What is Morley’s Miracle?

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Overview

Let \( \triangle ABC \) be a triangle, and let the radian measure of these angles be \( m(\angle CAB) = 3\alpha, \) \( m(\angle ABC) = 3\beta, \) and \( m(\angle BCA) = 3\gamma, \) so that \( \alpha + \beta + \gamma = \pi/3. \) We trisect each (interior) angle of the triangle, and take the intersections of the \textit{proximal} or adjacent trisectors; for example, to choose the intersection between a trisector of the angle at \( A \) and the angle at \( B, \) we choose the trisector from \( A \) closer to \( B, \) and the trisector from \( B \) closer to \( A. \) This gives us three points: one between a trisector from \( B \) and a trisector from \( C \) (labeled \( P \)), one between a trisector from \( C \) and a trisector from \( A \) (labeled \( Q \)) and one between a trisector from \( A \) and a trisector from \( B \) (labeled \( R \)). See Figure 1.

\[\text{Figure 1: The (First) Morley Triangle, } \triangle PQR. \text{ Source: [2]}\]
Theorem 0.1 (Morley’s Theorem) \( \triangle PQR \) is equilateral.

This theorem has many, many proofs and extensions ([8] has over 100 references), which would make fine material for a future “What is . . . ?” talk.

1 Basic Trigonometric Proof

Let \( \triangle ABC \) be the triangle, with \( m(\angle CAB) = 3\alpha \), \( m(\angle ABC) = 3\beta \), and \( m(\angle BCA) = 3\gamma \), so that \( \alpha + \beta + \gamma = \pi/3 \). Our ultimate goal is to create formulae for the side-lengths \( PR \) that is symmetric in \( \alpha \), \( \beta \), and \( \gamma \), so that the corresponding constructions for \( PQ \) and \( QR \), using the corresponding sides, will give the same answer; therefore, the side-lengths of \( \triangle PQR \) will be equal. Our first goal is to construct expressions for the side lengths \( BC \), \( CA \), \( AB \) in terms of \( \alpha \), \( \beta \), \( \gamma \) respectively. To do so, we use the circumcircle of the triangle.

**Fact 1.1** There is a circumcircle of \( \triangle ABC \); that is, a circle touching the triangle \( \triangle ABC \) at only the three points \( A \), \( B \), \( C \), and such that \( \triangle ABC \) is contained inside the given circle.

Having constructed the circumcircle with center \( O \) and radius \( r \), we might as well scale the whole figure by the factor of \( \frac{1}{r} \), so that the rescaled circle has radius 1. We know that \( m(\angle BCA) = 3\gamma \), and this angle encompasses the arc \( \overarc{AB} \). By basic circle geometry, the measure of a central angle encompassing an arc is twice the measure of an angle encompassing the same arc and whose vertex is placed on the circumference of the circle; in other words,

\[
m(\angle BOA) = 2m(\angle BCA) = 6\gamma.
\]

See Figure 2. By the Law of Cosines,

\[
(AB)^2 = (AO)^2 + (AB)^2 - 2(AO)(AB) \cos(m(\angle BOA)).
\]

Yet by our scaling, the circle has radius 1, and \( AO \) and \( AB \) are radii; hence,

\[
(AB)^2 = 1 + 1 - 2 \cos(6\gamma)
\]

\[
= 4 \left( \frac{1 - \cos^2(3\gamma)}{2} \right)
\]

\[
= 4 \sin^2(3\gamma) \text{ (by the Half-Angle Formula)}
\]

\[
AB = 2 \sin(3\gamma) \text{ (since } AB \geq 0 \text{ and } 3\gamma < \pi).\]

---

\(^2\)This section is adapted from [2].

