What is Morley's Miracle?

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Overview

Let ΔABC be a triangle, and let the radian measure of these angles be $m(\measuredangle CAB) = 3\alpha$, $m(\measuredangle ABC) = 3\beta$, and $m(\measuredangle BCA) = 3\gamma$, so that $\alpha + \beta + \gamma = \pi/3$.[1] We trisect each (interior) angle of the triangle, and take the intersections of the *proximal* or adjacent trisectors; for example, to choose the intersection between a trisector of the angle at A and the angle at B, we choose the trisector from A closer to B, and the trisector from B closer to A. This gives us three points: one between a trisector from B and a trisector from C (labeled P), one between a trisector from C and a trisector from A (labeled Q) and one between a trisector from A and a trisector from B (labeled R). See Figure 1.

[1] For the duration of this article, all angle measure will be in radians.

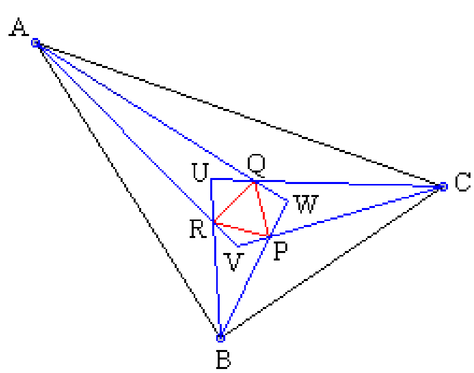

Figure 1: The (First) Morley Triangle, ΔPQR. Source: [2]

Theorem 0.1 (Morley's Theorem) *ΔPQR is equilateral.*

This theorem has many, many proofs and extensions ([8] has over 100 references), which would make fine material for a future "What is . . . ?" talk.

1 Basic Trigonometric Proof

Let ΔABC be the triangle, with $m(\measuredangle CAB) = 3\alpha$, $m(\measuredangle ABC) = 3\beta$, and $m(\measuredangle BCA) = 3\gamma$, so that $\alpha + \beta + \gamma = \pi/3$. Our ultimate goal is to create formulae for the side-lengths PR that is symmetric in α, β, and γ, so that the corresponding constructions for PQ and QR, using the corresponding sides, will give the same answer; therefore, the side-lengths of ΔPQR will be equal.[2] Our first goal is to construct expressions for the side lengths BC, CA, AB in term of α, β, γ respectively. To do so, we use the circumcircle of the triangle.

Fact 1.1 *There is a circumcircle of ΔABC; that is, a circle touching the triangle ΔABC at only the three points A, B, C, and such that ΔABC is contained inside the given circle.*[3]

Having constructed the circumcircle with center O and radius r, we might as well scale the whole figure by the factor of $\frac{1}{r}$, so that the rescaled circle has radius 1. We know that $m(\angle BCA) = 3\gamma$, and this angle encompasses the arc $\overset{\frown}{AB}$. By basic circle geometry, the measure of a central angle encompassing an arc is twice the measure of an angle encompassing the same arc and whose vertex is placed on the circumference of the circle; in other words,

$$m(\angle BOA) = 2m(\angle BCA) = 6\gamma.$$

See Figure 2. By the Law of Cosines,

$$(AB)^2 = (AO)^2 + (AB)^2 - 2(AO)(AB)\cos(m(\angle BOA)).$$

Yet by our scaling, the circle has radius 1, and AO and AB are radii; hence,

$$\begin{aligned}
(AB)^2 &= 1 + 1 - 2\cos(6\gamma) \\
&= 4\left(\frac{1-\cos^2(3\gamma)}{2}\right) \\
&= 4\sin^2(3\gamma) \text{ (by the Half-Angle Formula)} \\
AB &= 2\sin(3\gamma) \text{ (since } AB \geq 0 \text{ and } 3\gamma < \pi).
\end{aligned}$$

[2]This section is adapted from [2].

[3]This assumes that the points are not collinear.

