Worksheet #3 Answers

I. Arclength

1. $s = \frac{8}{775}\left[(226)^{3/2} - 1\right]$
2. $s = \ln(1+\sqrt{2})$
3. $s = 361/32$
4. $s = \frac{1}{6}\left[e^{3} - e^{-3}\right]$.

5 a) $m = \frac{1}{2}L^2$

b) $L = \sqrt{80}$ (or $4\sqrt{5}$)

6 a) $m = 10000 + 4a - a^4$

b) $a = 9.989$

7 a) $F = 50N$

b) $W = 1.25\,J$

c) $W = 3.75\,J$

8 a) $W = 80\,J$

b) $W = 240\,J$

c) $a = -4 + \sqrt{41}$

9 < see solution >

10. $W = 138544 24\,J$

11. $a = 20 - \frac{20}{7}\sqrt{35} \approx 3.097$.

12. $W = 1484401\,J$

13 a) $V = \frac{\pi h^2}{6}$

b) $W = \frac{9800\pi}{3}\left[6h^2 - \frac{1}{3}h^3\right]$

c) $h = 6$

14. The segment of $y = x^2$ is longer

15 a) $P_1 = P_2$

b) $a < L/2$.

16 a) Equal

b) Greater to stretch it the additional meter.

17 a) $h_2 > h_1$

b)

Worksheet # 3 Solutions

I. Arclength

1. $y = 1 + 5x^{3/2}$, $x = 1$ to $x = 3$:

$y' = \frac{15}{2} x^{1/2}$

$(y')^2 = \frac{225}{4} x.$

So: $s = \int_{x=0}^{x=4} \sqrt{1+(y')^2}\, dx$

$= \int_0^4 \sqrt{1 + \frac{225}{4}x}\, dx$

-OR-

$u = 1 + \frac{225}{4}x$

$du = \frac{225}{4} dx$

$\frac{4}{225} du = dx$

So $\int (1+\frac{225}{4}x)^{1/2} dx = \int u^{1/2} \frac{4}{225} du$

$= \frac{4}{225} \cdot \frac{2}{3} u^{3/2} + C$

$= \frac{4}{225} \cdot \frac{2}{3} \left(1 + \frac{225}{4}x\right)^{3/2} \Big|_0^4$

$= \frac{8}{775} \left[(1+225)^{3/2} - (1+0)^{3/2} \right]$

$$\boxed{s = \frac{8}{775}\left[(226)^{3/2} - 1\right]}$$

2. $x = \ln|\sec y|$, $y = 0$, $y = \frac{\pi}{4}$.

$x' = \frac{1}{\sec y} (\sec y)'$

$= \frac{1}{\sec y} \sec y \tan y$

$x' = \tan y$

So: $(x')^2 = \tan^2 y$

$s = \int_{y=0}^{y=\pi/4} \sqrt{1+(x')^2}\, dy$

$= \int_0^{\pi/4} \sqrt{1+\tan^2 y}\, dy$ ← Recall: $1+\tan^2 y = \sec^2 y$

$= \int_0^{\pi/4} \sqrt{\sec^2 y}\, dy$

Since $\sec y > 0$ on $0 \le y \le \frac{\pi}{4}$, $\sqrt{\sec^2 y} = \sec y$ and:

$$s = \int_0^{\pi/4} \sec y \, dy$$

$$= \ln|\sec y + \tan y| \Big|_0^{\pi/4}$$

$$= \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln|1+0|$$

$$\boxed{s = \ln(1+\sqrt{2})}$$

3. $y = \frac{3}{4}x^4 + \frac{1}{24x^2}$, $x=1$, $x=2$.

$$y = \frac{3}{4}x^4 + \frac{1}{24} \cdot \frac{1}{x^2}$$

$$y = \frac{3}{4}x^4 + \frac{1}{24}x^{-2}$$

$$\Rightarrow y' = 3x^3 - \frac{1}{12}x^{-3}$$

$$(y')^2 = \left(3x^3 - \frac{1}{12}x^{-3}\right)^2 \longleftarrow$$

YOU MUST do the algebra correctly! If you don't this problem will not work out!