\(^3\)This assumes that the points are not collinear.
Figure 2: Central Angle of the Circumcircle
Thus, the side opposite $\angle BCA$ has measure $2 \sin(3\gamma) = 2 \sin(m(\angle BCA))$; similarly, we have

$$BC = 2 \sin(3\alpha) = 2 \sin(m(\angle CAB))$$
$$AC = 2 \sin(3\beta) = 2 \sin(m(\angle ABC))$$

Thus, we have the side-lengths of the triangle in terms of the angles of the triangle. Referring back to Figure 1, our next goal is to find expressions for the lengths of the intermediate sides $BP$, $CP$, $BR$, etc. To start, we will concentrate our attention on $\triangle BPC$. See Figure 3. By the Law of Sines,

$$\frac{BP}{\sin(\gamma)} = \frac{BC}{\sin(m(\angle BPC))}.$$ 

Yet by $BP$ and $CP$ trisectors of angles $\angle ABC$, $\angle BCA$, respectively, we have that $m(\angle PBC) = \beta$ and $m(\angle BCP) = \gamma$, so $m(\angle BPC) = \pi - \beta - \gamma$. Yet since $\sin(\pi - \delta) = \sin(\delta)$ and $\alpha + \beta + \gamma = \frac{\pi}{3}$, we have that

$$\frac{BP}{\sin(\gamma)} = \frac{BC}{\sin(\pi - \beta - \gamma)}$$
$$\frac{BP}{\sin(\gamma)} = \frac{2 \sin(3\alpha)}{\sin(\beta + \gamma)}$$
$$\frac{BP}{\sin(\gamma)} = \frac{2 \sin(3\alpha)}{\sin(\pi/3 - \alpha)}$$
$$BP = \frac{2 \sin(3\alpha) \sin(\gamma)}{\sin(\pi/3 - \alpha)}$$

This is a difficult expression, but it can be simplified by unpacking $\sin(3\alpha)$. We use the triple-angle formula, then factorize and use the expression $\cos^2(\alpha) + \sin^2(\alpha) = 1$ in reverse.
\[
\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha) = 4\sin(\alpha) \left[ \frac{3}{4} - \sin^2(\alpha) \right]
\]

\[
= 4\sin(\alpha) \left[ \frac{3}{4}(\cos^2(\alpha) + \sin^2(\alpha)) - \sin^2(\alpha) \right]
\]

\[
= 4\sin(\alpha) \left[ \frac{3}{4} \cos^2(\alpha) - \frac{1}{4} \sin^2(\alpha) \right]
\]

\[
= 4\sin(\alpha) \left[ \left( \frac{\sqrt{3}}{2} \cos(\alpha) \right)^2 - \left( \frac{1}{2} \sin(\alpha) \right)^2 \right]
\]

We may now use difference-of-squares factorization, and get

\[
4\sin(\alpha) \left( \frac{\sqrt{3}}{2} \cos(\alpha) + \frac{1}{2} \sin(\alpha) \right) \left( \frac{\sqrt{3}}{2} \cos(\alpha) - \frac{1}{2} \sin(\alpha) \right)
\]

Yet note that \( \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) \) and \( \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \). Therefore, the above formula becomes

\[
4\sin(\alpha) \left( \sin\left(\frac{\pi}{3}\right) \cos(\alpha) + \cos\left(\frac{\pi}{3}\right) \sin(\alpha) \right) \left( \sin\left(\frac{\pi}{3}\right) \cos(\alpha) - \cos\left(\frac{\pi}{3}\right) \sin(\alpha) \right)
\]

Using the sum and difference formulas for sine, this becomes

\[
4\sin(\alpha) \sin\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{3} - \alpha\right)
\]

Plugging back into our formula for \( BP \), we cancel the \( \sin(\pi/3 - \alpha) \) term in the denominator and get

\[
BP = 8\sin(\alpha) \sin\left(\frac{\pi}{3} + \alpha\right) \sin(\gamma).
\]

(1)

By treating \( \Delta BRA \) in similar fashion to \( \Delta BPC \) we get that

\[
BR = 8\sin(\gamma) \sin\left(\frac{\pi}{3} + \gamma\right) \sin(\alpha).
\]

(2)