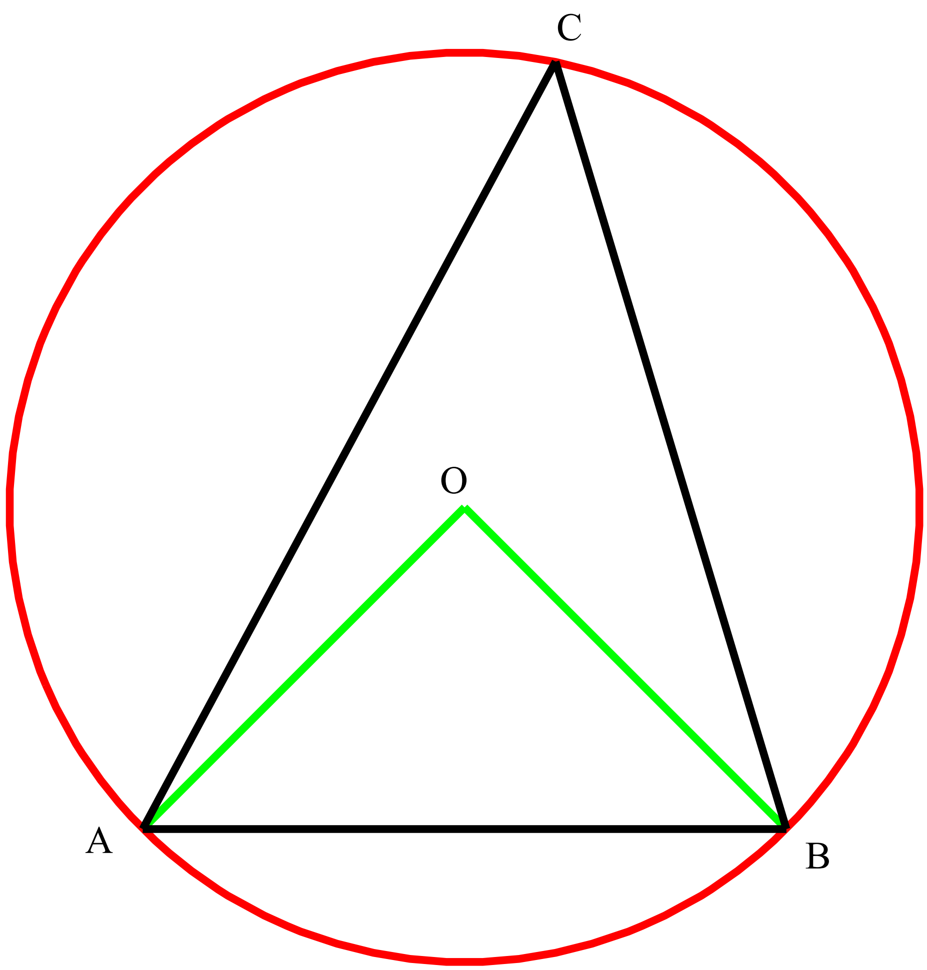

Figure 2: Central Angle of the Circumcircle

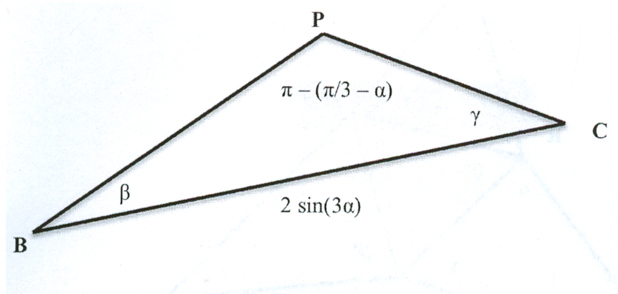

Figure 3: ΔBPC

Thus, the side opposite $\angle BCA$ has measure $2\sin(3\gamma) = 2\sin(m(\measuredangle BCA))$; similarly, we have

$$\begin{aligned} BC &= 2\sin(3\alpha) = 2\sin(m(\measuredangle CAB)) \\ AC &= 2\sin(3\beta) = 2\sin(m(\measuredangle ABC)) \end{aligned}$$

