Worksheet #11 Solutions

1 a) $\vec{r}\,'(t) = \langle -6\sin 3t, -6\cos 3t \rangle$

$|\vec{r}\,'(t)| = \sqrt{36\sin^2 3t + 36\cos^2 3t} = \underline{6.}$

So: $s = \int_0^\pi |\vec{r}\,'(t)|\,dt = \int_0^\pi 6\,dt = 6t\Big|_0^\pi = \boxed{6\pi}$

b) $\vec{r}\,'(t) = \langle 3t^2, 2t, 2t \rangle$

$|\vec{r}\,'(t)| = \sqrt{9t^4 + 4t^2 + 4t^2}$

$= \sqrt{t^2(9t^2+8)}$

$= |t|\sqrt{9t^2+8}$; $\leftarrow = t\sqrt{9t^2+8}$ when $t \geq 0$

So: $s = \int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 t\sqrt{9t^2+8}\,dt$

$= \int_0^1 t\,(9t^2+8)^{1/2}\,dt.$

$u = 9t^2+8$

$du = 18t\,dt$

$\frac{du}{18t} = dt.$

$= \int_{t=0}^{t=1} \cancel{t}\, u^{1/2} \frac{du}{18\cancel{t}}$

$= \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_{t=0}^{t=1}$

$= \frac{1}{27}\left[9t^2+8\right]^{3/2}\Big|_0^1$

$= \boxed{\frac{1}{27}\left[17^{3/2} - 8^{3/2}\right]}$

c) $\vec{r}'(t) = \langle 5,\ 6t^{1/2},\ -1 \rangle$

$|\vec{r}'(t)| = \sqrt{25 + 36t + 1} = \sqrt{26+36t}$.

$$s = \int_0^4 |\vec{r}'(t)|\, dt = \int_0^4 \sqrt{26+36t}\, dt$$

$$= \frac{1}{36} \cdot \frac{2}{3} (26+36t)^{3/2} \Big|_0^4$$

$$= \boxed{\frac{1}{54} \left[160^{3/2} - 26^{3/2} \right]}$$

2. In each of the problems above, $|\vec{r}'(t)| \neq 1$, so none of these curves is parameterized by arclength.

a) $s = \int_0^t |\vec{r}'(\tau)|\, d\tau = \int_0^t 6\, d\tau = 6\tau \Big|_0^t = 6t$.

So: $s = 6t \iff \underline{t = \frac{1}{6} s}$. and $\boxed{\vec{r}(s) = \langle 2\cos\frac{s}{2}, -2\sin\frac{s}{2} \rangle}$

b) $|\vec{r}(t)| = \tau(9\tau+8)^{3/2}$

$s = \int_0^t |\vec{r}'(\tau)|\, d\tau = \left[\frac{1}{27} (9\tau+8)^{3/2} \right|_0^t$ ← found in 1b)

$$s = \frac{1}{27} (9t+8)^{3/2} - \frac{1}{27} (8^{3/2})$$

$$27s - 8^{3/2} = (9t+8)^{3/2}$$

$$t = \frac{\left[27s - 8^{3/2} \right]^{2/3} - 8}{9}$$

So: $\boxed{\vec{r}(s) = \langle t^3, t^2, t^2 \rangle, \text{ where } t \text{ is as above}}$

c) $|\vec{r}\,'(\tau)| = \sqrt{26+36\tau}$

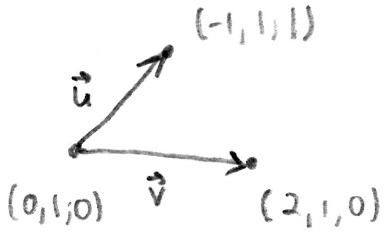
$$s = \int_0^t |\vec{r}\,'(\tau)|\,d\tau = \frac{1}{54}(26+36\tau)^{3/2}\Big|_0^t \quad \leftarrow \text{found in 1c)}$$

$$s = \frac{1}{54}(26+36t)^{3/2} - \frac{1}{54}(26^{3/2})$$

$$54s - (26)^{3/2} = (26+36t)^{3/2}$$

$$t = \frac{\left[54s-(26)^{3/2}\right]^{2/3} - 26}{36}$$

3.