We now have expressions for the sides \( BP \) and \( BR \); similarly, we can find equations for \( CP, CQ, AQ, \) and \( AR \). We are now ready to tackle the next-innermost layer of subtriangles, namely \( \Delta BPR, \Delta CQP, \) and \( \Delta ARQ \). For example, for \( \Delta BPR \), consider Figure \[4\] Sides \( BP \) and \( BR \) have already been discussed, and by the Law of Cosines,

\[
(PR)^2 = (BP)^2 + (BR)^2 - 2(BP)(BR)\cos(m(\angle PBR)).
\]

\[4\]Note that we choose this ordering of the vertices—we use the “reflective” rather than the “rotational” correspondence
Yet by \( BP \) and \( BR \) trisectors of \( \angle ABC \), we see that \( m(\angle PBR) = \beta \). Therefore, we get

\[
(PR)^2 = 64\sin^2(\alpha) \sin^2(\gamma) \left[ \sin^2\left(\frac{\pi}{3} + \alpha\right) + \sin^2\left(\frac{\pi}{3} + \gamma\right) - 2\sin\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{3} + \gamma\right) \cos(\beta) \right]
\]

This is the side length of one of the sides of our proposed equilateral triangle, which should be an expression symmetric in \( \alpha, \beta, \) and \( \gamma \) if we are going to use the corresponding constructions in \( \triangle CQP \) and \( \triangle ARQ \) to find the lengths of \( QP \) and \( RQ \), respectively. To see that we indeed do have symmetry, however hidden, in the above expression, note that the term in square brackets in Equation 1 involves the angles \( \left(\frac{\pi}{3}+\alpha\right), \left(\frac{\pi}{3}+\gamma\right), \) and \( \beta \), and that \( \left(\frac{\pi}{3}+\alpha\right) + \left(\frac{\pi}{3}+\gamma\right) + \beta = \pi \). Therefore, we can construct an auxiliary triangle \( XYZ \) such that \( m(\angle ZXY) = \left(\frac{\pi}{3}+\alpha\right), \) \( m(\angle XYZ) = \beta \), and \( m(\angle YZX) = \left(\frac{\pi}{3}+\gamma\right) \). As in the first part of the proof, we take the circumcircle of \( XYZ \) and scale so that the radius of the circumcircle is 1. Then as before,

\[
\begin{align*}
YZ &= 2\sin\left(\frac{\pi}{3} + \alpha\right) \\
XZ &= 2\sin(\beta) \\
XY &= 2\sin\left(\frac{\pi}{3} + \gamma\right)
\end{align*}
\]

Therefore, by one more application of the law of cosines, we have

\[
\begin{align*}
(XZ)^2 &= (YZ)^2 + (XZ)^2 - 2 * (YZ)(XZ) \cos(\theta) \\
4\sin^2(\beta) &= 4 \left[ \sin^2\left(\frac{\pi}{3} + \alpha\right) + \sin^2\left(\frac{\pi}{3} + \gamma\right) - 2\sin\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{3} + \gamma\right) \cos(\beta) \right] \\
\sin^2(\beta) &= \sin^2\left(\frac{\pi}{3} + \alpha\right) + \sin^2\left(\frac{\pi}{3} + \gamma\right) - 2\sin\left(\frac{\pi}{3} + \alpha\right) \sin\left(\frac{\pi}{3} + \gamma\right) \cos(\beta)
\end{align*}
\]

Thus, the bracketed expression in Equation 1 is equal to \( \sin^2(\beta) \), so we get

\[
(PR)^2 = 64\sin^2(\alpha) \sin^2(\beta) \sin^2(\gamma) \quad \text{and hence}
\]

\[
PR = 8\sin(\alpha) \sin(\beta) \sin(\gamma), \quad (3)
\]
since \( PR \) is positive, and since the sines of the various constants are positive by \( \alpha, \beta, \) and \( \gamma \) being in \([0, \pi]\).