$$(y')^2 = 9x^6 - \frac{1}{2} + \frac{1}{144}x^{-6}.$$

$$\left(3x^3 - \frac{1}{12}x^{-3}\right)\left(3x^3 - \frac{1}{12}x^{-3}\right)$$

$$= 9x^6 - \frac{1}{4} - \frac{1}{4} + \frac{1}{144}x^{-6}$$

So: $$s = \int_{x=1}^{x=2} \sqrt{1+(y')^2}\, dx$$

$$= \int_1^2 \sqrt{1 + 9x^6 - \frac{1}{2} + \frac{1}{144}x^{-6}}\, dx$$

$$= \int_1^2 \sqrt{9x^6 + \frac{1}{2} + \frac{1}{144}x^{-6}}\, dx$$

$$= \int_1^2 \sqrt{\left(3x^3 + \frac{1}{12}x^{-3}\right)^2}\, dx$$

$$= \int_1^2 \left(3x^3 + \frac{1}{12}x^{-3}\right) dx$$ since $3x^3 + \frac{1}{12}x^{-3} > 0$ on $1 \le x \le 2$.

$$= \left[\frac{3}{4}x^4 - \frac{1}{24}x^{-2}\right|_1^2$$

$$= \left[\frac{3}{4}(2)^4 - \frac{1}{24}(2)^{-2}\right] - \left[\frac{3}{4}(1)^4 - \frac{1}{24}(1)^{-2}\right]$$

$$\boxed{s = \frac{361}{32}}$$

4. $x = \frac{1}{6}\left[e^{3y} + e^{-3y}\right] = \frac{1}{6}e^{3y} + \frac{1}{6}e^{-3y}$

$x' = \frac{1}{2}e^{3y} - \frac{1}{2}e^{-3y}$

$(x')^2 = \frac{1}{4}e^{6y} - \frac{1}{2} + \frac{1}{4}e^{-3y} \leftarrow$ $\left(\frac{1}{2}e^{3y} - \frac{1}{2}e^{-3y}\right)\left(\frac{1}{2}e^{3y} - \frac{1}{2}\right)^{-3y} = \frac{1}{4}e^{3y} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}e^{-3y}$.

So: $s = \int_{y=0}^{y=1} \sqrt{1 + (x')^2}\, dy$

$$= \int_0^1 \sqrt{1 + \frac{1}{4}e^{6y} - \frac{1}{2} + \frac{1}{4}e^{-6y}}\, dy$$

$$= \int_0^1 \sqrt{\frac{1}{4}e^{6y} + \frac{1}{2} + \frac{1}{4}e^{6y}}\, dy$$

$$= \int_0^1 \sqrt{\left(\frac{1}{2}e^{3y} + \frac{1}{2}e^{-3y}\right)^2}\, dy$$

$$= \int_0^1 \left(\frac{1}{2}e^{3y} + \frac{1}{2}e^{-3y}\right) dy$$

$$= \left[\frac{1}{6}e^{3y} - \frac{1}{6}e^{-3y}\right|_0^1$$

$$= \left[\frac{1}{6}e^{3} - \frac{1}{6}e^{-3}\right] - \left[\frac{1}{6}e^{0} - \frac{1}{6}e^{0}\right].$$

$$= \boxed{\frac{1}{6}\left(e^3 - e^{-3}\right)}$$

II. Physical Applications

5 a) $m = \int_{x=0}^{x=L} \rho(x)\,dx = \int_0^L x\,dx$

$$= \frac{1}{2}x^2 \Big|_0^L$$

$$\boxed{m = \frac{1}{2}L^2}$$

b) If $m = 40$, then from a): $40 = \frac{1}{2}L^2$

$$80 = L^2$$

$$\boxed{L = \sqrt{80}} \text{ or } 4\sqrt{5}$$

6 a) $m = \int_{x=0}^{x=10} \rho(x)\,dx = \int_0^a 4\,dx + \int_a^{10} 4x^3\,dx$

$$= 4x\Big|_0^a + x^4\Big|_a^{10}$$

$$= 4a - 0 + (10^4 - a^4)$$

$$\boxed{m = 10000 + 4a - a^4}$$

b) When $a = 5$, $\rho(5) = 4(5)^3 = 625$. When $x < 5$, $\rho = 4$. Thus, the right side of the rod is significantly more massive, so a should be < 5.
To find a:

$$\int_0^a 4\,dx = \int_a^{10} 4x^3\,dx$$

$$4x\Big|_0^a = x^4\Big|_a^{10}$$

$$4a = 10000 - a^4$$

Using a calculator/computer:

$$a = 9.989,\ -10.01$$

While these are all acceptable values of a that solve the equation, the only value that is consistent to the confines of this problem is $\boxed{a = 9.989}$