Thus, we have the side-lengths of the triangle in terms of the angles of the triangle. Referring back to Figure 1, our next goal is to find expressions for the lengths of the intermediate sides BP, CP, BR, etc. To start, we will concentrate our attention on ΔBPC. See Figure 3. By the Law of Sines,

$$\frac{BP}{\sin(\gamma)} = \frac{BC}{\sin(m(\measuredangle BPC))}.$$

Yet by BP and CP trisectors of angles $\angle ABC$, $\angle BCA$, respectively, we have that $m(\measuredangle PBC) = \beta$ and $m(\measuredangle BCP) = \gamma$, so $m(\measuredangle BPC) = \pi - \beta - \gamma$. Yet since $\sin(\pi - \delta) = \sin(\delta)$ and $\alpha + \beta + \gamma = \frac{\pi}{3}$, we have that

$$\begin{aligned} \frac{BP}{\sin(\gamma)} &= \frac{BC}{\sin(\pi - \beta - \gamma)} \\ \frac{BP}{\sin(\gamma)} &= \frac{2\sin(3\alpha)}{\sin(\beta + \gamma)} \\ \frac{BP}{\sin(\gamma)} &= \frac{2\sin(3\alpha)}{\sin(\pi/3 - \alpha)} \\ BP &= \frac{2\sin(3\alpha)\sin(\gamma)}{\sin(\pi/3 - \alpha)} \end{aligned}$$

This is a difficult expression, but it can be simplified by unpacking $\sin(3\alpha)$. We use the triple-angle formula, then factorize and use the expression $\cos^2(\alpha) + \sin^2(\alpha) = 1$ in reverse.

$$
\begin{aligned}
\sin(3\alpha) &= 3\sin(\alpha) - 4\sin^3(\alpha) \\
&= 4\sin(\alpha)\left[\frac{3}{4} - \sin^2(\alpha)\right] \\
&= 4\sin(\alpha)\left[\frac{3}{4}(\cos^2(\alpha) + \sin^2(\alpha)) - \sin^2(\alpha)\right] \\
&= 4\sin(\alpha)\left[\frac{3}{4}\cos^2(\alpha) - \frac{1}{4}\sin^2(\alpha)\right] \\
&= 4\sin(\alpha)\left[\left(\frac{\sqrt{3}}{2}\cos(\alpha)\right)^2 - \left(\frac{1}{2}\sin(\alpha)\right)^2\right]
\end{aligned}
$$

We may now use difference-of-squares factorization, and get

$$4\sin(\alpha)\left(\frac{\sqrt{3}}{2}\cos(\alpha) + \frac{1}{2}\sin(\alpha)\right)\left(\frac{\sqrt{3}}{2}\cos(\alpha) - \frac{1}{2}\sin(\alpha)\right)$$

Yet note that $\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$ and $\frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$. Therefore, the above formula becomes

$$4\sin(\alpha)\left(\sin\left(\frac{\pi}{3}\right)\cos(\alpha) + \cos\left(\frac{\pi}{3}\right)\sin(\alpha)\right)\left(\sin\left(\frac{\pi}{3}\right)\cos(\alpha) - \cos\left(\frac{\pi}{3}\right)\sin(\alpha)\right)$$

. Using the sum and difference formulas for sine, this becomes

$$4\sin(\alpha)\sin\left(\frac{\pi}{3} + \alpha\right)\sin\left(\frac{\pi}{3} - \alpha\right)$$

. Plugging back into our formula for BP, we cancel the $\sin(\pi/3 - \alpha)$ term in the denominator and get

$$BP = 8\sin(\alpha)\sin\left(\frac{\pi}{3} + \alpha\right)\sin(\gamma). \tag{1}$$