Thus, using the corresponding logic for the triple of triangles \( \triangle CQA, \triangle CPB, \) and \( \triangle CQP \), we have that \( PQ \) has the same value as \( PR \); similarly, \( \triangle ARB, \triangle AQC, \) and \( \triangle ARQ \) ensure that \( QR \) has the same value as \( PR \). Thus, \( \triangle PQR \) is triangular.

2 John Conway’s Proof: only Euclidean Geometry

To motivate John Conway’s proof, we note that from the given information, we can actually make a guess at to what all the angle measures are. For example, by the Law of Sines,

\[
\frac{PR}{\sin(\beta)} = \frac{BR}{\sin(m(\angle BPR))} = \frac{8 \sin(\alpha) \sin(\gamma)}{8 \sin(\gamma) \sin(\alpha) \sin(\pi/3 + \gamma)} = \frac{\sin(m(\angle BPR))}{\sin(\pi/3 + \gamma)},
\]

so it is plausible that \( m(\angle BPR) = \frac{\pi}{3} + \gamma \). Similarly, it is plausible that \( m(\angle BRP) = \frac{\pi}{3} + \alpha \). If these statements actually hold, we can attempt to replace some of the numerical calculations of the Laws of Sines and Cosines by similarity and congruence arguments based solely on the angle measures and equality of certain sides. This indeed is what we do; but the interesting part of Conway’s proof is that we work backwards, starting with an equilateral triangle and building the other triangles outward. The following proof is adapted from [3] and [6].

First, one piece of notation: if \( \delta \) is a number (usually an angle measure), \( \delta^* := \delta + \frac{\pi}{3} \) and \( \delta^{**} := \delta + \frac{2\pi}{3} \). Note that an equilateral triangle has angles all \( \frac{\pi}{3} = 0^* \).

**Proof.** Fix \( \alpha, \beta, \gamma \) such that \( \alpha + \beta + \gamma = \frac{\pi}{3} \). Our objective is to show that for any triangle with angle-measures \( 3\alpha, 3\beta, \) and \( 3\gamma \), the Morley Triangle is equilateral; since for any triangle the appropriate values of \( \alpha, \beta, \gamma \) are discovered by dividing the corresponding angle-measures by three, this suffices.

Yet by \( \alpha + \beta + \gamma = \frac{\pi}{3} \), if we distribute two stars on the left-hand-side, we will increase the sum of the terms to \( \pi \), and hence the three numbers can be the angle measures of a triangle. Since we can choose to distribute the two stars to a single term or to two different terms, we find six triples of numbers that can be the angles of a triangle. Considering the equilateral triangle, with angles equal to
0*, we have seven such triples:\footnote{Table adapted from \cite{3}.}

\[
\begin{array}{ccc}
0*,0*,0* & \alpha*,\beta*,\gamma* & \alpha*,\beta*,\gamma \\
\alpha**,\beta,\gamma & \alpha,\beta**,\gamma & \alpha,\beta,\gamma**
\end{array}
\]

Our objective is to show that any triangle with angle measures $3\alpha$, $3\beta$, and $3\gamma$ is built out of seven triangles, one from each of the classes given above. To make sure all the sizes match up, however, we must determine a particular size of each triangle, reducing the angle-only information to congruence information. It suffices to determine the length of one side; then by the ASA Congruence theorem, any triangle with the same angle measures and the same corresponding side length is congruent to the model triangles we shall determine. As the above table suggests, we have three types of triangle.

The equilateral triangle Just set it to have side-length 1.

Two single-starred angle measures Set the triangles so that the side opposite the unstarred angle (by the decomposition stated) has side-length one.