$\vec{u} = \langle -1, 0, 1\rangle$

$\vec{v} = \langle 2, 0, 0\rangle$

$$\vec{u}\times\vec{v} = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\hat{\jmath}.$$

The plane is $\quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Using (0,1,0): $\quad 0(x-0) + 2(y-1) + 0(z-0) = 0$

$$2y - 2 = 0$$

$$\boxed{y = 1}$$

4. A vector normal to the plane is $\langle 2,-3,1\rangle$ since parallel planes have parallel normal vectors. The eqn is:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$2(x-1) + (-3)(y-2) + 1(z-3) = 0$$

$$2x - 2 - 3y + 6 + z - 3 = 0$$

$$\boxed{2x - 3y + z = -1}$$

5. We consider the normal vectors. Let the first plane listed be P_1, the second be P_2.

a) $\vec{n}_1 = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{n}_2 = \hat{i} - 2\hat{j} - 4\hat{k}$

$\vec{n}_1 \cdot \vec{n}_2 = 2 - 6 + 4 = \underline{0}$.

Hence, $\boxed{\text{the planes are } \perp}$

b) $\vec{n}_1 = \hat{i} + 7\hat{j} - 3\hat{k}$, $\vec{n}_2 = 2\hat{i} + 14\hat{j} - 6\hat{k}$

Since $2\vec{n}_1 = \vec{n}_2$ by inspection, $\boxed{\text{the planes are parallel}}$

c) $\vec{n}_1 = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{n}_2 = 2\hat{i} + \hat{j} - \hat{k}$

- $\vec{n}_1 \cdot \vec{n}_2 = 2 - 2 - 1 = -1 \neq 0 \rightarrow$ The planes aren't $\perp$.
- $\vec{n}_1 \neq (\text{const})\, \vec{n}_2$ so the planes aren't $\parallel$.

$\Rightarrow \boxed{\text{The planes are neither } \perp \text{ or } \parallel}$

6 a) The normal vectors are $\vec{n}_1 = 2\hat{i} - 3\hat{j} + \hat{k}$

$\vec{n}_2 = 5\hat{i} + \hat{j} + 2\hat{k}$

These are not multiples of each other, so the planes aren't parallel.

b) $\underline{P_1}$: $2x - 3y + z = 1 \rightarrow z = 1 - 2x + 3y$

$2z = 2 - 4x + 6y$

$\underline{P_2}$: $5x + y + 2z = 0 \rightarrow 2z = -5x - y$

$\underline{\text{Hence}}$: $2 - 4x + 6y = -5x - y$

$\underline{x = -2 - 7y.}$

Letting $y = t$, $x = -2 - 7t$ and $z = 1 - 2x - 3y$

$\rightarrow$ $z = 1 - 2(-2-7t) - 3(t)$

$z = 1 + 4 + 14t - 3t$

$z = 5 + 11t$

A parametric eqn for the line is: $\boxed{\vec{r}(t) = \langle -2-7t, t, 5+11t \rangle}$

7. a) $\lim_{(x,y)\to(1,2)} \dfrac{3x^2 - y - 1}{4x^2 y} = \dfrac{3(1)^2 - 2 - 1}{4(1)^2(2)} = \dfrac{0}{8} = \boxed{0}$

b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + 4xy + 4y^2}{2x + 4y} = \lim_{(x,y)\to(0,0)} \dfrac{(x+2y)^2}{2(x+2y)}$

$= \lim_{(x,y)\to(0,0)} \dfrac{x+2y}{2} = \boxed{0}$

Testing $(x,y) = (0,0)$ gives $\frac{0}{0}$.

c) Along $x = 0$: $f(0,y) = \dfrac{2(0) + 3(0)y}{4y^2 - 3(0)} = 0 \leftarrow f(0,y) \to 0$ as $(x,y) \to (0,0)$