7 a) $F = kx$

$F = 10(.5) = \boxed{50\text{ N}}$

b) $W = \int_0^{.5} kx\,dx = \int_0^{.5} 10x\,dx$

$= 5x^2 \Big|_0^{.5}$

$= 5(.5)^2 - 5(0)^2$

$\boxed{W = 1.25\text{ J}}$

c) $W = \int_{.5}^{1} kx\,dx = \int_{.5}^{1} 10x\,dx$

$= 5x^2 \Big|_{.5}^{1}$

$= 5(1)^2 - 5(.5)^2$

$= 5 - 1.25$

$\boxed{W = 3.75\text{ J}}$

8 a) First, find k:

$F = kx \Rightarrow 80 = k(2) \Rightarrow \underline{k = 40\text{ N/m}}$

Now: $W = \int_0^2 kx = \int_0^2 40x\,dx$

$= 20x^2 \Big|_0^2$

$= 20(2)^2 - 20(0)^2$

$\boxed{W = 80\text{ J}}$

b) $W = \int_2^4 kx\,dx = \int_2^4 40x\,dx$

$= 20x^2 \Big|_2^4$

$= 20(4)^2 - 20(2)^2$

$= 320 - 80$

$\boxed{W = 240\text{ J}}$

c) $W = \int_4^{4+a} kx\,dx$

$500 = \int_4^{4+a} 40x\,dx$

$500 = 20x^2 \Big|_4^{4+a}$

$500 = 20(4+a)^2 - 20(4)^2$

$25 = (4+a)^2 - 16$

$41 = (4+a)^2$

$\pm\sqrt{41} = 4 + a$

$a = -4 \pm \sqrt{41}$

$\boxed{a = -4 + \sqrt{41}}$

9 a) work to displace a meters: $W = \int_0^a kx\,dx$

b meters: $W = \int_0^b kx\,dx$

b) (Work from 0 to b) = (Work from 0 to a) + $\underbrace{\text{(work from a to b)}}_{}$ call this W

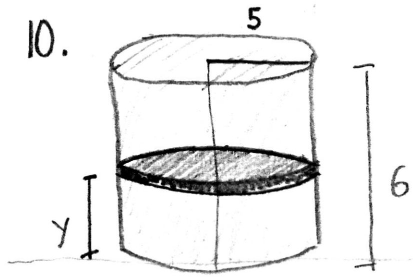
$$\int_0^b kx\,dx = \int_0^a kx\,dx + W$$

$$W = \int_0^b kx\,dx - \int_0^a kx\,dx$$

$$= \int_0^b kx\,dx + \int_a^0 kx\,dx$$

$$\boxed{W = \int_a^b kx\,dx}$$

10.



$b = 6$

$h = 6$

$A(y)$: area of a disc

$= \pi r^2$

$= \pi(5)^2$

$A(y) = 25\pi$

$$W = \int_{y=0}^{y=b} \rho g A(y)(h-y)\,dy$$

$$= \int_0^6 1000(9.8)\cdot 25\pi\,(6-y)\,dy$$

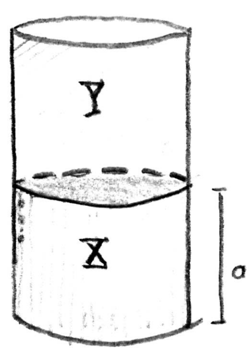
$$= 1000(9.8)\cdot 25\pi \int_0^6 (6-y)\,dy$$

$$= 1000(9.8)\cdot 25\pi \left[6y - \tfrac{1}{2}y^2\right|_0^6$$

$$= 1000(9.8)(25\pi)\left[6(6) - \tfrac{1}{2}(6)^2 - 0\right]$$

$$\boxed{W = 13854423.6\text{J}}$$

11.