One double-starred angle measures We handle the case of angle measures $\alpha**$, $\beta$, $\gamma$, and the rest are handled similarly. We denote this triangle $\triangle PB'C'$, because it will eventually take the place of $\triangle PBC$ in our proof, so that $m(\angle B'PC') = \alpha**$, $m(\angle PB'C') = \beta$, and $m(\angle B'C'P) = \gamma$. Fix $Y$ on the line $\overrightarrow{B'C'}$ such that $m(\angle B'YP) = \alpha*$; in other words, $Y$ is the point on the line $\overrightarrow{B'C'}$ passing through $P$ and making an angle of $\alpha*$ with the line segment $\overrightarrow{BC}$, when viewed from $B$\footnote{Y lies inside the segment $\overrightarrow{BC}$, since $\alpha* > \pi/3 > \gamma$ and $\alpha* < \frac{\pi}{3} < \pi - \beta$, so that the line $YP$ must be more orthogonal than $BP$ and $CP$. This helps visualization, but is not actually necessary for the proof.}. Similarly, let $Z$ be the point on the line $\overrightarrow{B'C'}$ such that $m(\angle B'Z') = \alpha*$. See Figure\footnote{Figure\footref{fig:5}.} 5. If $\alpha* = \pi/2$, then $Y = Z$, and of course $PY = PZ$; for all other angles, $P'YZ$ forms an isosceles triangle with repeated angle $\alpha*$ if $\alpha* < \pi/2$, and $(\pi - \alpha*)$ if $\alpha* > \pi/2$. Therefore, $PY = PZ$ in this case as well. We scale the triangle so that this length $PY = PZ$ is always 1.

Now we assemble the seven triangles into a complete triangle. First, take the equilateral triangle with side-length 1 to be triangle $\triangle PQR$. Then, we attach the three triangles with two single-starred angle-measures, so that the following conditions hold.

\begin{align*}
\end{align*}
Figure 5: $\Delta PB'C'$ with auxiliary points $Y, Z$. 
• Each triangle’s length-1 side is a side of ∆PQR.

• The triangles only overlap at the common edges or vertices (elementary inequalities with sums of angle-measures ensure that this is possible).

• The triangle with angle-measures α, β*, γ* is placed opposite P and shares side QR, and its third angle is labeled A, so that \( m(\angle QAR) = \alpha, m(\angle AQR) = \gamma^*, m(\angle ARQ) = \beta^* \).

• The triangle with angle-measures α*, β, γ* is placed opposite Q and shares side PR, and its third angle is labeled B, so that \( m(\angle PBR) = \beta, m(\angle BPR) = \gamma^*, m(\angle BRP) = \alpha^* \).

• The triangle with angle-measures α*, β*, γ is placed opposite R and shares side PQ, and its third angle is labeled C, so that \( m(\angle PCQ) = \alpha, m(\angle CPQ) = \beta^*, m(\angle CQP) = \alpha^* \).

A picture is clearest; see Figure 6.

Now, our objective is to show that the other three triangles “fill in” the picture. To do so, we first calculate the sum of the angles at P: \( m(\angle RPQ) = \pi/3 \) by ∆RPQ equilateral, and by the above, \( m(\angle BPR) = \gamma^* \), and \( m(\angle CPQ) = \beta^* \). The total of these angle sums is \( \pi + \beta + \gamma = \frac{4\pi}{3} - \alpha \), so that the remaining angle measure is \( 2\pi - (\frac{4\pi}{3} - \alpha) = \frac{2\pi}{3} + \alpha = \alpha^{**} \). Therefore, the remaining opening exactly accommodates the triangle with angle-measurements \( \alpha^{**}, \beta, \gamma \), so we can find \( B' \) on ray \( \overrightarrow{PB} \) and \( C' \) on ray \( \overrightarrow{PC} \) such that \( m(\angle PB'C') = \beta \) and \( m(\angle PC'B') = \gamma \). Thus, the “model triangle” \( PB'C' \) slides into our figure by angle considerations, with \( B' \) and \( C' \) on the rays \( \overrightarrow{PB} \) and \( \overrightarrow{PC} \), respectively. We wish to show that the point \( B' \) actually is \( B \) and the point \( C' \) actually is \( C \), so that the sides match up.
To do so, we take advantage of the points $Y$ and $Z$ that we constructed; recall that $\angle PBY = \angle P'BY'$, so that $m(\angle P'BY) = \beta$, and $m(\angle PYB) = \alpha^*$ by the definition of $Y$. By $\Delta PBY$ a triangle, $m(\angle B'YP) = \pi - \beta - \alpha^* = \pi/3 + (\pi/3 - \beta - \alpha) = \pi/3 + \gamma = \gamma^*$. Therefore, $\Delta B'PY$ has angle measures $\alpha^*, \beta, \gamma^*$, and hence is similar to $\Delta BPR$, since they share the same angle-measures. Further, in both triangles, the side opposite the angle with measure $\beta$ has length 1; in $\Delta BPR$ this was by direct statement, and in $\Delta B'PY$ we defined $PY = 1$. Therefore, $\Delta B'PY \cong \Delta BPR$, and hence, since congruent sides of congruent figures are equal, $B'P = BP$. Yet $B'$ and $B$ are both on the same ray emanating from $P$, so $B' = B$.