By treating ΔBRA in similar fashion to ΔBPC,[4], we get that

$$BR = 8\sin(\gamma)\sin\left(\frac{\pi}{3} + \gamma\right)\sin(\alpha). \tag{2}$$

We now have expressions for the sides BP and BR; similarly, we can find equations for CP, CQ, AQ, and AR. We are now ready to tackle the next-innermost layer of subtriangles, namely ΔBPR, ΔCQP, and ΔARQ. For example, for ΔBPR, consider Figure 4. Sides BP and BR have already been discussed, and by the Law of Cosines,

$$(PR)^2 = (BP)^2 + (BR)^2 - 2(BP)(BR)\cos(m(\measuredangle PBR)).$$

[4]Note that we choose this ordering of the vertices–we use the “reflective” rather than the “rotational” correspondence

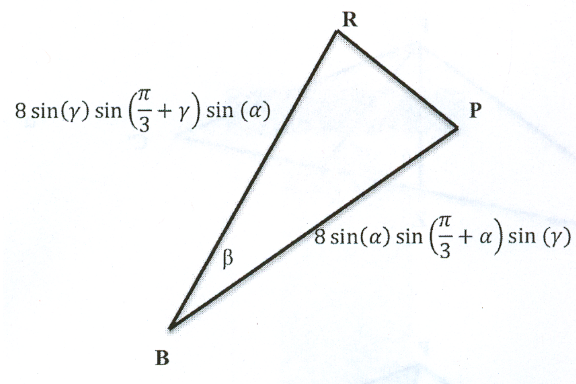

Figure 4: ΔBPR

Yet by BP and BR trisectors of $\angle ABC$, we see that $m(\measuredangle PBR) = \beta$. Therefore, we get

$$(PR)^2 = 64\sin^2(\alpha)\sin^2(\gamma)\left[\sin^2(\pi/3+\alpha)+\sin^2(\pi/3+\gamma)-2\sin(\pi/3+\alpha)\sin(\pi/3+\gamma)\cos(\beta)\right]$$

This is the side length of one of the sides of our proposed equilateral triangle, which should be an expression symmetric in α, β, and γ if we are going to use the corresponding constructions in ΔCQP and ΔARQ to find the lengths of $\overline{QP}$ and $\overline{RQ}$, respectively. To see that we indeed do have symmetry, however hidden, in the above expression, note that the term in square brackets in Equation 1 involves the angles $(\pi/3+\alpha)$, $(\pi/3+\gamma)$, and β, and that $(\pi/3+\alpha)+(\pi/3+\gamma)+\beta = \pi$. Therefore, we can construct an auxillary triangle XYZ such that $m(\measuredangle ZXY) = (\pi/3+\alpha)$, $m(\measuredangle XYZ) = \beta$, and $m(\measuredangle YZX) = (\pi/3+\gamma)$. As in the first part of the proof, we take the circumcircle of XYZ and scale so that the radius of the circumcircle is 1. Then as before,

$$\begin{aligned} YZ &= 2\sin\left(\frac{\pi}{3}+\alpha\right) \\ XZ &= 2\sin(\beta) \\ XY &= 2\sin\left(\frac{\pi}{3}+\gamma\right) \end{aligned}$$

Therefore, by one more application of the law of cosines, we have

$$\begin{aligned} (XZ)^2 &= (YZ)^2+(XZ)^2-2*(YZ)(XZ)\cos(\theta) \\ 4\sin^2(\beta) &= 4\left[\sin^2\left(\frac{\pi}{3}+\alpha\right)+\sin^2\left(\frac{\pi}{3}+\gamma\right)-2\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}+\gamma\right)\cos(\beta)\right] \\ \sin^2(\beta) &= \sin^2\left(\frac{\pi}{3}+\alpha\right)+\sin^2\left(\frac{\pi}{3}+\gamma\right)-2\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}+\gamma\right)\cos(\beta) \end{aligned}$$

Thus, the bracketed expression in Equation 1 is equal to $\sin^2(\beta)$, so we get $(PR)^2 = 64\sin^2(\alpha)\sin^2(\beta)\sin^2(\gamma)$ and hence

$$PR = 8\sin(\alpha)\sin(\beta)\sin(\gamma), \tag{3}$$

since PR is postitve, and since the sines of the various constants are positive by α, β, and γ being in $[0, \pi]$.