Along $y = x$: $f(x,x) = \dfrac{2x^2 + 3x^2}{4x^2 - 3x^2} = 5 \leftarrow f(x,x) \to 5$ as $(x,y) \to (0,0)$

as $(x,y) \to (0,0)$

Since $f(x,y)$ tends to two different values along the 2 different paths above, the $\boxed{\text{limit DNE}}$.

d) Along $x = 0$: $f(0,y) = \dfrac{0 - y}{0 + y} = 1$ so $f(0,y) \to 1$ as $(x,y) \to (0,0)$

Along $y = 0$: $f(x,0) = \dfrac{x^3 - 0}{4x^3 - 0} = \dfrac{1}{4}$ so $f(x,0) \to \frac{1}{4}$ as $(x,y) \to (0,0)$

Since $f(x,y)$ tends to different values along different paths, $\boxed{\text{the limit DNE}}$

e) Along $x=0$, $f(0,y) \to \frac{1}{4}$

$y=0$, $f(x,0) \to \frac{2}{3}$.

Since the function approaches different values along different paths, $\boxed{\text{the limit DNE}}$

f) $\lim\limits_{(x,y)\to(0,0)} \frac{x+3e^y}{2x-e^y} = \frac{0+3}{0-1} = \boxed{-3}$.

9 a) $f(x,mx) = \frac{x-3(mx)^3}{2x+(mx)^2} = \frac{x-3m^3x^3}{2x+m^2x^2}$

$= \frac{1-3m^3x^2}{2+m^2x}$.

As $(x,y)\to(0,0)$, $f(x,mx) \to \frac{1}{2}$.

b) $f(my^2, y) = \frac{my^2-3y^3}{2my^2+y^2} = \frac{m-3y}{2m+1}$.

As $(x,y)\to(0,0)$, $f(my^2,y) \to \frac{m}{2m+1}$.

c) The limit DNE since the function approaches different values along the paths (my^2, y) as $y\to 0$.

10 a) $\boxed{f_x = 1, \quad f_y = 2}$

b) $f_x = \frac{\partial}{\partial x}(2xy) = 2y\frac{\partial}{\partial x}(x) \Rightarrow \boxed{2y = f_x}$

$f_y = \frac{\partial}{\partial y}(2xy) = 2x\frac{\partial}{\partial y}(y) \Rightarrow \boxed{2x = f_y}$

c) $f_x = \frac{\partial}{\partial x}(y^4 \sin 3xy^5) = y^4 \frac{\partial}{\partial x}(\sin 3xy^5)$

$f_x = y^4 \cos 3xy^5 \cdot \frac{\partial}{\partial x}(3xy^5)$

$\boxed{f_x = 3y^9 \cos 3xy^5}$

$\boxed{f_y = 4y^3 \sin 3xy^5 + y^4 \cos 3xy^5 \cdot 15xy^4}$

d) $f_x = 4e^{x^2y^3} \frac{\partial}{\partial x}(x^2y^3) = 4e^{x^2y^3} \cdot 2xy^3$

$\boxed{f_x = 8xy^3 e^{x^2y^3}}$

$\boxed{f_y = 12y^2 e^{x^2y^3}}$

e) $\boxed{\begin{array}{l} f_x = 3(5x+2y^8)^2 \cdot 5 \\ f_y = 3(5x+2y^8)^2 \cdot 16y^7 \end{array}}$

f) $\boxed{\begin{array}{l} f_x = \ln(3x-y) + x \cdot \frac{1}{3x-y} \cdot 3 \\ f_y = \frac{x}{3x-y}(-1) \end{array}}$

11. $f_x = 12x^2 + 3y e^{xy} \rightarrow \boxed{f_x(0,0) = 0}$

$f_y = 3xe^{3xy} - 2y \rightarrow \boxed{f_y(0,0) = 0}$

12. $f_x = 6x^2 + 4y$, $f_y = 4x$.

$f_{xy} = f_{yx}$!

