^2 = r^2 \sin\theta \cos\theta$$

$$r^2 = (r\sin\theta)(r\cos\theta)$$

Want to force $r\cos\theta$, $r\sin\theta$ to appear!

$$\boxed{x^2+y^2 = yx.}$$

d) $r = 5$

$$\sqrt{x^2+y^2} = 5$$

$$\rightarrow \boxed{x^2+y^2 = 25}$$

10. $r = 2\cos\theta.$

a) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}.$

Note: • $x = r\cos\theta = (2\cos\theta)\cos\theta = 2\cos^2\theta$

$\rightarrow \dfrac{dx}{d\theta} = 4\cos\theta\sin\theta$

• $y = r\sin\theta = (2\cos\theta)\sin\theta$

$\rightarrow \dfrac{dy}{d\theta} = 2\cos^2\theta - 2\sin^2\theta$

So: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \boxed{\dfrac{2\cos^2\theta - 2\sin^2\theta}{4\cos\theta\sin\theta}}$

b) Horizontal Tangents: $2\cos^2\theta - 2\sin^2\theta = 0$

$$2 - 2\sin^2\theta - 2\sin^2\theta = 0$$

$$2 = 4\sin^2\theta$$

$$\pm\sqrt{\tfrac{1}{2}} = \sin\theta.$$

Noting $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, we find $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

We need the x,y-values.

θ	$r = 2\cos\theta$	$x = r\cos\theta$	$y = r\sin\theta$
$\pi/4$	$\sqrt{2}$	1	1
$3\pi/4$	$-\sqrt{2}$	1	-1
$5\pi/4$	$-\sqrt{2}$	1	1
$7\pi/4$	$\sqrt{2}$	1	-1

← ex: $\theta = \frac{\pi}{4}$: • $r = 2\cos\frac{\pi}{4} = \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}$

• $x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1$

The horizontal tangents are: $\boxed{\begin{array}{l} y = 1 \text{ at } (1,1) \\ y = -1 \text{ at } (1,-1) \end{array}}$

Vertical tangents: Need: $\frac{dy}{dx}$ is undefined → $4\cos\theta\sin\theta = 0$

→ $\sin\theta = 0$ or $\cos\theta = 0$

⇒ $\theta = 0, \pi$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

θ	$r = 2\cos\theta$	$x = r\cos\theta$	$y = r\sin\theta$
0	2	2	0
π	-2	2	0
$\pi/2$	0	0	0
$3\pi/2$	0	0	0

The vertical tangents are $\boxed{\begin{array}{l} x = 2 \text{ at } (2,0) \\ x = 0 \text{ at } (0,0) \end{array}}$

Note: Graphically, $r = \cos\theta$ is a circle of radius $\frac{1}{2}$ centered at $(1,0)$:

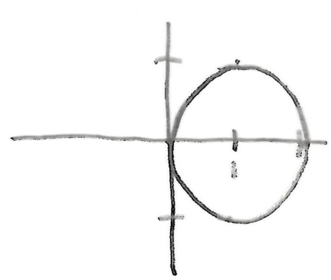
$r = 2\cos\theta$

$r^2 = 2r\cos\theta$

$x^2 + y^2 = 2x$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$.



← vertical tangents and horiz. tangents are what they should be!

c) • Find slope: $\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{2\cos^2\frac{\pi}{6} - 2\sin^2\frac{\pi}{6}}{4\cos\frac{\pi}{6}\sin\frac{\pi}{6}}$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2}{4\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}$$

$$= \frac{2\cdot\frac{3}{4} - 2\left(\frac{1}{4}\right)}{\sqrt{3}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{1}{\sqrt{3}}$$

• Find r: $r = 2\cos\theta = 2\cos\frac{\pi}{6} = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} \Rightarrow r = \sqrt{3}$.

• Find x, y: $x = r\cos\theta = \sqrt{3}\cos\frac{\pi}{6} = \sqrt{3}\cdot\frac{\sqrt{3}}{2} = \frac{3}{2}$

$y = r\sin\theta = \sqrt{3}\sin\frac{\pi}{6} = \sqrt{3}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$.