Thus, using the corresponding logic for the triple of triangles ΔCQA, ΔCPB, and ΔCQP, we have that PQ has the same value as PR; similarly, ΔARB, ΔAQC, and ΔARQ ensure that QR has the same value as PR. Thus, ΔPQR is triangular.

2 John Conway's Proof: only Euclidean Geometry

To motivate John Conway's proof, we note that from the given information, we can actually make a guess at to what all the angle measures are. For example, by the Law of Sines,

$$\begin{aligned}
\frac{PR}{\sin(\beta)} &= \frac{BR}{\sin(m(\measuredangle BPR))} \\
8\sin(\alpha)\sin(\gamma) &= \frac{8\sin(\gamma)\sin(\alpha)\sin(\pi/3+\gamma)}{\sin(m(\measuredangle BPR))} \\
\sin(m(\measuredangle BPR)) &= \sin(\pi/3+\gamma),
\end{aligned}$$

so it is plausible that $m(\measuredangle BPR) = \frac{\pi}{3}+\gamma$. Similarly, it is plausible that $m(\measuredangle BRP) = \frac{\pi}{3}+\alpha$. If these statements actually hold, we can attempt to replace some of the numerical calculations of the Laws of Sines and Cosines by similarity and congruence arguments based solely on the angle measures and equality of certain sides. This indeed is what we do; but the interesting part of Conway's proof is that we work backwards, starting with an equilateral triangle and building the other triangles outward. The following proof is adapted from [3] and [6].

First, one piece of notation: if δ is a number (usually an angle measure), $\delta^* := \delta + \frac{\pi}{3}$ and $\delta^{**} := \delta + \frac{2\pi}{3}$. Note that an equilateral triangle has angles all $\frac{\pi}{3} = 0^*$.

Proof. Fix α, β, γ such that $\alpha+\beta+\gamma = \frac{\pi}{3}$. Our objective is to show that for any triangle with angle-measures 3α, 3β, and 3γ, the Morley Triangle is equilateral; since for any triangle the appropriate values of α, β, γ are discovered by dividing the corresponding angle-measures by three, this suffices.

Yet by $\alpha+\beta+\gamma = \frac{\pi}{3}$, if we distribute two stars on the left-hand-side, we will increase the sum of the terms to π, and hence the three numbers can be the angle measures of a triangle. Since we can choose to distribute the two stars to a single term or to two different terms, we find six triples of numbers that can be the angles of a triangle. Considering the equilateral triangle, with angles equal to

0^*, we have seven such triples[5]

$$\begin{array}{ccc} & 0^*, 0^*, 0^* & \\ \alpha, \beta^*, \gamma^* & \alpha^*, \beta, \gamma^* & \alpha^*, \beta^*, \gamma \\ \alpha^{**}, \beta, \gamma & \alpha, \beta^{**}, \gamma & \alpha, \beta, \gamma^{**} \end{array}$$

Our objective is to show that any triangle with angle measures 3α, 3β, and 3γ is built out of seven triangles, one from each of the classes given above. To make sure all the sizes match up, however, we must determine a particular size of each triangle, reducing the angle-only information to congruence information. It suffices to determine the length of one side; then by the ASA Congruence theorem, any triangle with the same angle measures and the same corresponding side length is congruent to the model triangles we shall determine. As the above table suggests, we have three types of triangle.

The equilateral triangle Just set it to have side-length 1.

Two single-starred angle measures Set the triangles so that the side opposite the unstarred angle (by the decomposition stated) has side-length one.