$f_{xx} = \frac{\partial}{\partial x}(f_x) \rightarrow \boxed{f_{xx} = 12x}$ $f_{yx} = \frac{\partial}{\partial y}(f_x) \rightarrow \boxed{f_{xy} = 4}$

$f_{xy} = \frac{\partial}{\partial y}(f_x) \Rightarrow \boxed{f_{yx} = 4}$ $f_{yy} = \frac{\partial}{\partial y}(f_y) \rightarrow \boxed{f_{yy} = 0}$

13. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$. It is a pain to find f_x

So we compute $f_{yx} = \frac{\partial}{\partial x}(f_y)$ instead since Clairaut's Thm guarantees $f_{xy}(\frac{\pi}{4},1) = f_{yx}(\frac{\pi}{4},1)$.

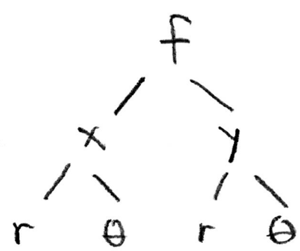
$$f_y = \cancel{\tan x} \cdot 4(1+y\cot x)^3 \cdot \cancel{\cot x}$$

$$\frac{\partial}{\partial x}(f_y) = 12(1+y\cot x)\cdot(-y\csc^2 x)$$

So $f_{yx}(\frac{\pi}{4},1) = 12(1+1\cdot 1)\cdot(-1\cdot(\sqrt{2})^2)$

$$= \boxed{-48}$$

14.



$r=1,\ \theta=\frac{\pi}{2}$: $x = r\cos\theta = 1\cos\frac{\pi}{2} = 0$

$y = r\sin\theta = 1\sin\frac{\pi}{2} = 1$

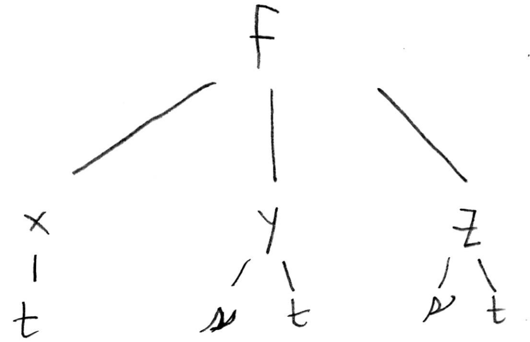
- $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 3x^2y\cdot\cos\theta + x^3\sin\theta$

$\rightarrow \frac{\partial f}{\partial r}(r=1,\theta=\frac{\pi}{2}) = \boxed{0}$

- $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = 3x^2y\,(-r\sin\theta) + x^3(r\cos\theta)$

When $r=1,\ \theta=\frac{\pi}{2}$: $\boxed{\frac{\partial f}{\partial \theta} = 0}$

15.



When $s=1,\ t=-2$:

$x = 3t = \underline{-6}$

$y = 3ts = \underline{-6}$

$z = \sin(2s+t) = \underline{0}$.

- $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial s}$

$$= 8xyz \cdot 3t + 4xy^2 \cdot \cos(2s+t)\cdot 2.$$

$$\Rightarrow f_s(1,-2) = 8(-6)(-6)(0)\cdot 3(-2) + 4(-6)(-6)^2 \cdot \cos(0)\cdot 2$$

$$\boxed{f_s(1,-2) = -1728}$$

- $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial t}$.

$$= \left(3x^2 + 4y^2z\right)\cdot 3 + 8xyz \cdot 3s + 4xy^2 \cdot \cos(2s+t)\cdot 1$$

$$\Rightarrow f_t(1,-2) = 3(-6)^2 + 0 + 0 + 4(-6)(-6)^2(1)(1).$$

$$\boxed{f_t(1,-2) = -864}$$

16. a) $f(x,y) = 4xy - 3y^2$

$f_x = 4y$

$f_y = 4x - 6y$ $\Rightarrow$ $\boxed{\nabla f(x,y) = \langle 4y,\ 4x-6y \rangle}$

b) $f(x,y) = e^{4x+8y}$

$f_x = 4e^{4x+8y}$

$f_y = 8e^{4x+8y}$ $\Rightarrow$ $\boxed{\nabla f(x,y) = \langle 4e^{4x+8y},\ 8e^{4x+8y} \rangle}$