$A(y) = \pi r^2$

$= \pi(5)^2$

$= 25\pi$

$$W = \int_0^{20} \rho g A(y)(h-y)\,dy$$

$$W = \underbrace{\int_0^a \rho_X g A(y)(20-y)\,dy}_{\text{work to move X out}} + \underbrace{\int_a^{20} \rho_Y g A(y)(20-y)\,dy}_{\text{work to move Y out}}$$

So we want $\int_0^a \rho_X g A(y)(20-y)\,dy = \int_a^{20} \rho_Y g A(y)(20-y)\,d$

$$\int_0^a 5000\,\cancel{g}\cdot\cancel{25\pi}\,(20-y)\,dy = \int_a^{20} 2000\,\cancel{g}\,\cancel{25\pi}\,(20-y)\,d$$

$$5000\int_0^a (20-y)\,dy = 2000\int_a^{20} 20-y\,dy$$

$$5\left[20y - \tfrac{1}{2}y^2\right|_0^a = 2\left[20y - \tfrac{1}{2}y^2\right|_a^{20}$$

$$5\left[20a - \tfrac{1}{2}a^2\right] - 0 = 2\left[20(20) - \tfrac{1}{2}(20)^2\right] - 2\left[20(a) - \tfrac{1}{2}a^2\right].$$

$$100a - \tfrac{5}{2}a^2 = 400 - 40a + a^2$$

$$0 = \tfrac{7}{2}a^2 - 140a + 400$$

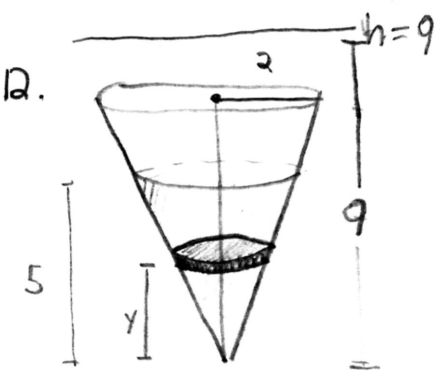
$$a = \frac{140 \pm \sqrt{19600 - 5600}}{7}$$

$$a = 20 \pm \tfrac{1}{7}\sqrt{14000}$$ ← Need to use "−" so $0 < a < 20$!

$$a = 20 - \tfrac{1}{7}\sqrt{14000}$$

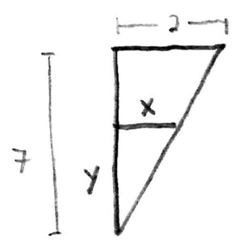
$$\boxed{\begin{aligned} a &= 20 - \tfrac{20}{7}\sqrt{35} \\ a &= 3.097 \end{aligned}}$$

12.



$$W = \int_{y=0}^{y=b} \rho g\, A(y)\,(h-y)\, dy.$$

$$W = \int_0^5 \rho g\, A(y)(9-y)\, dy.$$



$$A(y) = \pi r^2(y)$$
$$= \pi x^2$$
$$= \pi\left(\tfrac{2}{7}y\right)^2$$
$$A(y) = \tfrac{4\pi}{49}y^2$$

$$\frac{2}{7} = \frac{x}{y}$$

$$x = \tfrac{2}{7}y$$

$$W = \int_0^5 2700(9.8)\cdot \tfrac{4\pi}{49}\, y^2(9-y)\, dy.$$

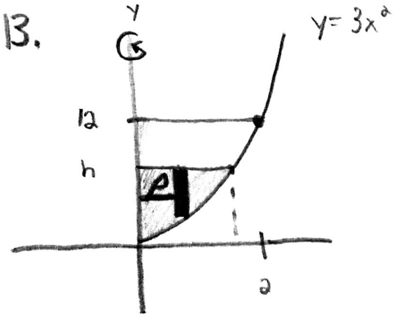
$$W = 2700(9.8)\,\tfrac{4\pi}{49}\int_0^5 (9y^2 - y^3)\, dy$$

$$= 2700(9.8)\,\tfrac{4\pi}{49}\left[3y^3 - \tfrac{1}{4}y^4\right|_0^5$$

$$= 2700(9.8)\,\tfrac{4\pi}{49}\left[3(5)^3 - \tfrac{1}{4}(5)^4 - 0\right].$$

$$\boxed{W = 1484401 \text{ J}}$$

13.