• Find tan line: $y - y_0 = m(x - x_0)$

$$\boxed{y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x - \frac{3}{2}\right)}$$

10. $r = 2 + 2\sin\theta$.

• $x = r\cos\theta = (2 + 2\sin\theta)\cos\theta = 2\cos\theta + 2\sin\theta\cos\theta$.

$dx/d\theta = -2\sin\theta + 2\cos^2\theta - 2\sin^2\theta$.

• $y = r\sin\theta = (2 + 2\sin\theta)\sin\theta = 2\sin\theta + 2\sin^2\theta$

$dy/d\theta = 2\cos\theta + 4\sin\theta\cos\theta$.

$$\rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta + 4\sin\theta\cos\theta}{-2\sin\theta + 2\cos^2\theta - 2\sin^2\theta}$$

Noting $\cos^2\theta = 1 - \sin^2\theta$, we can write after some algebra:

$$\frac{dy}{dx} = -\frac{\cos\theta(1 + 2\sin\theta)}{2\sin^2\theta + \sin\theta - 1} = -\frac{\cos\theta(1 + 2\sin\theta)}{(2\sin\theta - 1)(1 + \sin\theta)}.$$

$$\boxed{\frac{dy}{dx} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}}$$

b) <u>Horizontal Tangents:</u> $\frac{dy}{dx} = 0 \Rightarrow \cos\theta = 0$, or $1+2\sin\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Note when $\theta = \frac{3\pi}{2}$, the denominator is 0 as well!

- If $\lim_{\theta\to 3\pi/2} \frac{\cos\theta}{1+\sin\theta}$ exists, the limit laws tell us

$$\lim_{\theta\to 3\pi/2} \frac{\cos\theta}{1+\sin\theta}\frac{1+2\sin\theta}{1-2\sin\theta} = \lim_{\theta\to 3\pi/2} \frac{\cos\theta}{1+\sin\theta} \lim_{\theta\to 3\pi/2} \frac{1+2\sin\theta}{1-2\sin\theta}.$$

Note $\lim_{\theta\to 3\pi/2} \frac{\cos\theta}{1+\sin\theta} \overset{LH}{=} \lim_{\theta\to 3\pi/2} \frac{-\sin\theta}{\cos\theta} = -\frac{(-1)}{0} \leftarrow$ DNE!

<u>Thus, $\lim_{\theta\to 3\pi/2} \frac{dy}{dx}$ DNE and this corresponds to a vertical tangent!</u>

θ	$r = 2+2\sin\theta$	$x = r\cos\theta$	$y = r\sin\theta$
$\pi/2$	4	0	4
$7\pi/6$	1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$11\pi/6$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$.

<u>Horizontal Tangents:</u>

$y = 4$ at $(x,y) = (0,4)$

$y = -\frac{1}{2}$ at $(x,y) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$y = \frac{1}{2}$ at $(x,y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Vertical Tangents : $\frac{dy}{dx}$ is undefined → $1+\sin\theta=0$ or $1-2\sin\theta=0$.

$\theta=\frac{3\pi}{2}$ or $\theta=\frac{\pi}{6}, \frac{5\pi}{6}$.

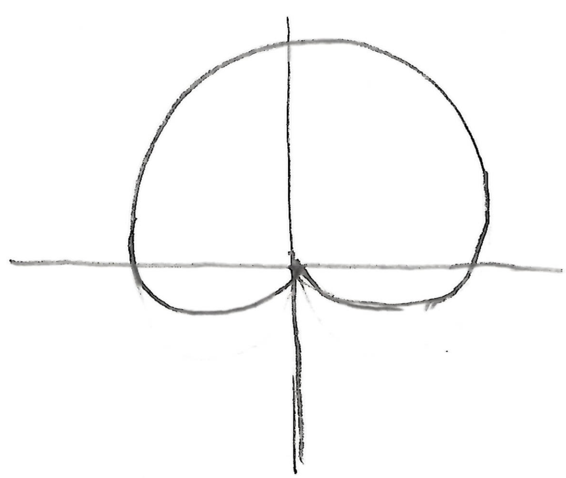
When $\theta=\frac{3\pi}{2}$, the numerator is also 0, but we showed previously the limit of $\frac{dy}{dx}$ as $\theta\to\frac{3\pi}{2}$ DNE, so $\theta=\frac{3\pi}{2}$ → a vertical tangent

θ	$r=2+2\sin\theta$	$x=r\cos\theta$	$y=r\sin\theta$
$3\pi/2$	0	0	0
$\pi/6$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{2}$
$5\pi/6$	3	$-\frac{3\sqrt{3}}{2}$	$\frac{3}{2}$.