One double-starred angle measures We handle the case of angle measures $\alpha^{**}, \beta, \gamma$, and the rest are handled similarly. We denote this triangle $\Delta PB'C'$, because it will eventually take the place of ΔPBC in our proof, so that $m(\measuredangle B'PC') = \alpha^{**}$, $m(\measuredangle PB'C') = \beta$, and $m(\measuredangle B'C'P) = \gamma$. Fix Y on the line $\overleftrightarrow{B'C'}$ such that $m(\measuredangle B'YP) = \alpha^*$; in other words, Y is the point on the line $\overleftrightarrow{B'C'}$ passing through P and making an angle of α^* with the line segment $\overline{B'C'}$, when viewed from B.[6] Similarly, let Z be the point on the line $\overline{B'C'}$ such that $m(\measuredangle B'Z') = \alpha^*$. See Figure 5. If $\alpha^* = \pi/2$, then $Y = Z$, and of course $PY = PZ$; for all other angles, $P'YZ$ forms an isoceles triangle with repeated angle α^* if $\alpha^* < \pi/2$, and $(\pi - \alpha^*)$ if $\alpha* > \pi/2$. Therefore, $PY = PZ$ in this case as well. We scale the triangle so that this length $PY = PZ$ is always 1.

Now we assemble the seven triangles into a complete triangle. First, take the equilateral triangle with side-length 1 to be triangle ΔPQR. Then, we attach the three triangles with two single-starred angle-measures, so that the following conditions hold.

[5] Table adapted from [3].

[6] Y lies inside the segment $\overline{B'C'}$, since $\alpha^* > \pi/3 > \gamma$ and $\alpha^* < \frac{2\pi}{3} < \pi - \beta$, so that the line YP must be more orthogonal than BP and CP. This helps visualization, but is not actually necessary for the proof.

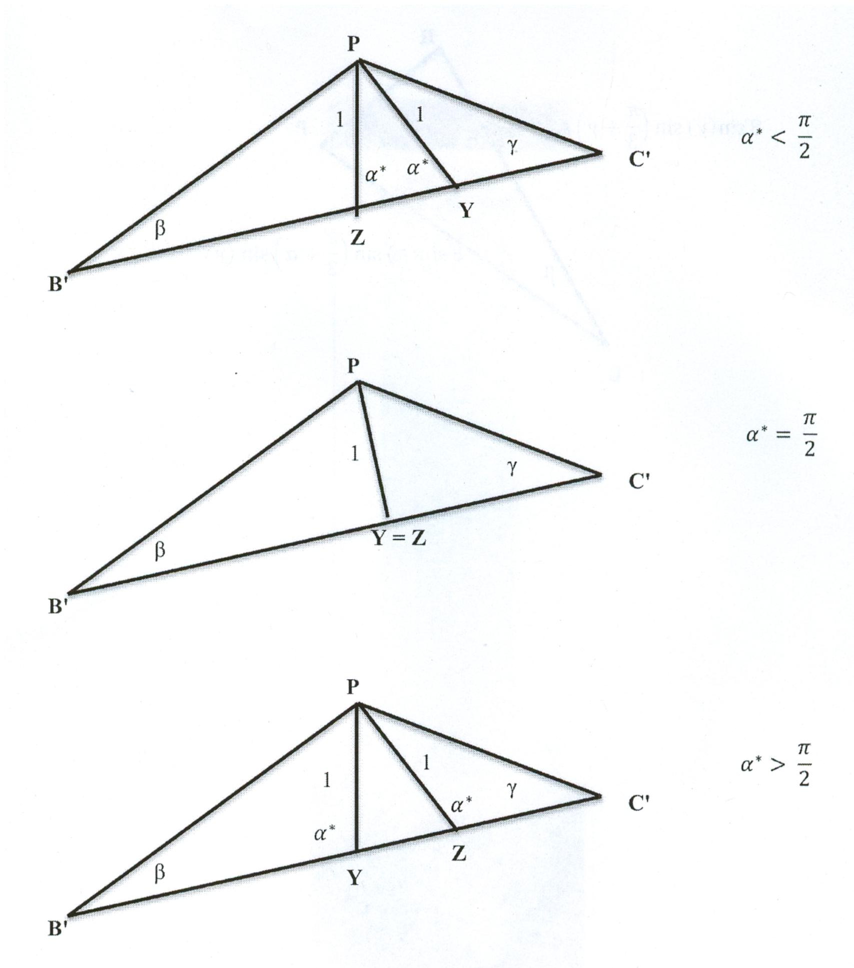

Figure 5: $\Delta PB'C'$ with auxillary points Y, Z.