$h = 3x^2 \Rightarrow x = \sqrt{\frac{h}{3}}$

a)

$$V = 2\pi \int_{x=0}^{x=\sqrt{\frac{h}{3}}} \rho h \, dx$$

$$= 2\pi \int_0^{\sqrt{h/3}} x(h - 3x^2)\, dx$$

shell method!

$$= 2\pi \int_0^{\sqrt{h/3}} (hx - 3x^3)\, dx$$

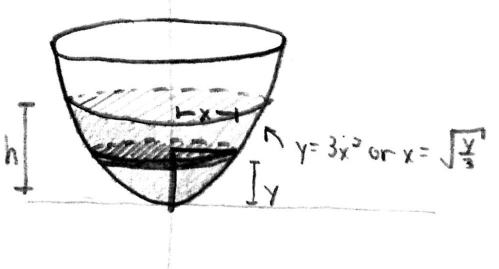
$$= 2\pi \cdot \left[\frac{1}{2}hx^2 - \frac{3}{4}x^4 \right|_0^{\sqrt{h/3}}$$

$$= 2\pi \left[\frac{1}{2}h\left(\frac{h}{3}\right) - \frac{3}{4}\left(\frac{h}{3}\right)^2\right]$$

$$= 2\pi \left[\frac{1}{6}h^2 - \frac{3}{4}\cdot\frac{h^2}{9_3}\right]$$

$$\boxed{V = \frac{\pi}{6}h^2}$$

b)



$$W = \int_0^b \rho g A(y)(h-y)\, dy$$

$$= \int_0^h 1000(9.8)\cdot\frac{\pi}{3}y(12-y)\, dy$$

$$= \frac{1000(9.8)\pi}{3}\int_0^h (12y - y^2)\, dy$$

$$= \frac{1000(9.8)\pi}{3}\left[6y^2 - \frac{1}{3}y^3\right|_0^h$$

$$\boxed{W = \frac{9800\pi}{3}\left[6h^2 - \frac{1}{3}h^3\right]}$$

$$A(y) = \pi r^2$$
$$= \pi\left(\sqrt{\tfrac{y}{3}}\right)^2$$
$$= \frac{\pi}{3}y$$

c) The tank is narrower at the bottom, but the distance the liquid must be moved is greater. We'll do the computations to find which effect is more important.

The work required to move the full tank is given when $h = 12$:

Full tank: $W = \frac{9800\pi}{3}\left[6(12)^2 - \frac{1}{3}(12)^3\right] =$

Now, set $W = \frac{1}{2}\left(\frac{9800\pi}{3}\left[6(12)^2 - \frac{1}{3}(12)^3\right]\right) = \frac{9800\pi}{3}\left[6h^2 - \frac{1}{3}h^3\right]$

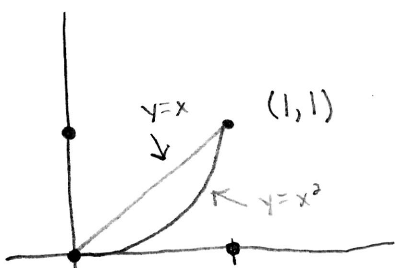
$$\Rightarrow \quad 144 = 6h^2 - \frac{1}{3}h^3$$

Use a calculator or a computer to find:

$$h = -4.392,\ 6,\ 16.392$$

We need $0 \le h \le 12$, so choose $\boxed{h=6}$.

14. The length of the curve segment of $y=x^2$ from $x=0$ to $x=1$ is longer;



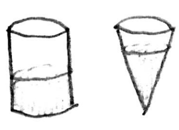
15a) $\rho_1 = \rho_2$; if the value of a is in the middle of the rod, the only way the rod to the left is the same mass as to the right is if $\rho_1 = \rho_2$.

b) $a < \frac{L}{2}$; if $\rho_1 > \rho_2$ the left end is more massive

16a) Equal; $F = kx$ so the force required to displace the spring 1m is equal no matter how far from equilibrium the spring starts.

b) Greater to stretch the additional meter

17 a) $h_2 > h_1$; there is more liquid in the cylinder's base. If $\rho_1 = \rho_2$, the cone must be filled higher.



b) $\rho_2 > \rho_1$; there is more liquid in the base of the cylinder. The density in the cone must be greater to compensate for this.