Vertical Tangents:

$$\boxed{\begin{array}{ll} x=0 & \text{at } (0,0) \\ x=\frac{3\sqrt{3}}{2} & \text{at } \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \\ x=-\frac{3\sqrt{3}}{2} & \text{at } \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \end{array}}$$

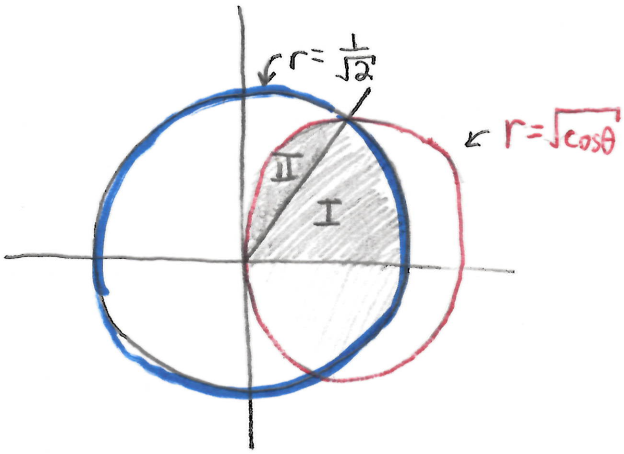
Note: The graph of $r=2+2\sin\theta$ is shown below:



The positions of the horizontal and vertical tangents are again justified graphically!

* For 12-15, you **MUST** draw a picture!!! *

12.



- We have 2 curves
- If we draw a ray extending from the origin into the region, the curve it hits changes! ⇒ mult integrals
- From symmetry, the area of the shaded region is twice the area of the dark region.

Int Pts: $\frac{1}{\sqrt{2}} = \sqrt{\cos\theta}$

$\frac{1}{2} = \cos\theta$

$\theta = \frac{\pi}{3}, \cancel{-\frac{\pi}{3}}$

$\rightarrow A_I = \frac{1}{2}\int_0^{\pi/3} r_I^2 \, d\theta = \frac{1}{2}\int_0^{\pi/3} \left(\frac{1}{\sqrt{2}}\right)^2 d\theta = \frac{\pi}{12}$

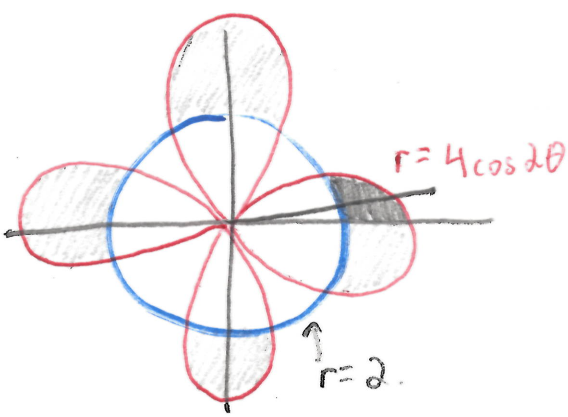
$A_{II} = \frac{1}{2}\int_{\pi/3}^{\pi/2} r_{II}^2 \, d\theta = \frac{1}{2}\int_{\pi/3}^{\pi/2} \cos\theta \, d\theta = \frac{1}{2}\sin\theta \Big|_{\pi/3}^{\pi/2}$

$= \frac{1}{2} - \frac{1}{2}\frac{\sqrt{3}}{2}$.

$$A = 2(A_I + A_{II}) = 2\left[\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}\right].$$

$$\boxed{A = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}}$$

13.



- We have 2 curves
- From symmetry, the total area is 8 times the area of the darkly shaded region
- In this region there is an inner and outer curve!