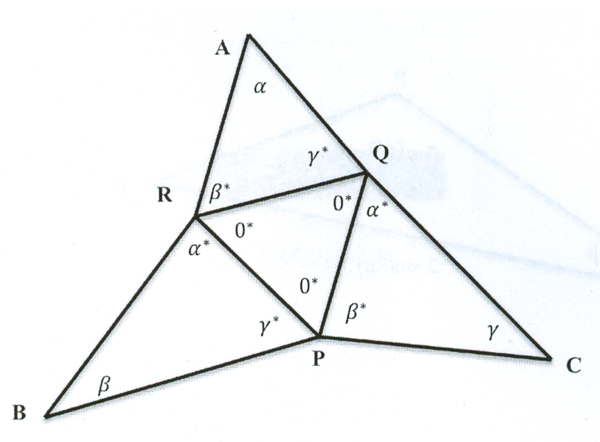

Figure 6: Central Star of ΔABC

- Each triangle's length-1 side is a side of ΔPQR.
- The triangles only overlap at the common edges or vertices (elementary inequalities with sums of angle-measures ensure that this is possible).
- The triangle with angle-measures $\alpha, \beta^*, \gamma^*$ is placed opposite P and shares side QR, and its third angle is labeled A, so that $m(\measuredangle QAR) = \alpha$, $m(\measuredangle AQR) = \gamma^*$, $m(\measuredangle ARQ) = \beta^*$.
- The triangle with angle-measures $\alpha^*, \beta, \gamma^*$ is placed opposite Q and shares side PR, and its third angle is labeled B, so that $m(\measuredangle PBR) = \beta$, $m(\measuredangle BPR) = \gamma^*$, $m(\measuredangle BRP) = \alpha^*$.
- The triangle with angle-measures $\alpha^*, \beta^*, \gamma$ is placed opposite R and shares side PQ, and its third angle is labeled C, so that $m(\measuredangle PCQ) = \alpha$, $m(\measuredangle CPQ) = \beta^*$, $m(\measuredangle CQP) = \alpha^*$.

A picture is clearest; see Figure 6.

Now, our objective is to show that the other three triangles "fill in" the picture. To do so, we first calculate the sum of the angles at P: $m(\angle RPQ) = \pi/3$ by ΔRPQ equilateral, and by the above, $m(\measuredangle BPR) = \gamma^*$, and $m(\measuredangle CPQ) = \beta^*$. The total of these angle sums is $\pi + \beta + \gamma = \frac{4\pi}{3} - \alpha$, so that the remaining angle measure is $2\pi - \left(\frac{4\pi}{3} - \alpha\right) = \frac{2\pi}{3} + \alpha = \alpha^{**}$. Therefore, the remaining opening exactly accomodates the triangle with angle-measurements $\alpha^{**}, \beta, \gamma$, so we can find B' on ray $\overrightarrow{PB}$ and C' on ray $\overrightarrow{PC}$ such that $m(\measuredangle PB'C') = \beta$ and $m(\measuredangle PC'B') = \gamma$. Thus, the "model triangle" $PB'C'$ slides into our figure by angle considerations, with B' and C' on the rays $\overrightarrow{PB}$ and $\overrightarrow{PC}$, respectively. We wish to show that the point B' actually is B and the point C' actually is C, so that the sides match up.