c) $f(x,y) = x \sin y^2$

$f_x = \sin y^2$

$f_y = x \cos y \cdot 2y$ $\Rightarrow$ $\boxed{\nabla f(x,y) = \langle \sin y^2,\ 2xy \cos y \rangle}$

d) $f(x,y) = \frac{1}{x^2+y^2} = (x^2+y^2)^{-1}$

$f_x = -\frac{1}{(x^2+y^2)^2} \cdot 2x$

$f_y = -\frac{1}{(x^2+y^2)^2} \cdot 2y$ $\Rightarrow$ $\boxed{\nabla f(x,y) = \left\langle -\frac{2x}{(x^2+y^2)^2},\ \frac{-2y}{(x^2+y^2)^2} \right\rangle}$

17a) $f(x,y) = 8x^2 - 3xy$

$f_x = 16x - 3y$

$f_y = -3x$ $\Rightarrow$ $\nabla f(x,y) = \langle 16x - 3y, -3x \rangle$

$\boxed{\nabla f(1,0) = \langle 16, -3 \rangle}$

b) $\vec{u} = 3\hat{\imath} - 2\hat{\jmath}$

$|\vec{u}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

So $\hat{u} = \frac{3}{\sqrt{13}} \hat{\imath} - \frac{2}{\sqrt{13}} \hat{\jmath}$

$D_{\hat{u}} f(0,1) = \nabla f(0,1) \cdot \hat{u}$

$= \langle 16, -3 \rangle \cdot \left\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right\rangle$

$= \boxed{\frac{54}{\sqrt{13}}}$

c) The max rate of increase is $|\nabla f(0,1)| = \sqrt{16^2 + 3^2} = \boxed{\sqrt{265}}$ and occurs when $\hat{u}$ is in the direction of $\nabla f(0,1) \rightarrow$ $\boxed{\hat{u} = \frac{1}{\sqrt{265}} \langle 16, -3 \rangle}$

d) The level curve is $z = f(1,0) = 8(1)^2 - 3(0)(1) = 8$. so we have

$8 = 8x^2 - 3xy \rightarrow y = \frac{8}{3} x - \frac{8}{3} \frac{1}{x}$

This curve is parameterized by $x = t$, $y = \frac{8}{3}t - \frac{8}{3}$ $\rightarrow$ $\vec{r}(t) = \langle t, \frac{8}{3}t - \frac{8}{3t} \rangle$.

To show $\nabla f(Q, 1)$ is orthogonal to the level curve there, we must find a tangent vector: $\vec{r}\,'(t) = \langle 1, \frac{8}{3} + \frac{8}{3}\frac{1}{t^2} \rangle$.

So, when $(x, y) = (1, 0)$, $t = 1$ from our parameterization $\vec{r}(t)$

and at this point: $\nabla f(1,0) \cdot \vec{r}\,'(t=1) = \langle 16, -3 \rangle \cdot \langle 1, \frac{16}{3} \rangle$

$= 16 - 16$

$= \boxed{0}$.

18 a) $f(x,y) = \sin(4x^3 y)$

- $\nabla f = \langle f_x, f_y \rangle = \langle 12x^2 y \cos(4x^3 y), 4x^3 \cos(4x^3 y) \rangle$

$\nabla f(2,0) = \langle 0, 32 \rangle$.

- $|\vec{u}| = \sqrt{4^2 + 3^2} = 5$ so $\hat{u} = \langle \frac{4}{5}, \frac{3}{5} \rangle$.
- $D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u} = \boxed{\frac{96}{5}}$.

b) The max rate of decrease is $-|\nabla f(2,0)| = -\sqrt{0^2 + 32^2}$

$= \boxed{-32}$

it occurs in the direction of $-\nabla f(2,0) = \langle 0, -32 \rangle$, so $\boxed{\hat{u} = \langle 0, -1 \rangle}$

c) $\nabla f(2,0)$ will be orthogonal to the level curve $z = f(2,0) = 0$.

$0 = \sin 4x^3 y \Rightarrow 4x^3 y = n\pi$ for integers n.