Similar logic with triangles $\Delta C'PZ$ and $\Delta CPQ$ assures us that $C' = C$. Thus, $\Delta PB'C' = \Delta PBC$, so that the triangle with the correct side-lengths to fill in the picture also has the desired angle measure. Similarly, $\Delta CQA$ is congruent to our model triangle with angle-measures $\alpha, \beta^*, \gamma$, and $\Delta ARB$ is congruent to our model triangle with angle-measures $\alpha, \beta, \gamma^**$.

Therefore, we can finally consider the triangle $\triangle ABC$. This triangle does have angle-measures $3\alpha, 3\beta$, and $3\gamma$; for example, since $\angle ABC = \angle ABR + \angle RBP + \angle PBC$, $m(\angle ABC) = \beta + \beta + \beta = 3\beta$. Therefore, we also have that $BP$ and $BR$ are the trisectors of the (interior) angle $\angle ABC$. Similarly, $AQ$ and $AR$ are the trisectors of $\angle CAB$, and $CP$ and $CQ$ are the trisectors of $\angle BCA$. The proximal intersections of the trisectors are $P, Q,$ and $R$, and $\Delta PQR$ is equilateral by definition. Therefore, at least for this triangle, the trisectors’ proximal intersections form an equilateral triangle.

If we have any other triangle $\triangle A'B'C'$ with the same angle-measures as $\triangle ABC$, let $\lambda$ be the linear scaling constant: i.e., $\lambda := \frac{AB}{A'B'}$. Note that by taking the trisectors of this new triangle, with corresponding labeling, each subtriangle is similar to the corresponding triangle. This is easiest to see for the three outer triangles $\triangle ARB, BPC$ and $\triangle CQA$ and their counterparts $\triangle A'R'B', etc., since each has two angles that are trisectors of corresponding angles of the original triangle.

Since $AB$, $BC$, and $AC$ scale with the same scaling constant, then, for example, $\frac{AR}{A'R'} = \lambda$. With that knowledge, we can get that $\Delta AQR$ is similar to triangle $\Delta A'R'B'$ with the same proportionality constant $\lambda$ by the SAS similarity rule, since $m(\angle QAR) = m(\angle Q'A'R') = \alpha$, and $\frac{AR}{A'R'} = \frac{AQ}{A'Q'} = \lambda$. Thus, $\frac{QR}{Q'R'} = \frac{AR}{A'R'} = \lambda$. Similar work with the triangles $\triangle CQP$ and $\triangle BPR$, and their associated triangles, shows that $\frac{PQ}{P'Q'} = \frac{PR}{P'R'} = \frac{QR}{Q'R'} = \lambda$; since $PQ = PR = QR = 1$, $\frac{PQ}{P'Q'} = \frac{PR}{P'R'} = \frac{QR}{Q'R'} = 1$, so the new triangle is equilateral.

Therefore, for any triangle, we have that the (first) Morley triangle is equilateral.
References