To do so, we take advantage of the points Y and Z that we constructed; recall that $\angle PB'Y = \angle PB'C'$, so that $m(\measuredangle PB'Y) = \beta$, and $m(\measuredangle PYB') = \alpha^*$ by the definition of Y. By ΔPBY a triangle, $m(\measuredangle B'PY) = \pi - \beta - \alpha^* = \pi/3 + (\pi/3 - \beta - \alpha) = \pi/3 + \gamma = \gamma^*$. Therefore, $\Delta B'PY$ has angle measures $\alpha^*, \beta, \gamma^*$, and hence is similar to ΔBPR, since they share the same angle-measures. Further, in both triangles, the side opposite the angle with measure β has length 1; in ΔBPR this was by direct statement, and in $\Delta B'PY$ we defined $PY = 1$. Therefore, $\Delta B'PY \cong \Delta BPR$, and hence, since congruent sides of congruent figures are equal, $B'P = BP$. Yet B' and B are both on the same ray emanating from P, so $B' = B$.

Similar logic with triangles $\Delta C'PZ$ and ΔCPQ assures us that $C' = C$. Thus, $\Delta PB'C' = \Delta PBC$, so that the triangle with the correct side-lengths to fill in the picture also has the desired angle measure. Similarly, ΔCQA is congruent to our model triangle with angle-measures $\alpha, \beta^{**}, \gamma$, and ΔARB is congruent to our model triangle with angle-measures $\alpha, \beta, \gamma^{**}$.

Therefore, we can finally consider the triangle ABC. This triangle does have angle-measures $3\alpha, 3\beta$, and 3γ; for example, since $\angle ABC = \angle ABR + \angle RBP + \angle PBC$, $m(\measuredangle ABC) = \beta + \beta + \beta = 3\beta$. Therefore, we also have that BP and BR are the trisectors of the (interior) angle $\angle ABC$. Similarly, AQ and AR are the trisectors of $\angle CAB$, and CP and CQ are the trisectors of $\angle BCA$. The proximal intersections of the trisectors are P, Q, and R, and ΔPQR is equilateral by definition. Therefore, at least for this triangle, the trisectors' proximal intersections form an equilateral triangle.

If we have any other triangle $A^\dagger B^\dagger C^\dagger$ with the same angle-measures as ABC, let λ be the linear scaling constant: i.e., $\lambda := \frac{AB}{A^\dagger B^\dagger}$. Note that by taking the trisectors of this new triangle, with corresponding labeling, each subtriangle is similar to the corresponding triangle. This is easiest to see for the three outer triangles ARB, BPC and CQA and their counterparts $A^\dagger R^\dagger B^\dagger$, etc., since each has two angles that are trisectors of corresponding angles of the original triangle. Since AB, BC, and AC scale with the same scaling constant, then, for example, $\frac{AR}{A^\dagger R^\dagger} = \lambda$. With that knowledge, we can get that ΔAQR is similar to triangle $\Delta A^\dagger Q^\dagger R^\dagger$ with the same proportionality constant λ by the SAS similarity rule, since $m(\measuredangle QAR) = m(\measuredangle Q^\dagger A^\dagger R^\dagger) = \alpha$, and $\frac{AR}{A^\dagger R^\dagger} = \frac{AQ}{A^\dagger Q^\dagger} = \lambda$. Thus, $\frac{QR}{Q^\dagger R^\dagger} = \frac{AR}{A^\dagger R^\dagger} = \lambda$. Similar work with the triangles CQP and BPR, and their associated triangles, shows that $\frac{PQ}{P^\dagger Q^\dagger} = \frac{PR}{P^\dagger R^\dagger} = \frac{QR}{Q^\dagger R^\dagger} = \lambda$; since $PQ = PR = QR = 1$, $P^\dagger Q^\dagger = P^\dagger R^\dagger = Q^\dagger R^\dagger = \frac{1}{\lambda}$, so the new triangle is equilateral.

Therefore, for any triangle, we have that the (first) Morley triangle is equilateral.

■

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