So: $\quad y = \frac{n\pi}{4x^3}$ along these curves, which are thus parameterized

by $\quad \vec{r}(t) = \langle t, \frac{n\pi}{4} t^{-3} \rangle.$

Along these curves, $\quad \nabla f(x(t), y(t)) = \langle 12x^2 y \cos 4xy^3,\ 4x^3 \cos 4xy^3 \rangle \Big|_{4xy^3 = n\pi}$

$$= \langle 12t^2 \cdot \tfrac{n\pi}{4} t^{-3} \cos n\pi,\ 4t^3 \cos n\pi \rangle$$

$$= \langle 3n\pi t^{-1} \cos n\pi,\ 4t^3 \cos n\pi \rangle$$

and $\quad \vec{r}\,'(t) = \langle 1, -\frac{3n\pi}{4} t^{-4} \rangle$ so:

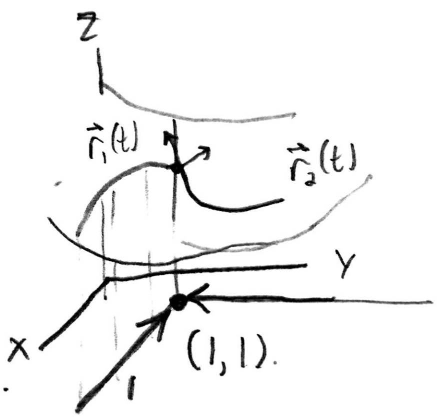
$$\nabla f \cdot \vec{r}\,' = 3n\pi t^{-1} \cos n\pi - 3n\pi t^{-1} \cos n\pi$$
$$= \boxed{0} \checkmark$$

19a) Curve 1: $\quad \vec{r}_1(x) = \langle x, 1, f(x, \ ) \rangle$

$\quad \vec{r}_1\,'(x) = \langle 1, 0, f_x(x, 1) \rangle$

Curve 2: $\quad \vec{r}_2(y) = \langle 1, y, f(1, y) \rangle.$

$\quad \vec{r}_2\,'(y) = \langle 0, 1, f_y(1, y) \rangle.$



when $(x,y) = (1,1)$:

$$\vec{r}_1\,' \times \vec{r}_2\,' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x(1,1) \\ 0 & 1 & f_y(1,1) \end{vmatrix} = -f_x(1,1)\hat{i} - f_y(1,1)\hat{j} + \hat{k}$$

So, the tangent plane is $\quad a(x-1) + b(y-1) + c(z-1) = 0$

$$-f_x(1,1)(x-1) \ \ f_y(1,1)(y-1) + 1(z-1) = 0$$

Since $f(x,y) = 3x^2 - 4y^2 + 2xy$

$f_x' = 6x + 2y \rightarrow f_x(1,1) = 6 + 2 = \underline{8}$

$f_y = -8y + 2x \rightarrow f_y(1,1) = -8 + 2 = \underline{-6}$

So the tan plane is $-8(x-1) - (-6)(y-1) + z - 1 = 0$

$-8x + 8 + 6y - 6 + z - 1 = 0$

$\boxed{-8x + 6y + z = -1}$

b) Using: $z = f(1,1) + \nabla f(1,1) \cdot \langle x-1, y-1 \rangle$

$= 1 + \langle 8, -6 \rangle \cdot \langle x-1, y-1 \rangle$

$z = 1 + 8(x-1) - 6(y-1)$

$z = 1 + 8x - 8 - 6y + 6.$

$\boxed{-8x + 6y + z = -1}$

20. We'll use $z = f(1,0) + \nabla f(1,0) \cdot \langle x-1, y-0 \rangle$

- $f(1,0) = 4 \sin 0 + 1 - 3(0) = \underline{1}$
- $\nabla f = \langle 4y \cos xy + 1, 4x \cos xy - 3 \rangle$

$\nabla f(1,0) = \langle 1, 1 \rangle.$

$\Rightarrow z = 1 + \langle 1, 1 \rangle \cdot \langle x-1, y \rangle$

$z = 1 + 1(x-1) + y$

$z = \cancel{1} + x - \cancel{1} + y \longrightarrow \boxed{x + y - z = 0}$