Worksheet #3

I. Arclength

1. Find the length of the segment of the curve $y = 1 + 5x^{3/2}$ from $x = 0$ to $x = 4$.

2. Find the length of the segment of the curve $x = \ln|\sec y|$ from $y = 0$ to $y = \frac{\pi}{4}$. (Hint: $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$).

3. Find the length of the segment of the curve $y = \frac{3}{4}x^4 + \frac{1}{24x^2}$ from $x = 1$ to $x = 2$.

4. Find the length of the segment of the curve $x = \frac{1}{6}\left[e^{3y} + e^{-3y}\right]$ from $y = 0$ to $y = 1$.

II. Physical Applications

5. A thin rod from $x = 0$ to $x = L$ has density $\rho(x) = x$.

   a) What is the mass of the rod?

   b) If the rod has mass 40, what is L?

6. A thin rod from $x = 0$ to $x = 10$ has density

$$\rho(x) = \begin{cases} 4, & 0 \le x \le a \\ 4x^3, & a \le x \le 10. \end{cases}$$

   a) Find the mass of the rod in terms of a.

   b) What value of a ensures that the mass of the portion of the rod left of $x = a$ has the same mass as the part of the rod right of a? Before you begin, deduce whether a should be greater than 5.

7. A spring with spring constant 10 N/m is stretched .5 m from its equilibrium position. Find:

a) the magnitude of the force required to do this

b) the work done to stretch the spring.

c) the work required to stretch the spring an additional .5m

8. A spring with spring constant k requires 80 N to stretch 2 m.

a) How much work is required to stretch the spring?

b) How much work is required to stretch it an additional 2m?

c) Suppose the spring (which is now 4 m from its equilibrium position) is stretched an additional a meters. What should a be so that the total work stretching the spring to $(4+a)$ m is 500 J?

9. (Motivation for $W = \int_a^b kx\,dx$). As shown in class, the amount of work required to stretch a spring a distance of d meters from its equilibrium position is given by:

$$W = \int_0^d kx\,dx.$$

Suppose the spring is displaced $a > 0$ meters from its equilibrium position and we want to find the additional work required to stretch it to $x = b > a$.

a) Write down the integrals that compute the total work required to displace the spring a meters and b (total) meters

b) Using the fact that:

(Work from 0 to b) = (Work from 0 to a) + (Work from a to b).

deduce that the additional work required to stretch the spring from $x=a$ to $x=b$ is:

$$W = \int_a^b kx\,dx.$$

10. A cylindrical tank has base diameter 10m and height 6 m. If the tank is filled with water ($\rho = 1000\ kg/m^3$), find the work required to pump the water out of the tank.

11. A cylindrical tank has base radius 5 m and height 20m. The tank is filled up to a height a with Liquid X ($\rho = 5000\ kg/m^3$) and then filled the rest of the way with Liquid Y ($\rho = 2000\ kg/m^3$). Assuming that the liquids do not mix, find a value for a so the work required to pump Liquid X from the tank is equal to the amount of work required to pump Liquid Y from the tank.

12. A conical tank has base radius 2 m and height 7m. If the tank is filled with liquid aluminum ($2700\ kg/m^3$) up to a height of 5 m, find the amount of work needed to bring the liquid 2 m above the tank.

13. A tank is formed by revolving the parabolic segment $y = 3x^2$ from $x=0$ to $x=2$ about the y-axis. (x in meters). The tank is filled to a height h meters with water. ($\rho = 1000\ kg/m^3$)

a) Find the volume of water in the tank in terms of h.

b) Find the work required to pump the liquid to the top of the tank.

c) What value of h will ensure the work required to pump the liquid out of the tank will be half of the work required to pump the liquid out of the tank if it were filled to the top? Should h be less than, equal to, or greater than half the height of the tank?

III. Conceptual Questions.

14. Is the length of the curve segment of $y = x^2$ from $x = 0$ to $x = 1$ greater than, equal to, or less than the curve $y = x$ from $x = 0$ to $x = 1$?

15. A thin rod of length L is made from 2 materials of const. density and its density is given by $\rho(x) = \begin{cases} \rho_1, & 0 \le x \le a \\ \rho_2, & a \le x \le L \end{cases}$

a) If $a = \frac{L}{2}$ and the mass of the rod to the left of $x = a$ is equal to the mass of the rod to the right of $x = a$, what can be said of ρ_1 and ρ_2?

b) Suppose now $\rho_1 > \rho_2$ and the mass to the left of $x = a$ is the same as the mass to the right. Is $a < \frac{1}{2}$, $a = \frac{1}{2}$, or $a > \frac{1}{2}$?

16. A spring requires 60 N of force to be applied to stretch it 1m from equilibrium. To stretch and additional 1m:

a) Is the force required less than, equal to, or greater than 60N?

b) Is the work required to stretch it the additional 1m less than, equal to, or greater than the work required to stretch it the first 1m?

17. A cylindrical tank and a conical tank both have height 1m and base radius 2m. The cylindrical tank is filled to h_1 meters with liquid X ($\rho = \rho_1$) and the conical tank is filled to h_2 m with Liquid Y ($\rho = \rho_2$)

Suppose the amount of work required to pump the liquid from each tank is the same.

a) If $\rho_1 = \rho_2$, is $h_1 > h_2$, $h_1 = h_2$, or $h_1 < h_2$?

b) If $h_1 = h_2$, is $\rho_1 > \rho_2$, $\rho_1 = \rho_2$, or $\rho_1 < \rho_2$?