Limits: $\theta = 0$ (by inspection)

$4\cos 2\theta = 2$

$\cos 2\theta = \frac{1}{2} \rightarrow \quad 2\theta = \frac{\pi}{3}$

$\underline{\theta = \frac{\pi}{6}}$

So: $A = 8\left[\frac{1}{2}\int_0^{\pi/6} (r_{outer}^2 - r_{inner}^2)\, d\theta\right]$

$= 4\int_0^{\pi/6} (16\cos^2 2\theta - 4)\, d\theta.$

$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$

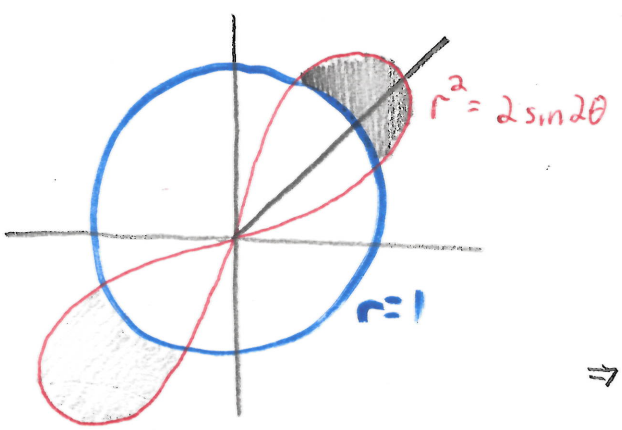
$= 4\int_0^{\pi/6} (8 + 8\cos 4\theta - 4)\, d\theta$

$= 4\left[\, 4\theta + 2\sin 4\theta \Big|_0^{\pi/6}\right].$

$= 4\left[\left(4\cdot\frac{\pi}{6} + 2\sin\frac{4\pi}{6}\right) - 0\right]$

$= \boxed{\frac{8\pi}{3} + 4\sqrt{3}}$

11.



- From symmetry, we know the total area is twice the shaded area
- There is an inner and outer curve.

$\Rightarrow \quad A = 2\left[\frac{1}{2}\int_{\pi/12}^{5\pi/12} (r_{outer}^2 - r_{inner}^2)\, d\theta\right]$

$\boxed{A = \int_{\pi/12}^{5\pi/12} (2\sin 2\theta - 1)\, d\theta}$

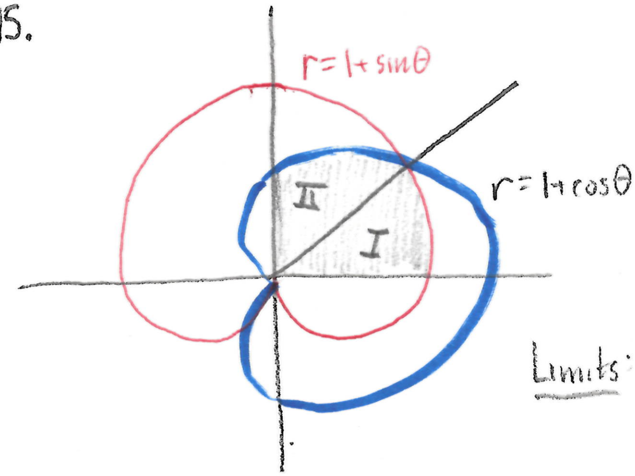
$2\sin 2\theta = 1$

$\sin 2\theta = \frac{1}{2}$

$\rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\underline{\theta = \frac{\pi}{12}, \frac{5\pi}{12}.}$

15.



· A curve in RI strikes only the red line

· A curve in RII strikes only the blue line

$\Rightarrow$ 2 integrals!

Limits: $1+\sin\theta = 1+\cos\theta$

$\sin\theta = \cos\theta$

$\tan\theta = 1$

$\rightarrow \theta = \frac{\pi}{4}$

So: $A_I = \frac{1}{2}\int_0^{\pi/4} r_I^2 \, d\theta \rightarrow A_I = \frac{1}{2}\int_0^{\pi/4} (1+\sin\theta)^2 \, d\theta.$

$A_{II} = \frac{1}{2}\int_{\pi/4}^{\pi/2} r_{II}^2 \, d\theta \rightarrow A_{II} = \frac{1}{2}\int_{\pi/4}^{\pi/2} (1+\cos\theta)^2 \, d\theta$

and $\boxed{A = A_I + A_{II}}$