Worksheet #1

I. Arclength

1. Given the following vector-valued functions, find the length of the curve from $t=a$ to $t=b$.

   a) $\vec{r}(t) = \langle 2\cos 3t, -2\sin 3t \rangle$, $[a,b] = [0,\pi]$.

   b) $\vec{r}(t) = \langle t^3, t^2, t^2 \rangle$, $[a,b] = [0,1]$.

   c) $\vec{r}(t) = \langle 5t, 4t^{3/2}, -t \rangle$, $[a,b] = [0,4]$

2. For each of the vector-valued functions in 1., determine if the curve is parameterized by arclength. If it is not, find a description of the curve in terms of arclength.

II. Lines and Planes.

3. Find the equation of a plane passing through $(0,1,0)$, $(2,1,0)$, $(-1,1,1)$.

4. Find the equation of a plane parallel to $2x-3y+z=6$ that passes through $(1,2,3)$

5. Determine whether the following sets of planes are parallel, perpendicular, or neither.

   a) $2x+3y-z=4$, $x-2y-4z=5$.

   b) $x+7y-3z=6$, $2x+14y-6z=5$.

   c) $x-2y+z=5$, $2x+y-z=4$.

6. Consider the planes $2x-3y+z=1$, $5x+y+2z=0$.

a) Verify that the planes are not parallel

b) Find a parametric description of the line of intersection

III. Limits.

7. Determine whether or not the following limits exist. If they do, compute their value. If they do not, clearly explain why.

a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{3x^2-y-1}{4x^2y}$

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+4xy+4y^2}{2x+4y}.$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+3xy}{4y^2-3xy}.$

d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y}{4x^3+y}.$

e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x+y^2}{3x+4y^2}$

f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+3e^y}{2x-e^y}$

8. Show that $f(x,y)=\dfrac{4xy+y^2}{x^2+3y^2}$ depends on the choice of m if $y=mx$. Conclude $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

9. Let $f(x,y)=\dfrac{x-3y^3}{2x+y^2}$.

a) Show that along any path $y=mx$, $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$

b) Show that along a path $x=my^2$, the value of $f(x,y)$ as $(x,y)\to(0,0)$ depends on the choice of m.

c) Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

IV. Partial Derivatives

10. Given the following functions $f(x,y)$, find f_x and f_y.

a) $f(x,y) = x + 2y$

b) $f(x,y) = 2xy$

c) $f(x,y) = y^4 \sin(3xy^5)$

d) $f(x,y) = 4e^{x^2y^3}$

e) $f(x,y) = (5x + 2y^8)^3$

f) $f(x,y) = x\ln(3x - y)$

11. Given $f(x,y) = 4x^3 + e^{3xy} - y^2$, find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

12. Given $f(x,y) = 6x^3 + 4xy - 7$, calculate f_{xx}, f_{xy}, f_{yx}, f_{yy}. Verify $f_{xy} = f_{yx}$.

13. Find $f_{xy}(\frac{\pi}{4}, 1)$ if $f(x,y) = \tan x(1 + y\cot x)^4$.

V. Chain Rule

14. Given $f(x,y) = x^3y$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$ when $r = 1$, $\theta = \frac{\pi}{2}$.

15. Given $f(x,y,z) = x^3 + 4xy^2z$, $x = 3t$, $y = 3ts$, $z = \sin(2s+t)$ find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ when $(s,t) = (1,-2)$

VI. Gradients and Directional Derivatives

16. Given the following functions $f(x,y)$, compute $\nabla f(x,y)$.

a) $f(x,y) = 4xy - 3y^2$

b) $f(x,y) = e^{4x+8y}$

c) $f(x,y) = x\sin y^2$

d) $f(x,y) = \frac{1}{x^2+y^2}$

17. Given $f(x,y) = 8x^2 - 3xy$, find:

a) $\nabla f(1,0)$.

b) $D_{\hat{u}} f(1,0)$ when $\hat{u}$ is parallel to $\vec{u} = 3\hat{i} - 2\hat{j}$.

c) The maximum rate of increase for $f(x,y)$ at $(1,0)$ and the direction in which it occurs

d) Verify that $\nabla f(1,0)$ is orthogonal to the level curve $z = f(1,0)$, when $x=1$, $y=0$.

18. Given $f(x,y) = \sin(4x^3y)$, $\vec{u} = \langle 4,3 \rangle$, find:

a) $D_{\hat{u}} f(2,0)$, where $\hat{u}$ is in the direction of $\vec{u}$.

b) The maximum rate of decrease for $f(x,y)$ at $(2,0)$ and the direction in which it occurs.

c) To what level curve should $\nabla f(2,0)$ be orthogonal? Verify that $\nabla f(x,y)$ is orthogonal to this level curve!

VII. Tangent Planes

19. Let $f(x,y) = 3x^2 - 4y^2 + 2xy$.

Find the equation of the plane tangent to the surface at $(1,1,1)$ in 2 ways:

a) i) Find a vector $\vec{r}_1$ that is tangent to the curve given by $f(x,0)$ and a vector $\vec{r}_2$ that is tangent to $f(0,y)$. at the point $(x,y) = (1,1)$.

ii) These tangent vectors generate the tangent plane at $(1,1,1)$; $\vec{r}_1 \times \vec{r}_2$ will be normal to this plane! Using this find the equation of the plane.

b) Using the gradient.

20. Let $f(x,y) = 4\sin(xy) + x - 3y$. Find the tangent plane at $(x,y) = (1,0)$.